Introduction to Mathematical Finance

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Subsection 1

First-Order Stochastic Dominance

Stochastic Domination

- Let X and Y be random variables.
- X **stochastically dominates** Y, or X is **stochastically larger than** Y, written as $X \geq_{\text{st}} Y$, if, for all t,

$$P(X > t) \geq P(Y > t).$$

- That is, $X \geq_{\text{st}} Y$ if for every constant t, it is at least as likely that X will exceed t as it is that Y will.

  Remark: Because a probability is always a continuous function on events, an equivalent definition would be that $X \geq_{\text{st}} Y$ if

$$P(X \geq t) \geq P(Y \geq t), \quad \text{for all } t.$$

Characterization of Stochastic Dominance: Lemma 1

Lemma

If X is a nonnegative random variable, then

$$E[X] = \int_0^\infty P(X > t)dt.$$

- For $t > 0$, define the random variable $I(t)$ by

$$I(t) = \begin{cases} 1, & \text{if } t < X, \\ 0, & \text{if } t \geq X. \end{cases}$$

Now, $\int_0^\infty I(t)dt = \int_0^X I(t)dt + \int_X^\infty I(t)dt = X$.
Consequently,

$$E[X] = E\left[\int_0^\infty I(t)dt\right] = \int_0^\infty E[I(t)]dt = \int_0^\infty P(X > t)dt.$$

Characterization of Stochastic Dominance: Lemma 2

Lemma

If $X \geq_{\text{st}} Y$, then $E[X] \geq E[Y]$.

- Suppose first that X and Y are nonnegative random variables.

  By the preceding lemma and the definition of stochastic dominance,

  $$E[X] = \int_0^\infty P(X > t)dt \geq \int_0^\infty P(Y > t)dt = E[Y].$$

  Hence, the result is true when the random variables are nonnegative.

  To prove the result in general, note that any number a can be expressed as the difference of its positive and negative parts: $a = a^+ - a^-$, where $a^+ = \max(a, 0)$, $a^- = \max(-a, 0)$.

  This is a consequence of the fact that:
  - If $a \geq 0$, then $a^+ = a$ and $a^- = 0$;
  - If $a < 0$, then $a^+ = 0$ and $a^- = -a$.

Proof of Lemma 2 (Cont'd)

- Assume that $X \geq_{\text{st}} Y$.
  Express X, Y as the difference of their positive and negative parts:
  $$X = X^+ - X^- \quad \text{and} \quad Y = Y^+ - Y^-$$
  Now, for any $t \geq 0$,
  $$\begin{aligned} P(X^+ > t) &= P(X > t) \geq P(Y > t) = P(Y^+ > t); \\ P(X^- > t) &= P(-X > t) = P(X < -t) \leq P(Y < -t) \\ &= P(-Y > t) = P(Y^- > t). \end{aligned}$$
  Hence, $X^+ \geq_{\text{st}} Y^+$ and $X^- \leq_{\text{st}} Y^-$.
  These random variables are all nonnegative.
  So we have $E[X^+] \geq E[Y^+]$ and $E[X^-] \leq E[Y^-]$.
  The result now follows because
  $$E[X] = E[X^+] - E[X^-] \geq E[Y^+] - E[Y^-] = E[Y].$$

Characterization of Stochastic Dominance

Proposition

$X \geq_{\text{st}} Y$ if and only if $E[h(X)] \geq E[h(Y)]$ for all increasing functions h.

- Suppose that $X \geq_{\text{st}} Y$ and that h is an increasing function.
  To show that $E[h(X)] \geq E[h(Y)]$, we first show that $h(X) \geq_{\text{st}} h(Y)$.
  Since h is increasing, for any t, there is some value, say $h^{-1}(t)$, such that $h(X) > t$ is equivalent to either $X \geq h^{-1}(t)$ or $X > h^{-1}(t)$.
  Assume there is a unique value y such that $h(y) = t$.
  Then the latter case holds and $y = h^{-1}(t)$.
  Assuming the latter case, we have

  $$P(h(X) > t) = P(X > h^{-1}(t)) \geq P(Y > h^{-1}(t)) = P(h(Y) > t).$$

  A similar argument holds if $h(X) > t$ is equivalent to $X \geq h^{-1}(t)$.
  Therefore, $h(X) \geq_{\text{st}} h(Y)$.
  By the preceding lemma, $E[h(X)] \geq E[h(Y)]$.

Characterization of Stochastic Dominance (Converse)

- Conversely, assume $E[h(X)] \geq E[h(Y)]$, for all increasing functions h.

  For fixed t, define the function h_t by

$$h_t(x) = \begin{cases} 0, & \text{if } x \leq t, \\ 1, & \text{if } x > t. \end{cases}$$

  Then $h_t(x)$ is increasing.

  So we have

$$E[h_t(X)] \geq E[h_t(Y)].$$

  But $E[h_t(X)] = P(X > t)$ and $E[h_t(Y)] = P(Y > t)$.

  This shows that that $X \geq_{\text{st}} Y$.

Subsection 2

Using Coupling to Show Stochastic Dominance

Using Coupling to Show Stochastic Dominance

- One way to show that $X \geq_{\text{st}} Y$ is to find random variables X' and Y' such that:
  - X' has the same distribution as X;
  - Y' has the same distribution as Y;
  - It is always the case that $X' \geq Y'$.
- Assume that we have found such random variables.

  Then $Y' > t$ implies that $X' > t$. So $P(Y' > t) \leq P(X' > t)$.

  But

  $$P(X' > t) = P(X > t) \quad \text{and} \quad P(Y' > t) = P(Y > t).$$

  Therefore, $X \geq_{\text{st}} Y$.
- This method of establishing that one random variable is stochastically larger than another is called **coupling**.

Example

- a Poisson random variable is stochastically increasing in its mean.
- That is, a Poisson random variable with mean $\lambda_1 + \lambda_2$ is stochastically larger than a Poisson random variable with mean λ_1 when $\lambda_i > 0$, $i = 1, 2$.
- For a Poisson random variable X with mean λ,

$$P(X \geq j) = \sum_{i=j}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!}.$$

Direct verification that it is increasing in λ for any j is not easy.

So we use coupling.

Let X_1, X_2 be independent Poisson with means λ_1, λ_2.

The sum of independent Poisson random variables is also Poisson.

So $X_1 + X_2$ is Poisson with mean $\lambda_1 + \lambda_2$.

But $X_1 + X_2 \geq X_1$. So the result follows by coupling.

Continuous Distribution and Uniform Random Variable

Lemma

If F is a continuous distribution function and U a uniform $(0,1)$ random variable, then the random variable $F^{-1}(U)$ has distribution function F, where $F^{-1}(u)$ is defined to be that value such that $F(F^{-1}(u)) = u$.

- A distribution function is increasing.

  So the inequalities $a \leq x$ and $F(a) \leq F(x)$ are equivalent.

  Hence,

$$\begin{aligned} P(F^{-1}(U) \leq x) &= P(F(F^{-1}(U)) \leq F(x)) \\ &= P(U \leq F(x)) \\ &= F(x). \end{aligned}$$

Stochastic Dominance and Coupling

Proposition

If $X \geq_{\text{st}} Y$, then there are random variables X' having the same distribution as X, and Y' having the same distribution as Y, such that $X' \geq Y'$.

- Assume that X and Y are continuous, with respective distribution functions F and G, and that $X \geq_{\text{st}} Y$.

  $X \geq_{\text{st}} Y$ implies that $F(x) \leq G(x)$, for all x.

  So we get

  $$F(G^{-1}(u)) \leq G(G^{-1}(u)) = u = F(F^{-1}(u)).$$

  Since F is increasing, we get $G^{-1}(u) \leq F^{-1}(u)$.

  Let U be uniform $(0, 1)$ and set $X' = F^{-1}(U)$ and $Y' = G^{-1}(U)$.

  We showed $X' \geq Y'$. The result follows from the preceding lemma.

An Application

Theorem

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be vectors of independent random variables, and suppose that $X_i \geq_{\text{st}} Y_i$ for each $i = 1, \ldots, n$. If $g(x_1, \ldots, x_n)$ is increasing in each component,

$$g(X_1, \ldots, X_n) \geq_{\text{st}} g(Y_1, \ldots, Y_n).$$

- Let $g(x_1, \ldots, x_n)$ be increasing in each component.
  For $i = 1, \ldots, n$, let:
  - F_i be the distribution function of X_i;
  - G_i be the distribution function of Y_i.

An Application (Cont'd)

- Let $U_1, \ldots, U_n$ be independent uniform $(0,1)$ random variables. Set, for all $i = 1, \ldots, n$,

$$X_i' = F_i^{-1}(U_i) \quad \text{and} \quad Y_i' = G_i^{-1}(U_i).$$

  We have $X_i' \geq Y_i'$, for all i.

  So $g(X_1', \ldots, X_n') \geq g(Y_1', \ldots, Y_n')$.

  By the proposition, for all i, we have:
  - X_i' has the same distribution as X_i;
  - Y_i' has the same distribution as Y_i.

  It follows that:
  - $g(X_1', \ldots, X_n')$ has the same distribution as $g(X_1, \ldots, X_n)$;
  - $g(Y_1', \ldots, Y_n')$ has the same distribution as $g(Y_1, \ldots, Y_n)$.

  Now the result follows by coupling.

Subsection 3

Likelihood Ratio Ordering

Likelihood Ratio Ordering

- Suppose X and Y are continuous random variables, with:
  - X having density function f;
  - Y having density function g.
- We say that X is **likelihood ratio larger than** Y if $\frac{f(x)}{g(x)}$ is increasing in x over the region where either $f(x)$ or $g(x)$ is greater than 0.
- Suppose, next, that X and Y are discrete random variables.
- We say that X is **likelihood ratio larger than** Y if $\frac{P(X=x)}{P(Y=x)}$ is increasing in x over the region where either $P(X=x)$ or $P(Y=x)$ is greater than 0.

Likelihood Ratio Ordering and Stochastic Order

Proposition

If X is likelihood ratio larger than Y, then X is stochastically larger than Y.

- Let X and Y have probability density functions f and g.
  Suppose that $\frac{f(x)}{g(x)} \uparrow x$.
  We must show that, for any a, $\int_{x>a} f(x)dx \geq \int_{x>a} g(x)dx$.
  There are two cases:
  Case 1 $f(a) \geq g(a)$: If $x > a$, then $\frac{f(x)}{g(x)} \geq \frac{f(a)}{g(a)} \geq 1$. Hence, if $x \geq a$, $f(x) \geq g(x)$. This gives the result.
  Case 2 $f(a) < g(a)$: If $x \leq a$ then $\frac{f(x)}{g(x)} \leq \frac{f(a)}{g(a)} < 1$. So we get $\int_{x\leq a} f(x)dx < \int_{x\leq a} g(x)dx$. This gives
  $$\int_{x>a} f(x)dx = 1 - \int_{x\leq a} f(x)dx \geq 1 - \int_{x\leq a} g(x)dx = \int_{x>a} g(x)dx.$$

Tilted Density Function

- Let X be a random variable with density function $f(x)$.
- Define
$$C = \frac{1}{\int e^{ty} f(y) dy}.$$
- The t-**tilted density with regard to** f is the density function f_t given by
$$f_t(x) = Ce^{tx} f(x).$$

Tilted Density Function and Likelihood Ratio Ordering

- Let X be a random variable with density function $f(x)$.
- Note that
$$\frac{f_t(x)}{f(x)} = \frac{e^{tx}}{\int e^{ty} f(y) dy}$$
is:
  - Increasing in x when $t > 0$;
  - Decreasing in x when $t < 0$.
- It follows that a random variable X_t having density function f_t is:
  - Likelihood ratio (and thus also stochastically) larger than X when $t > 0$;
  - Likelihood ratio (and thus also stochastically) smaller than X when $t < 0$.

Subsection 4

A Single-Period Investment Problem

A Single-Period Investment Problem

- Suppose we have an initial fortune w.
- We must decide on an amount y, $0 \leq y \leq w$, to invest.
- After one period, an investment of size y returns the amount

$$yX + (1 + r)(w - y),$$

  where
  - X is a nonnegative random variable having a known distribution;
  - r is a specified interest rate earned by the uninvested amount.
- Let u be an increasing, concave utility function.
- We maximize the expected utility of the end-of-period wealth.
- That is, with $\beta = 1 + r$, the objective is to find

$$M = \max_{0 \leq y \leq w} E[u(yX + \beta(w - y))].$$

Maximizing the Expected Utility

- Suppose X is a continuous random variable with density f.

$$\begin{aligned} M &= \max_{0 \le y \le w} E[u((X-\beta)y+\beta w)] \\ &= \max_{0 \le y \le w} \int_{-\infty}^{\infty} u((x-\beta)y+\beta w) f(x)dx. \end{aligned}$$

- Differentiating the term inside the maximum yields that

$$\begin{aligned} &\frac{d}{dy}\int_0^\infty u((x-\beta)y+\beta w)f(x)dx \\ &= \int_0^\infty u'((x-\beta)y+\beta w)(x-\beta)f(x)dx \\ &= \int_0^\infty h(y,x)f(x)dx, \end{aligned}$$

where $h(y,x) = u'((x-\beta)y+\beta w)(x-\beta)$.

- So the maximizing value y_f of y is such that

$$\int_0^\infty h(y_f,x)f(x)dx = 0.$$

Properties of $h(y,x)$

Lemma

For fixed x, $h(y,x)$ is decreasing in y. In addition:

- $h(y,x) \leq 0$, if $x \leq \beta$;
- $h(y,x) \geq 0$, if $x \geq \beta$.

Case 1 $x \leq \beta$: We have the following implications.

$$\begin{aligned} x \leq \beta \quad & \Rightarrow \quad (x-\beta)y + \beta w \downarrow y \\ & \Rightarrow \quad u'((x-\beta)y+\beta w) \uparrow y \quad (u \text{ concave} \Rightarrow \ u'(v) \downarrow v) \\ & \Rightarrow \quad h(y,x) = (x-\beta)u'((x-\beta)y+\beta w) \downarrow y. \end{aligned}$$

We also have:

- $x - \beta \leq 0$;
- $u' \geq 0$ (since u is increasing).

It follows that $h(y,x) \leq 0$.

Properties of $h(y,x)$ (Case 2)

Case 2 $x \geq \beta$: We have the following implications.

$$\begin{array}{rcl} x \geq \beta & \Rightarrow & (x-\beta)y + \beta w \uparrow y \\ & \Rightarrow & u'((x-\beta)y+\beta w) \downarrow y \quad (u'(v) \downarrow v) \\ & \Rightarrow & h(y,x) = (x-\beta)u'((x-\beta)y+\beta w) \downarrow y. \end{array}$$

We also have:

- $x - \beta \geq 0$;
- $u' \geq 0$ (since u is increasing).

It follows that $h(y,x) \geq 0$.

Comparison Between Investments

- Now consider two scenarios for an investor with initial wealth w.
  - The multiplicative random variable is X_1 with density function f;
  - The multiplicative random variable is X_2, with density function g.
- We explore conditions on f and g under which the optimal amount invested in the first scenario is at least as large as the optimal amount invested in the second scenario, for every increasing, concave utility function.
- Equivalently, we want to ensure $y_f \geq y_g$.
- An initial guess may be that X_1 being stochastically larger than X_2 is sufficient.
- This, however, is not the case as is shown by the following example.

Stochastic Comparison and Size of Investment

- Suppose the utility function is
$$u(x) = \begin{cases} x, & \text{if } x \leq 100 \\ 100, & \text{if } x > 100 \end{cases}.$$
- Assume that:
  - $P(X_1 = 4) = P(X_1 = 0) = \frac{1}{2}$;
  - $P(X_2 = 3) = P(X_2 = 0) = \frac{1}{2}$.
- Then X_1 is stochastically larger than X_2.
- Suppose, further, that:
  - The initial wealth is $w = 30$;
  - The interest rate is $r = 0$.
- The optimal amount to invest in the X_1 factor problem $\leq \frac{70}{3}$. Investing more than $\frac{70}{3}$ would yield the same utility value (of 100) as investing $\frac{70}{3}$ if $X_1 = 4$ and a smaller utility if $X_1 = 0$.
- The optimal amount to invest in the X_2 factor problem is 30.

Likelihood Ratio and Size of Investment

- Having a stochastically larger investment return factor does not necessarily imply that a larger amount should be invested.
- We show that this result is true when the investment returns are likelihood ratio ordered.

Theorem

If f and g are density functions of nonnegative random variables, for which $\frac{f(x)}{g(x)}$ increases in x, then $y_f \geq y_g$. That is, when f is a likelihood ratio ordered larger density than g, then the optimal amount to invest when the multiplicative factor has density f is larger than when it has density g.

- We know that the optimal amount to invest is:
  - y_g, such that $\int_0^\infty h(y_g, x)g(x)dx = 0$, if X has density g;
  - y_f, such that $\int_0^\infty h(y_f, x)f(x)dx = 0$, if X has density f.

  We would like to show $y_f \geq y_g$.

Likelihood Ratio and Size of Investment (Cont'd)

- We know $h(y,x)$ is decreasing in y.

  So $y_f \geq y_g$ is equivalent to

$$\int_0^\infty h(y_g,x)f(x)dx \geq \int_0^\infty h(y_f,x)f(x)dx.$$

  So it suffices to prove that $\int_0^\infty h(y_g,x)f(x)dx \geq 0$.

  We have

$$\int_0^\infty h(y_g,x)f(x)dx = \int_0^\beta h(y_g,x)f(x)dx + \int_\beta^\infty h(y_g,x)f(x)dx.$$

  We distinguish two cases according to whether $x \leq \beta$ or $x \geq \beta$.

Likelihood Ratio and Size of Investment (Cont'd)

- Suppose, first, $x \leq \beta$.
  Then $\frac{f(x)}{g(x)} \leq \frac{f(\beta)}{g(\beta)}$. So $f(x) \leq \frac{f(\beta)}{g(\beta)} g(x)$.
  Moreover, by the preceding lemma, $h(y_g, x) \leq 0$.
  Hence,
  $$\int_0^\beta h(y_g, x) f(x) dx \geq \frac{f(\beta)}{g(\beta)} \int_0^\beta h(y_g, x) g(x) dx.$$
- Suppose, next, $x \geq \beta$.
  Then $\frac{f(x)}{g(x)} \geq \frac{f(\beta)}{g(\beta)}$. So $f(x) \geq \frac{f(\beta)}{g(\beta)} g(x)$.
  Moreover, by the preceding lemma, $h(y_g, x) \geq 0$.
  Hence,
  $$\int_\beta^\infty h(y_g, x) f(x) dx \geq \frac{f(\beta)}{g(\beta)} \int_\beta^\infty h(y_g, x) g(x) dx.$$

  Adding, we get $\int_0^\infty h(y_g, x) f(x) dx \geq \frac{f(\beta)}{g(\beta)} \int_0^\infty h(y_g, x) g(x) dx = 0$.

Subsection 5

Second-Order Dominance

Second-Order Dominance

Definition

We say that X **second order dominates** Y, written as $X \geq_{\text{icv}} Y$, if

$$E[h(X)] \geq E[h(Y)], \quad \text{for all functions } h \text{ that are both increasing and concave.}$$

Remarks:

1. The notation $X \geq_{\text{icv}} Y$ is used because equivalent terminology to X second-order dominating Y is that X is **stochastically larger than Y in the increasing, concave sense**.
2. If X has expected value $E[X]$, then by Jensen's inequality, the constant random variable $E[X]$ second order dominates X.

A Necessary Condition

- For a specified value of a, let

$$h_a(x) = \begin{cases} x, & \text{if } x \leq a \\ a, & \text{if } x > a \end{cases}.$$

- $h_a(x)$ is an increasing straight line that becomes flat when it hits a.
- So $h_a(x)$ is an increasing, concave function.
- We write

$$h_a(X) = a - (a - h_a(X)).$$

- Then $a - h_a(X)$ nonnegative random variable.

A Necessary Condition (Cont'd)

- By a previous lemma,

$$\begin{aligned} E[h_a(X)] &= a - E[a - h_a(X)] \\ &= a - \int_0^\infty P(a - h_a(X) > t)dt \\ &= a - \int_0^\infty P(h_a(X) < a - t)dt \\ &= a - \int_0^\infty P(X < a - t)dt \\ &= a - \int_{-\infty}^a P(X < y)dy. \end{aligned}$$

- If X second-order stochastically dominates Y, $E[h_a(X)] \geq E[h_a(Y)]$.
- So, if X second-order stochastically dominates Y,

$$\int_{-\infty}^a P(X < y)dy \leq \int_{-\infty}^a P(Y < y)dy, \text{ for all } a.$$

Sufficiency of the Condition

- In fact, it can be shown that the preceding is also a sufficient condition for $X \geq_{\text{icv}} Y$:

Theorem

X second-order stochastically dominates Y if and only if

$$\int_{-\infty}^{a} P(X < y)dy \leq \int_{-\infty}^{a} P(Y < y)dy, \text{ for all } a.$$

Increasing Random Variables and Correlation

Proposition

If $f(x)$ and $g(x)$ are both increasing functions of x, then for any random variable X,

$$E[f(X)g(X)] \geq E[f(X)]E[g(X)].$$

If one of f and g is an increasing function and the other is a decreasing function, then

$$E[f(X)g(X)] \leq E[f(X)]E[g(X)].$$

- Let X and Y be independent with the same distribution.

  Suppose $f(x)$ and $g(x)$ are both increasing functions of x.

  Then $f(X) - f(Y)$ and $g(X) - g(Y)$ both have the same sign:
  - They are both nonnegative if $X \geq Y$;
  - They are both nonpositive if $X \leq Y$.

  Consequently, $(f(X) - f(Y))(g(X) - g(Y)) \geq 0$.

Increasing Random Variables and Correlation (Cont'd)

- Equivalently,

$$f(X)g(X)+f(Y)g(Y)\geq f(X)g(Y)+f(Y)g(X).$$

Taking expectations gives

$$E[f(X)g(X)]+E[f(Y)g(Y)]\geq E[f(X)g(Y)]+E[f(Y)g(X)].$$

By the independence of X and Y,

$$E[f(X)g(X)]+E[f(Y)g(Y)]\geq E[f(X)]E[g(Y)]+E[f(Y)]E[g(X)].$$

Since X and Y have the same distribution, we get:

- $E[f(Y)g(Y)]=E[f(X)g(X)]$;
- $E[f(Y)]=E[f(X)]$;
- $E[g(Y)]=E[g(X)]$.

The preceding inequality yields the desired

$$2E[f(X)g(X)]\geq 2E[f(X)]E[g(X)].$$

Increasing Random Variables and Correlation (Cont'd)

- Suppose, finally that:
  - f is decreasing;
  - g is increasing.

  The preceding gives that

  $$E[-f(X)g(X)] \geq E[-f(X)]E[g(X)].$$

  Multiplying both sides by -1, we get

  $$E[f(X)g(X)] \leq E[f(X)]E[g(X)].$$

  This completes the proof.

Zero Expectation Random Variable and Constant

Lemma

If $E[X] = 0$ and $c \geq 1$ is a constant, then $X \geq_{\text{icv}} cX$.

- Let h be an increasing concave function, and let $c \geq 1$.

  The Taylor series expansion with remainder of $h(cx)$ about x gives that, for some w between x and cx,

$$\begin{aligned} h(cx) &= h(x) + h'(x)(cx - x) + \tfrac{h''(w)}{2!}(cx - x)^2 \\ &\leq h(x) + h'(x)(cx - x). \\ & \quad (h''(w) \leq 0 \text{ by concavity}) \end{aligned}$$

  Since the preceding holds for all x,

$$h(cX) \leq h(X) + (c - 1)Xh'(X).$$

Zero Expectation Random Variable and Constant (Cont'd)

- We got $h(cX) \leq h(X) + (c-1)Xh'(X)$.

  Note that we have:
  - $f(x) = x$ is an increasing function;
  - $h'(x)$ is a decreasing function of x, by the concavity of h.

  So, by the preceding proposition,

  $$E[Xh'(X)] \leq E[X]E[h'(X)].$$

  Taking expectations in the inequality at the top gives

  $$\begin{aligned} E[h(cX)] &\leq E[h(X)] + (c-1)E[Xh'(X)] \\ &\leq E[h(X)] + (c-1)E[X]E[h'(X)] \\ &= E[h(X)]. \quad (E[X] = 0, \text{ by hypothesis}) \end{aligned}$$

  We conclude that $X \geq_{\text{icv}} cX$.

Normal Random Variables and Second-Order Dominance

Theorem

If X_i, $i = 1, 2$, are normal random variables with respective means μ_i and variances σ_i^2, then

$$\mu_1 \geq \mu_2 \text{ and } \sigma_1 \leq \sigma_2 \text{ imply } X_1 \geq_{\text{icv}} X_.$$

- Assume that $\mu_1 \geq \mu_2$ and $\sigma_1 \leq \sigma_2$.

  Let Z be a normal random variable with mean 0 and variance 1.

  By the lemma, with $c = \frac{\sigma_2}{\sigma_1} \geq 1$,

$$\sigma_1 Z \geq_{\text{icv}} c\sigma_1 Z = \sigma_2 Z.$$

Normal Variables and Second-Order Dominance (Cont'd)

- Let $h(x)$ be a concave and increasing function of x.

  We obtain

  $$\begin{array}{rcll} E[h(\mu_1+\sigma_1 Z)] & \geq & E[h(\mu_2+\sigma_1 Z)] & (\mu_1 \geq \mu_2 \text{ and } h\uparrow) \\ & \geq & E[h(\mu_2+\sigma_2 Z)], & \end{array}$$

  where the final inequality follows because $g(x)=h(\mu_2+x)$ is a concave, increasing function of x, and $\sigma_1 Z \geq_{\text{icv}} \sigma_2 Z$.

  The result now follows because $\mu_i+\sigma_i Z$ is a normal random variable with mean μ_i and variance σ_i^2.

Second Order Dominance and Sum

Theorem

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ both be vectors of n independent random variables. If $X_i \geq_{\text{icv}} Y_i$ for each $i = 1, \ldots, n$, then $\sum_{i=1}^n X_i \geq_{\text{icv}} \sum_{i=1}^n Y_i$.

- Let h be an increasing concave function.

  We need to show that $E[h(\sum_{i=1}^n X_i)] \geq E[h(\sum_{i=1}^n Y_i)]$.

  The proof is by induction on n.

  The result is true when $n = 1$.

  Assume it is true whenever the random vectors are of size $n - 1$.

  Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two vectors of independent random variables. Without loss of generality, we may assume that these vectors are independent of each other.

  This is because independence of the vectors does not affect the values of $E[h(\sum_{i=1}^n X_i)]$ and $E[h(\sum_{i=1}^n Y_i)]$.

Second Order Dominance and Sum (Cont'd)

- We start by showing that $\sum_{i=1}^{n} X_i \geq_{\text{icv}} \sum_{i=1}^{n-1} Y_i + X_n$.
  For any x, define the function $h_x(a)$ by $h_x(a) = h(x+a)$.
  Note that h_x is an increasing concave function. We have

$$\begin{array}{rcl} E[h(\sum_{i=1}^{n} X_i)|X_n = x] & = & E[h(x + \sum_{i=1}^{n-1} X_i)|X_n = x] \\ & = & E[h(x + \sum_{i=1}^{n-1} X_i)] \\ & = & E[h_x(\sum_{i=1}^{n-1} X_i)] \\ & \geq & E[h_x(\sum_{i=1}^{n-1} Y_i)] \\ & = & E[h(x + \sum_{i=1}^{n-1} Y_i)] \\ & = & E[h(x + \sum_{i=1}^{n-1} Y_i)|X_n = x] \\ & = & E[h(X_n + \sum_{i=1}^{n-1} Y_i)|X_n = x]. \end{array}$$

  Hence, $E[h(\sum_{i=1}^{n} X_i)|X_n] \geq E[h(X_n + \sum_{i=1}^{n-1} Y_i)|X_n]$.
  Taking expectations $E[h(\sum_{i=1}^{n} X_i)] \geq E[h(X_n + \sum_{i=1}^{n-1} Y_i)]$.

Second Order Dominance and Sum (Cont'd)

- We now show $\sum_{i=1}^{n-1} Y_i + X_n \geq_{\text{icv}} \sum_{i=1}^{n} Y_i$.
  Note that

$$\begin{array}{rcl} E[h(\sum_{i=1}^{n-1} Y_i + X_n) | \sum_{i=1}^{n-1} Y_i = y] & = & E[h_y(X_n)] \\ & \geq & E[h_y(Y_n)] \\ & = & E[h(\sum_{i=1}^{n} Y_i) | \sum_{i=1}^{n-1} Y_i = y], \end{array}$$

  where the inequality followed because h_y is an increasing, concave function and the equalities from independence.

  But the preceding gives that

$$E\left[h\left(\sum_{i=1}^{n-1} Y_i + X_n\right) | \sum_{i=1}^{n-1} Y_i\right] \geq E\left[h\left(\sum_{i=1}^{n} Y_i\right) | \sum_{i=1}^{n-1} Y_i\right].$$

  Taking expectations, $E[h(\sum_{i=1}^{n-1} Y_i + X_n)] \geq E[h(\sum_{i=1}^{n} Y_i)]$.
  Hence, $\sum_{i=1}^{n-1} Y_i + X_n \geq_{\text{icv}} \sum_{i=1}^{n} Y_i$.

Remark

- The theorem along with the central limit theorem can be used to give another proof that a normal random variable decreases in second order dominance as its variance increases.
- Suppose $\sigma_2 > \sigma_1$.

  Let X be equally likely to be plus or minus σ_1.

  Let Y be equally likely to be plus or minus σ_2.

  It is easy to directly verify that $X \geq_{\text{icv}} Y$ by showing that

  $$h(-\sigma_1) + h(\sigma_1) \geq h(-\sigma_2) + h(\sigma_2)$$

  whenever h is an increasing, concave function.

  We consider two vectors of independent random variables:
  - X_i, $i \geq 1$, all having the same distribution as X;
  - Y_i, $i \geq 1$, all having the same distribution as Y.

Remark (Cont'd)

- It follows from the theorem that

$$\sum_{i=1}^{n} X_i \geq_{\text{icv}} \sum_{i=1}^{n} Y_i.$$

But $W \geq_{\text{icv}} V$ implies $cW \geq_{\text{icv}} cV$, for any positive constant c.

So we get

$$\frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \geq_{\text{icv}} \frac{\sum_{i=1}^{n} Y_i}{\sqrt{n}}.$$

The result now follows by letting $n \to \infty$:

  - The term on the left converges to a normal random variable with mean 0 and variance σ_1^2;
  - The term on the right converges to a normal random variable with mean 0 and variance σ_2^2.

To make this argument truly rigorous, we would need to show that second-order stochastic dominance is preserved when taking a limit.