Introduction to Mathematical Finance

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Subsection 1

A Deterministic Optimization Model

The Problem

- Suppose we have m dollars to invest among n projects.
- Investing x in project i yields a (present value) return of $f_i(x)$, $i = 1, \ldots, n$.
- The problem is to determine the integer amounts to invest in each project so as to maximize the sum of the returns.
- Let x_i denote the amount to be invested in project i.
- The problem (mathematically) is to:

$$\text{Choose nonnegative integers } x_1, \ldots, x_n,$$
$$\text{such that } \sum_{i=1}^{n} x_i = m,$$
$$\text{to maximize } \sum_{i=1}^{n} f_i(x_i).$$

Solution Based on Dynamic Programming

- Let $V_j(x)$ denote the maximal possible sum of returns when we have a total of x to invest in projects $1, \ldots, j$.
- $V_n(m)$ represents the maximal value of the problem.
- We determine $V_n(m)$, and the optimal investment amounts, by:
  - Finding first the values of $V_1(x)$, for $x = 1, \ldots, m$;
  - Finding next the values of $V_2(x)$, for $x = 1, \ldots, m$;

    $\vdots$
  - Ending with the values of $V_n(x)$, for $x = 1, \ldots, m$.

Solution Based on Dynamic Programming (Cont'd)

- The maximal return when x must be invested in project 1 is $f_1(x)$. So we have
$$V_1(x) = f_1(x).$$
- Suppose that x must be invested between projects 1 and 2.
  - Let y be invested in project 2.
  - Then $x - y$ is available to invest in project 1.
- The best return from investing $x - y$ in project 1 is $V_1(x - y)$.
- So the maximal sum of returns possible when the amount y is invested in project 2 is
$$f_2(y) + V_1(x - y).$$
- The maximal sum of returns is obtained by maximizing over y,
$$V_2(x) = \max_{0 \le y \le x} \{f_2(y) + V_1(x - y)\}.$$

Solution Based on Dynamic Programming (Cont'd)

- In general, suppose that x must be invested among projects $1, \ldots, j$.
  - Suppose we invest y in project j.
  - Then a total of $x - y$ is available to invest in projects $1, \ldots, j-1$.
- The best return from investing $x - y$ in projects $1, \ldots, j-1$ is $V_{j-1}(x-y)$.
- So the maximal sum of returns possible when the amount y is invested in project j is $f_j(y) + V_{j-1}(x-y)$.
- The maximal sum of returns possible is obtained by maximizing the preceding over y,

$$V_j(x) = \max_{0 \leq y \leq x} \{f_j(y) + V_{j-1}(x-y)\}.$$

- Let $y_j(x)$ denote the value (or a value if there is more than one) of y that maximizes the right side of the preceding equation.
- Then $y_j(x)$ is the optimal amount to invest in project j when we have x to invest among projects $1, \ldots, j$.

Solution Based on Dynamic Programming (Cont'd)

- The value of $V_n(m)$ can now be obtained by first determining $V_1(x)$, then $V_2(x)$, $V_3(x)$, ..., $V_{n-1}(x)$ and finally $V_n(m)$.
- The optimal amounts to invest are:
  - $y_n(m)$ in project n;
  - $y_{n-1}(m - y_n(m))$ in project $n-1$;

    $\vdots$
- This solution approach is called **dynamic programming**.

  It views the problem as involving n sequential decisions.

  It then analyzes it by determining:
  - The optimal last decision;
  - The optimal next to last decision;

    $\vdots$

Example

- Suppose that three investment projects with the following return functions are available:

$$\begin{array}{rcl} f_1(x) & = & \frac{10x}{1+x}, \quad x = 0, 1, \ldots, \\ f_2(x) & = & \sqrt{x}, \quad x = 0, 1, \ldots, \\ f_3(x) & = & 10(1 - e^{-x}), \quad x = 0, 1, \ldots. \end{array}$$

- We want to maximize our return when we have 5 to invest.
- We have

$$V_1(x) = f_1(x) = \frac{10x}{1+x}.$$

- Moreover,

$$y_1(x) = x.$$

Example (Cont'd)

- Now
$$\begin{aligned} V_2(x) &= \max_{0\le y\le x}\{f_2(y)+V_1(x-y)\} \\ &= \max_{0\le y\le x}\left\{\sqrt{y}+\tfrac{10(x-y)}{1+x-y}\right\}. \end{aligned}$$
- So we have
$$\begin{aligned} V_2(1) &= \max\left\{\tfrac{10}{2},1\right\}=5,\ y_2(1)=0; \\ V_2(2) &= \max\left\{\tfrac{20}{3},1+5,\sqrt{2}\right\}=\tfrac{20}{3},\ y_2(2)=0; \\ V_2(3) &= \max\left\{\tfrac{30}{4},1+\tfrac{20}{3},\sqrt{2}+5,\sqrt{3}\right\}=\tfrac{23}{3},\ y_2(3)=1; \\ V_2(4) &= \max\left\{\tfrac{40}{5},1+\tfrac{30}{4},\sqrt{2}+\tfrac{20}{3},\sqrt{3}+5,\sqrt{4}\right\}=8.5,\ y_2(4)=1; \\ V_2(5) &= \max\left\{\tfrac{50}{6},1+8,\sqrt{2}+7.5,\sqrt{3}+\tfrac{20}{3},\sqrt{4}+5,\sqrt{5}\right\}=9, \\ & \qquad y_2(5)=1. \end{aligned}$$

Example (Cont'd)

- Continuing, we get

$$\begin{aligned} V_3(x) &= \max_{0 \le y \le x} \{f_3(y) + V_2(x-y)\} \\ &= \max_{0 \le y \le x} \{10(1-e^{-y}) + V_2(x-y)\}. \end{aligned}$$

- We compute the values:
  - $1 - e^{-1} = 0.632$;
  - $1 - e^{-2} = 0.865$;
  - $1 - e^{-3} = 0.950$;
  - $1 - e^{-4} = 0.982$;
  - $1 - e^{-5} = 0.993$.
- So we obtain

$$\begin{aligned} V_3(5) &= \max \{9, 6.32 + 8.5, 8.65 + \tfrac{23}{3}, \\ &\qquad 9.50 + \tfrac{20}{3}, 9.82 + 5, 9.93\} = 16.32, \\ y_3(5) &= 2. \end{aligned}$$

Example (Cont'd)

- Thus, the maximal sum of returns from investing 5 is 16.32;
- The optimal amount to invest in project 3 is $y_3(5) = 2$;
- The optimal amount to invest in project 2 is

$$y_2(5 - 2) = y_2(3) = 1;$$

- The optimal amount to invest in project 1 is

$$y_1(5 - 2 - 1) = y_1(2) = 2.$$

Concave Return Functions

- A function $g(i)$, $i = 0, 1, \ldots$, is said to be **concave** if

$$g(i+1) - g(i) \text{ is nonincreasing in } i.$$

- We will consider concave return functions $f_i(x)$.
- This means that the additional (or marginal) gain from each additional unit invested becomes smaller as more has already been invested.

Solution for Concave Return Functions

- Assume that the functions $f_i(x)$, $i = 1, \ldots, n$, are all concave.
- Consider the problem of choosing nonnegative integers $x_1, \ldots, x_n$, whose sum is m, to maximize $\sum_{i=1}^n f_i(x_i)$.
- Suppose that $x_1^o, \ldots, x_n^o$ is an optimal vector for this problem.
- I.e., a vector of nonnegative integers that sum to m, with

$$\sum_{i=1}^n f_i(x_i^o) = \max \sum_{i=1}^n f_i(x_i),$$

  the maximum over all nonnegative integers $x_1, \ldots, x_n$ that sum to m.
- Now suppose that we have a total of $m + 1$ to invest.
- We argue that there is an optimal vector $y_1^o, \ldots, y_n^o$ with $\sum_{i=1}^n y_i^o = m + 1$ that satisfies $y_i^o \geq x_i^o$, $i = 1, \ldots, n$.

Solution for Concave Return Functions (Cont'd)

- Suppose we have $m+1$ to invest.
- Consider any investment strategy $y_1, \ldots, y_n$, such that:
  - $\sum_{i=1}^n y_i = m+1$;
  - For some value of k, $y_k < x_k^o$.
- We have $m+1 = \sum_i y_i > \sum_i x_i^o = m$.
- Hence, there must be a j such that $x_j^o < y_j$.
- Consider the investment strategy that invests:
  - $y_k + 1$ in project k;
  - $y_j - 1$ in project j;
  - y_i in project i for $i \neq k, j$.
- We argue that this strategy is at least as good as the strategy that invests y_i in project i for each i.

Solution for Concave Return Functions (Cont'd)

- We must show that $f_k(y_k+1)+f_j(y_j-1)\geq f_k(y_k)+f_j(y_j)$.
- Equivalently,
$$f_k(y_k+1)-f_k(y_k)\geq f_j(y_j)-f_j(y_j-1).$$
- Now $x_1^o,\ldots,x_n^o$ is optimal when there is m to invest.
- So
$$f_k(x_k^o)+f_j(x_j^o)\geq f_k(x_k^o-1)+f_j(x_j^o+1).$$
- Equivalently, we have
$$f_k(x_k^o)-f_k(x_k^o-1)\geq f_j(x_j^o+1)-f_j(x_j^o).$$
- Consequently,
$$\begin{array}{l} f_k(y_k+1)-f_k(y_k) \\ \geq f_k(x_k^o)-f_k(x_k^o-1) \quad \text{(by concavity, since } y_k+1\leq x_k^o) \\ \geq f_j(x_j^o+1)-f_j(x_j^o) \quad \text{(by the preceding inequlaity)} \\ \geq f_j(y_j)-f_j(y_j-1) \quad \text{(by concavity, since } x_j^o+1\leq y_j). \end{array}$$

Solution for Concave Return Functions (Cont'd)

- Thus, any strategy for investing $m+1$ that calls for investing less than x_k^o in some project k can be at least matched by one whose investment in project k is increased by 1 with a corresponding decrease in some project j whose investment was greater than x_j^o.
- Repeating this argument shows that, for any strategy of investing $m+1$, we can find another strategy that:
  - Invests at least x_i^o in project i, for all $i = 1, \ldots, n$;
  - Yields a return that is at least as large as the original strategy.
- This implies that we can find an optimal strategy $y_1^o, \ldots, y_n^o$ for investing $m+1$ that satisfies the inequality claimed.

Solution for Concave Return Functions (Cont'd)

- We argued that the optimal strategy for investing $m+1$ invests at least as much in each project as does the optimal strategy for investing m.
- It follows that the optimal strategy for $m+1$ can be found by using the optimal strategy for m and then investing the extra dollar in that project whose marginal increase is largest.
- Therefore, we can find the optimal investment (when we have m) by:
  - First solving the optimal investment problem when we have 1 to invest;
  - Then solving the optimal investment problem when we have 2 to invest;
  - Then solving the optimal investment problem when we have 3 to invest;

  $\vdots$

Example Revisited

- We reconsider the preceding example.
- We have 5 to invest among three projects, with return functions

$$f_1(x) = \frac{10x}{1+x}, \quad f_2(x) = \sqrt{x}, \quad f_3(x) = 10(1 - e^{-x}).$$

- Let $x_i(j)$ denote the optimal amount to invest in project i when we have a total of j to invest.
- We have

$$\max\{f_1(1), f_2(1), f_3(1)\} = \max\{5, 1, 6.32\} = 6.32.$$

- So

$$x_1(1) = 0, \quad x_2(1) = 0, \quad x_3(1) = 1.$$

Example (Cont'd)

- Now
$$\max_i \{f_i(x_i(1)+1) - f_i(x_i(1))\} = \max\{5, 1, 8.65 - 6.32\} = 5.$$
- So we have
$$x_1(2) = 1, \quad x_2(2) = 0, \quad x_3(2) = 1.$$
- Further,
$$\begin{aligned} \max_i \{f_i(x_i(2)+1) - f_i(x_i(2))\} &= \max\left\{\tfrac{20}{3} - 5, 1, 8.65 - 6.32\right\} \\ &= 2.33. \end{aligned}$$
- So we get
$$x_1(3) = 1, \quad x_2(3) = 0, \quad x_3(3) = 2.$$

Example (Cont'd)

- Continuing,

$$\begin{aligned} \max_i \{f_i(x_i(3)+1) - f_i(x_i(3))\} &= \max\left\{\frac{20}{3} - 5, 1, 9.50 - 8.65\right\} \\ &= 1.67. \end{aligned}$$

- Therefore,

$$x_1(4) = 2, \quad x_2(4) = 0, \quad x_3(4) = 2.$$

- Finally,

$$\begin{aligned} \max_i \{f_i(x_i(4)+1) - f_i(x_i(4))\} &= \max\left\{\frac{30}{4} - \frac{20}{3}, 1, 9.50 - 8.65\right\} \\ &= 1. \end{aligned}$$

- This gives

$$x_1(5) = 2, \quad x_2(5) = 1, \quad x_3(5) = 2.$$

- Thus, the maximal return is

$$6.32 + 5 + 2.33 + 1.67 + 1 = 16.32.$$

Algorithm

- The following algorithm can be used to solve the problem when m is to be invested among n projects, each with a concave return function.
- The quantity k will represent the current amount to be invested.
- x_i will represent the optimal amount to invest in project i when a total of k is to be invested.
  (1) Set $k = 0$ and $x_i = 0$, $i = 1, \ldots, n$.
  (2) $m_i = f_i(x_i + 1) - f_i(x_i)$, $i = 1, \ldots, n$.
  (3) $k = k + 1$.
  (4) Let J be such that $m_J = \max_i m_i$.
  (5) If $J = j$, then $x_j \to x_j + 1$, $m_j \to f_j(x_j + 1) - f_j(x_j)$.
  (6) If $k < m$, go to step (3).
- Step (5) means that if the value of J is j, then:
  (a) The value of x_j should be increased by 1;
  (b) The value of m_j should be reset to equal the difference of f_j evaluated at 1 plus the new value of x_j and f_j evaluated at the new value of x_j.

Remark

- When $g(x)$ is defined for all x in an interval, then g is concave if $g'(t)$ is a decreasing function of t (that is, if $g''(t) \leq 0$).
- Hence, for g concave
$$\int_i^{i+1} g'(s)ds \leq \int_{i-1}^{i} g'(s)ds.$$
- So
$$g(i+1) - g(i) \leq g(i) - g(i-1).$$
- This is the definition of concavity we used for g defined on the integers.

The Knapsack Problem

- Assume we can invest at most m in the n projects.
- Suppose one invests in project i by buying an integral number of shares in that project, with each share:
  - Costing c_i;
  - Returning v_i.
- Let x_i denote the number of shares of project i that are purchased.
- Then the problem is to:

$$\text{Choose nonnegative integers } x_1, \ldots, x_n,$$
$$\text{such that } \sum_{i=1}^{n} x_i c_i \leq m,$$
$$\text{to maximize } \sum_{i=1}^{n} v_i x_i.$$

- We will use a dynamic programming approach to solve this problem.

The Knapsack Problem (Cont'd)

- Let $V(x)$ be the maximal return possible when we have x to invest.
- If we start by buying one share of project i, then a return v_i will be received and we will be left with a capital of $x - c_i$.
- $V(x - c_i)$ is the maximal return from investing $x - c_i$.
- So the maximal return possible if we have x and begin investing by buying one share of project i is

$$\text{maximal return if start by purchasing one share of } i = v_i + V(x - c_i).$$

- Hence, the maximal return $V(x)$ that can be obtained from the investment capital x, satisfies

$$V(x) = \max_{i : c_i \leq x} \{v_i + V(x - c_i)\}.$$

The Knapsack Problem (Cont'd)

- Let $i(x)$ denote the value of i that maximizes $v_i + V(x - c_i)$.
- Starting with x, it is optimal to purchase one share of project $i(x)$.
- Starting with
$$V(1) = \max_{i:c_i \leq 1} v_i,$$
it is easy to determine the values of $V(1)$ and $i(1)$.
- This will then enable us to use
$$V(x) = \max_{i:c_i \leq x} \{v_i + V(x - c_i)\}$$
to determine $V(2)$ and $i(2)$.
- And so on.

The Name

- We introduced the problem:

$$\text{Choose nonnegative integers } x_1, \ldots, x_n,$$
$$\text{such that } \sum_{i=1}^{n} x_i c_i \leq m,$$
$$\text{to maximize } \sum_{i=1}^{n} v_i x_i.$$

- This problem is called a **knapsack** problem.

  It is mathematically equivalent to determining the set of items to be put in a knapsack that can carry a total weight of at most m when there are n different types of items, with each type i item having:
  - Weight c_i;
  - Value v_i.

Example

- Suppose you have 25 to invest among three projects whose cost and return values are as on the right.

Project	Cost/Share	Return/Share
1	5	7
2	9	12
3	15	22

$$
\begin{aligned}
&V(x) = 0,\ x \leq 4;\\
&V(x) = 7,\ i(x) = 1,\ x = 5, 6, 7, 8;\\
&V(9) = \max\{7 + V(4), 12 + V(0)\} = 12,\ i(9) = 2;\\
&V(x) = \max\{7 + V(x-5), 12 + V(x-9)\} = 14,\ i(x) = 1,\\
&\qquad\qquad\qquad\qquad x = 10, 11, 12, 13;\\
&V(14) = \max\{7 + V(9), 12 + V(5)\} = 19,\ i(x) = 1 \text{ or } 2;\\
&V(15) = \max\{7 + V(10), 12 + V(6), 22 + V(0)\} = 22,\ i(15) = 3;\\
&V(16) = \max\{7 + V(11), 12 + V(7), 22 + V(1)\} = 22,\ i(16) = 3;\\
&V(17) = \max\{7 + V(12), 12 + V(8), 22 + V(2)\} = 22,\ i(17) = 3;\\
&V(18) = \max\{7 + V(13), 12 + V(9), 22 + V(3)\} = 24,\ i(18) = 2;
\end{aligned}
$$

and so on.

Subsection 2

Probabilistic Optimization Problems

A Gambling Model with Unknown Win Probabilities

- Suppose that an investment's win probability can be one of three possible values: $p_1 = 0.45$, $p_2 = 0.55$ or $p_3 = 0.65$.
- Suppose also that it will be:
  - p_1 with probability $\frac{1}{4}$;
  - p_2 with probability $\frac{1}{2}$;
  - p_3 with probability $\frac{1}{4}$.
- An investor, without any information about which p_i has been chosen, will take the win probability to be

$$p = \frac{1}{4}p_1 + \frac{1}{2}p_2 + \frac{1}{4}p_3 = 0.55.$$

Gambling with Unknown Win Probabilities (Cont'd)

- Assume the investor has:
  - Initial fortune x;
  - A log utility function.
- By a previous example, we know that the investor:
  - Will invest $100(2p-1) = 10\%$ of her fortune;
  - Will have expected utility of her final fortune

$$\log(x) + 0.55\log(1.1) + 0.45\log(0.9) = \log(x) + 0.0050 = \log(e^{0.0050}x).$$

Gambling with Unknown Win Probabilities (Cont'd)

- Suppose now that the investor is able to learn, before making her investment, which p_i is the win probability.
  - If 0.45 is the win probability, then the investor will not invest. The conditional expected utility of her final fortune will be $\log(x)$.
  - If 0.55 is the win probability, the investor will do as shown previously. The conditional expected utility of her final fortune will be $\log(x) + 0.0050$.
  - If 0.65 is the win probability, the investor will invest 30% of her fortune. The conditional expected utility of her final fortune will be

$$\log(x) + 0.65\log(1.3) + 0.35\log(0.7) = \log(x) + .0456.$$

- Therefore, the expected final utility of an investor who will learn which p_i is the win probability before making her investment is

$$\begin{aligned}\frac{1}{4}\log(x) + \frac{1}{2}(\log(x) + 0.0050) + \frac{1}{4}(\log(x) + 0.0456) \\ = \log(x) + 0.0139 = \log(e^{0.0139}x).\end{aligned}$$

An Investment Allocation Model

- An investor has the amount D available to invest.
- During each of N time instants, an opportunity to invest will (independently) present itself with probability p.
- If the opportunity occurs, the investor must decide how much of her remaining wealth to invest.
- If y is invested in an opportunity then $R(y)$, a specified function of y, is earned at the end of the problem.
- Both the amount invested and the return from that investment become unavailable for future investment.
- The investor's final wealth is equal to the sum of all the investment returns and the amount that was never invested.
- We determine how much to invest at each opportunity so as to maximize the expected value of the investor's final wealth.

Notation

- Let $W_n(x)$ denote the maximal expected final wealth when:
  - The investor has x to invest;
  - There are n time instants in the problem.
- Let $V_n(x)$ denote the maximal expected final wealth when:
  - The investor has x to invest;
  - There are n time instants in the problem;
  - An opportunity is at hand.

Determining $V_n(x)$

- Suppose y is initially invested;
- Then the investor's maximal expected final wealth will be $R(y)$ plus the maximal expected amount that she can obtain in $n-1$ time instants when her investment capital is $x-y$.
- The latter quantity is $W_{n-1}(x-y)$.
- So the maximal expected final wealth when y is invested is

$$R(y)+W_{n-1}(x-y).$$

- The investor can now choose y to maximize this sum,

$$V_n(x)=\max_{0\leq y\leq x}\{R(y)+W_{n-1}(x-y)\}.$$

Determining $W_n(x)$

- Suppose the investor has x to invest.
- Suppose there are n time instants to go.
- One of the following two cases arises:
  - An opportunity occurs and the maximal expected final wealth is $V_n(x)$;
  - An opportunity does not occur and the maximal expected final wealth is $W_{n-1}(x)$.
- Each opportunity occurs with probability p.
- So we have
$$W_n(x) = pV_n(x) + (1-p)W_{n-1}(x).$$

Solution Method

- Start with $W_0(x) = x$.
  - We first use the former equation to obtain $V_1(x)$, for all $0 \leq x \leq D$;
  - Then use the latter equation to obtain $W_1(x)$, for all $0 \leq x \leq D$;
  - Then use the former equation to obtain $V_2(x)$ for all $0 \leq x \leq D$;
  - Then use the latter equation to obtain $W_2(x)$;

    $\vdots$
- Let $y_n(x)$ be the value of y that maximizes the right side of the former equation.
- The optimal policy is to invest the amount $y_n(x)$ if:
  - Our current investment capital is x;
  - There are n time instants remaining;
  - An opportunity is present.

Example

- We work under the following hypotheses:
  - We have 10 to invest;
  - There are two time instants;
  - An opportunity presents itself each instant with probability $p = 0.7$, and $R(y) = y + 10\sqrt{y}$.
- We find the maximal expected final wealth and the optimal policy.
- We start with $W_0(x) = x$.
- We then get

$$\begin{aligned} V_1(x) &= \max_{0 \leq y \leq x}\{y + 10\sqrt{y} + x - y\} \\ &= x + \max_{0 \leq y \leq x}\{10\sqrt{y}\} \\ &= x + 10\sqrt{x}. \end{aligned}$$

- Moreover, $y_1(x) = x$.
- Thus,

$$W_1(x) = 0.7(x + 10\sqrt{x}) + 0.3x = x + 7\sqrt{x}.$$

Example (Cont'd)

- Now we have

$$\begin{aligned} V_2(x) &= \max_{0 \leq y \leq x} \{y + 10\sqrt{y} + x - y + 7\sqrt{x-y}\} \\ &= x + \max_{0 \leq y \leq x} \{10\sqrt{y} + 7\sqrt{x-y}\} \\ &= x + \sqrt{149x}, \end{aligned}$$

where calculus gives the final equation, as well as

$$y_2(x) = \frac{100}{149}x.$$

- The preceding now yields

$$\begin{aligned} W_2(x) &= 0.7(x + \sqrt{149x}) + 0.3(x + 7\sqrt{x}) \\ &= x + 0.7\sqrt{149x} + 2.1\sqrt{x}. \end{aligned}$$

Example (Conclusion)

- Starting with 10, the maximal expected final wealth is

$$W_2(10) = 10 + 0.7\sqrt{1490} + 2.1\sqrt{10} = 43.66.$$

- The optimal policy is to invest:
  - $\frac{1000}{149} = 6.71$, if an opportunity presents itself at the initial time instant;
  - Whatever of your fortune remains, if an opportunity presents itself at the final time instant.

Properties of $W_n(x)$ and $V_n(x)$

Theorem

If $R(y)$ is a nondecreasing concave function, then:

(a) $V_n(x)$ and $W_n(x)$ are both nondecreasing concave functions;

(b) $y_n(x)$ is a nondecreasing function of x;

(c) $x - y_n(x)$ is a nondecreasing function of x;

(d) $y_n(x)$ is a nonincreasing function of n.

- Part (b) states that the more you have the more you should invest.
- Part (c) states that the more you have the more you should conserve.
- Part (d) says that the more time you have the less you should invest each time.