Introduction to Mathematical Finance

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Subsection 1

Continuous Random Variables

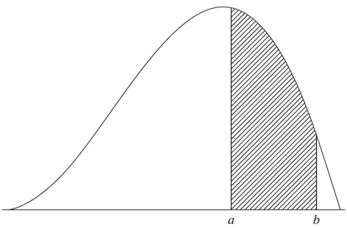
Continuous Random Variables

- We have looked at random variables whose possible values constituted discrete sets.
- There exist random variables whose sets of possible values are continuous regions.
- These continuous random variables can take on any value within some interval.

  Example: The following are continuous random variables:
  - The time it takes to complete an assignment;
  - The weight of a randomly chosen individual.

The Probability Density Function

- Every continuous random variable X has a function f associated with it.
- This function, called the **probability density function** of X, determines the probabilities associated with X in the following manner:
- For any numbers $a < b$, the area under f between a and b is equal to the probability that X assumes a value between a and b.
- That is, $P\{a \leq X \leq b\} =$ area under f between a and b.



$P\{a \leq X \leq b\}$ = area of shaded region

Subsection 2

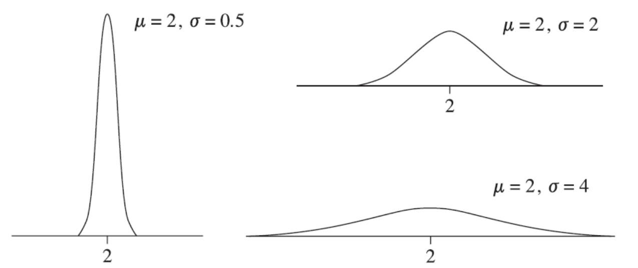
Normal Random Variables

Normal Random Variable

- The probability density function of a **normal random variable** X is determined by two parameters, denoted by μ and σ, and is given by the formula

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

- A plot of f gives a bell-shaped curve that is symmetric about the value μ, and with a variability that is measured by σ.



Standard Normal Random Variable

- The parameters μ and σ^2 are equal to the expected value and to the variance of X, respectively: $\mu = E[X]$, $\sigma^2 = \mathrm{Var}(X)$.
- A normal random variable having mean 0 and variance 1 is called a **standard normal random variable**.
- Let Z be a standard normal random variable.
- The function $\Phi(x)$, defined for all real numbers x by

$$\Phi(x) = P\{Z \leq x\},$$

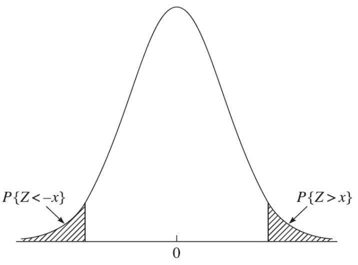
  is called the **standard normal distribution function**.
- Thus $\Phi(x)$, the probability that a standard normal random variable is less than or equal to x, is equal to the area under the **standard normal density function**

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty,$$

  between $-\infty$ and x.

Using a Table to Calculate Probabilities

- A table is used giving values of $\Phi(x)$ when $x > 0$.
- Probabilities for negative x can be obtained by using the symmetry of the standard normal density about 0.



$P\{Z < -x\} = P\{Z > x\}$

- We conclude that $P\{Z < -x\} = P\{Z > x\}$ or, equivalently, that
$$\Phi(-x) = 1 - \Phi(x).$$

Example

- Let Z be a standard normal random variable.

  For $a < b$, express $P\{a < Z \leq b\}$ in terms of Φ.

  We have

  $$P\{Z \leq b\} = P\{Z \leq a\} + P\{a < Z \leq b\}.$$

  Therefore

  $$\begin{array}{rcl} P\{a < Z \leq b\} & = & P\{Z \leq b\} - P\{Z \leq a\} \\ & = & \Phi(b) - \Phi(a). \end{array}$$

Example

- For a standard normal random variable Z, calculate, using the table, the values $P\{|Z| \leq 1\}$, $P\{|Z| \leq 2\}$ and $P\{|Z| \leq 3\}$.

  We have

$$\begin{aligned}
P\{|Z| \leq 1\} &= P\{-1 \leq Z \leq 1\} \\
&= \Phi(1) - \Phi(-1) \\
&= \Phi(1) - (1 - \Phi(1)) \\
&= 2\Phi(1) - 1 \\
&= 2 \cdot 0.8413 - 1 = 0.6826.
\end{aligned}$$

  Similarly,

$$\begin{aligned}
P\{|Z| \leq 2\} &= 2\Phi(2) - 1 = 2 \cdot 0.9772 - 1 = 0.9544; \\
P\{|Z| \leq 3\} &= 2\Phi(3) - 1 = 2 \cdot 0.9987 - 1 = 0.9974.
\end{aligned}$$

Subsection 3

Properties of Normal Random Variables

Linear Function of Normal Variable

- If X is a normal random variable then so is $aX + b$, when a and b are constants.
- This property enables us to transform any normal random variable X into a standard normal random variable.
- Suppose X is normal with mean μ and variance σ^2.
- Then
$$Z = \frac{X - \mu}{\sigma}$$
has expected value 0 and variance 1.
- So Z is a standard normal random variable.
- This allows computing probabilities for any normal random variable in terms of the standard normal distribution function Φ.

Example

- IQ examination scores for sixth-graders are normally distributed with mean value 100 and standard deviation 14.2.

  What is the probability that a randomly chosen sixth-grader has an IQ score greater than 130?

  Let X be the score of a randomly chosen sixth-grader.

  Then,

$$\begin{array}{rcl} P\{X > 130\} & = & P\{\frac{X-100}{14.2} > \frac{130-100}{14.2}\} \\ & = & P\{\frac{X-100}{14.2} > 2.113\} \\ & = & 1 - \Phi(2.113) \\ & = & 1 - 0.9834 \\ & = & 0.017. \end{array}$$

Sum of Normal Variables

- The sum of independent normal random variables is also a normal random variable.
- If X_1 and X_2 are independent normal random variables with means μ_1 and μ_2 and with standard deviations σ_1 and σ_2, then $X_1 + X_2$ is normal with mean

$$E[X_1 + X_2] = E[X_1] + E[X_2] = \mu_1 + \mu_2$$

and variance

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = \sigma_1^2 + \sigma_2^2.$$

Example

- The annual rainfall in Cleveland, Ohio, is normally distributed with mean 40.14 inches and standard deviation 8.7 inches.

  Find the probability that the sum of the next two years' rainfall exceeds 84 inches.

  Let X_i denote the rainfall in year i $(i = 1, 2)$.

  We assume that the rainfalls in successive years are independent. Then $X_1 + X_2$ is normal with:
  - Mean $40.14 + 40.14 = 80.28$;
  - Variance $(8.7)^2 + (8.7)^2 = 151.38$.

  Therefore, with Z denoting a standard normal random variable,

$$\begin{aligned} P\{X_1 + X_2 > 84\} &= P\{Z > \tfrac{84-80.28}{\sqrt{151.38}}\} \\ &= P\{Z > 0.3023\} \\ &= 1 - \Phi(0.3023) \\ &\approx 0.3812. \end{aligned}$$

Lognormal Random Variables

- The random variable Y is said to be a **lognormal random variable with parameters μ and σ** if $\log(Y)$ is a normal random variable with mean μ and variance σ^2.
- That is, Y is lognormal if it can be expressed as

$$Y = e^X,$$

  where X is a normal random variable.
- The mean and variance of a lognormal random variable are as follows:

$$\begin{aligned} E[Y] &= e^{\mu + \frac{\sigma^2}{2}}; \\ \mathrm{Var}(Y) &= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} \\ &= e^{2\mu + \sigma^2}(e^{\sigma^2} - 1). \end{aligned}$$

Example

- Starting at some fixed time, let $S(n)$ denote the price of a certain security at the end of n additional weeks, $n \geq 1$.
- A popular model for the evolution of these prices assumes that the price ratios $\frac{S(n)}{S(n-1)}$, for $n \geq 1$, are independent and identically distributed (i.i.d.) lognormal random variables.
- Assuming this model, with lognormal parameters $\mu = 0.0165$ and $\sigma = 0.0730$, what is the probability that:
  (a) The price of the security increases over each of the next two weeks;
  (b) The price at the end of two weeks is higher than it is today?

Example (Part (a))

- Let Z be a standard normal random variable.

  We have

$$\begin{array}{rcl} P\left\{\frac{S(2)}{S(1)} > 1, \frac{S(1)}{S(0)} > 1\right\} & = & P\left\{\frac{S(2)}{S(1)} > 1\right\} P\left\{\frac{S(1)}{S(0)} > 1\right\} \\ & = & P\left\{\frac{S(1)}{S(0)} > 1\right\}^2 \\ & = & P\left\{\log\left(\frac{S(1)}{S(0)}\right) > 0\right\}^2 \\ & = & P\left\{Z > \frac{-0.0165}{0.0730}\right\}^2 \\ & = & P\{Z > -0.2260\}^2 \\ & = & P\{Z < 0.2260\}^2 \\ & \approx & 0.5894^2 = 0.3474. \end{array}$$

Example (Part (b))

- To solve part (b), reason as follows:

$$
\begin{aligned}
P\left\{\frac{S(2)}{S(0)} > 1\right\} &= P\left\{\frac{S(2)}{S(1)}\frac{S(1)}{S(0)} > 1\right\} \\
&= P\left\{\log\left(\frac{S(2)}{S(1)}\right) + \log\left(\frac{S(1)}{S(0)}\right) > 0\right\} \\
&= P\left\{Z > \frac{-0.0165-0.0165}{\sqrt{0.0730^2+0.0730^2}}\right\} \\
&= P\left\{Z > \frac{-0.0330}{0.0730\sqrt{2}}\right\} \\
&= P\{Z > -0.31965\} \\
&= P\{Z < 0.31965\} \\
&\approx 0.6254.
\end{aligned}
$$

Subsection 4

The Central Limit Theorem

The Central Limit Theorem

- Suppose that $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables, each with expected value μ and variance σ^2.
- Define
$$S_n = \sum_{i=1}^{n} X_i.$$

Central Limit Theorem

For large n, S_n will approximately be a normal random variable with expected value $n\mu$ and variance $n\sigma^2$. As a result, for any x, we have

$$P\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right\} \approx \Phi(x),$$

with the approximation becoming exact as n becomes larger and larger.

Binomial Random Variables Revisited

- Suppose that X is binomial with parameters n and p.
- X represents the number of successes in n independent trials, each of which is a success with probability p.
- Thus, it can be expressed as

$$X = \sum_{i=1}^{n} X_i,$$

  where X_i is 1 if trial i is a success and is 0 otherwise.
- We know that

$$E[X_i] = p \quad \text{and} \quad \mathrm{Var}(X_i) = p(1-p).$$

- By the Central Limit Theorem, when n is large, X will approximately have a normal distribution with mean np and variance $np(1-p)$.

Example

- A fair coin is tossed 100 times. What is the probability that heads appears fewer than 40 times?

  Let X denote the number of heads.

  Then X is binomial with parameters $n = 100$ and $p = \frac{1}{2}$.

  So $np = 50$ and $np(1 - p) = 25$.

  Now we have

$$\begin{aligned} P\{X < 40\} &= P\left\{\frac{X-50}{\sqrt{25}} < \frac{40-50}{\sqrt{25}}\right\} \\ &= P\left\{\frac{X-50}{\sqrt{25}} < -2\right\} \\ &\approx \Phi(-2) = 0.0228. \end{aligned}$$

Example (Comments)

- The preceding is not quite as acccurate as we might like.
- We could improve the approximation by noting that, since X is an integral-valued random variable, the event that $X < 40$ is equivalent to the event that $X < 39 + c$ for any c, $0 < c \leq 1$.
- Consequently, a better approximation may be obtained by writing the desired probability as $P\{X < 39.5\}$.
- This gives

$$\begin{aligned} P\{X < 39.5\} &= P\left\{\frac{X-50}{\sqrt{25}} < \frac{39.5-50}{\sqrt{25}}\right\} \\ &= P\left\{\frac{X-50}{\sqrt{25}} < -2.1\right\} \\ &\approx \Phi(-2.1) = 0.0179. \end{aligned}$$

This is indeed a better approximation.