Introduction to Mathematical Finance

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LSSU Math 500

1. Interest Rates and Present Value Analysis
   - Interest Rates
   - Present Value Analysis
   - Rate of Return
   - Continuously Varying Interest Rates

Subsection 1

Interest Rates

Simple Interest

- Suppose we borrow the amount P (called the **principal**).
- It must be repaid after a time T.
- Added is simple interest at rate r per time T.
- Then the amount to be repaid at time T is

$$P + rP = P(1 + r).$$

- That is, we must repay both the principal P and the interest, equal to the principal times the interest rate.

  Example: Suppose we borrow \$100 to be repaid after one year with a simple interest rate of 5% per year.

  Then $r = 0.05$, and, at the end of the year, we have to repay

$$100(1 + 0.05) = \$105.$$

Compounded Interest

- Suppose we borrow the amount P.
- It is to be repaid after one year.
- Accrued is interest at a rate r per year **compounded** semiannually.
- Interest being compounded semiannually means that:
  - After half a year, we are charged simple interest at the rate of $\frac{r}{2}$ per half-year.
  - Interest is then added on to our principal.
  - This is again charged interest at rate $\frac{r}{2}$ for the second half-year period.
- After six months we owe $P(1+\frac{r}{2})$.
- This is then regarded as the new principal for another six-month loan at interest rate $\frac{r}{2}$.
- Hence, at the end of the year we owe

$$P\left(1+\frac{r}{2}\right)\left(1+\frac{r}{2}\right)=P\left(1+\frac{r}{2}\right)^2.$$

Example

- Suppose we borrow \$1,000 for one year at an interest rate of 8% per year compounded quarterly.
- We calculate how much we owe at the end of the year.
- Interest rate of 8% compounded quarterly is equivalent to paying simple interest at 2% per quarter-year, with each successive quarter charging interest not only on the original principal but also on the interest that has accrued up to that point.
- Thus:
  - After one quarter, we owe $1,000(1+0.02)$;
  - After two quarters, we owe $1,000(1+0.02)(1+0.02) = 1,000(1+0.02)^2$;
  - After three quarters, we owe $1,000(1+0.02)^2(1+0.02) = 1,000(1+0.02)^3$;
  - After four quarters, we owe $1,000(1+0.02)^3(1+0.02) = 1,000(1+0.02)^4 = \$1,082.40$.

Example

- Many credit-card companies charge interest at a yearly rate of 18% compounded monthly.
- If the amount P is charged at the beginning of a year, how much is owed at the end of the year if no previous payments have been made?
- Such a compounding is equivalent to paying simple interest every month at a rate of $\frac{18}{12} = 1.5\%$ per month, with the accrued interest then added to the principal owed during the next month.

  Hence, after one year, the amount owed is

$$P(1 + 0.015)^{12} = 1.1956P.$$

Nominal versus Effective Interest Rate

- If the interest rate r is compounded, then the amount of interest actually paid is greater than when paying simple interest at rate r.
- The reason is that in compounding we are being charged interest on the interest that has already been computed in previous compoundings.
- We call r the **nominal interest rate**.
- We define the **effective interest rate** by

$$r_{\text{eff}} = \frac{\text{amount repaid at the end of a year} - P}{P}.$$

Nominal versus Effective Interest Rate (Cont'd)

- If the loan is for one year at a nominal interest rate r that is to be compounded quarterly, then the effective interest rate for the year is

$$r_{\text{eff}} = \left(1 + \frac{r}{4}\right)^4 - 1.$$

- Note that the amount repaid at the end of a single year is

$$P\left(1 + \frac{r}{4}\right)^4 = P(1 + r_{\text{eff}}).$$

- Therefore, the payment made in a one-year loan with compound interest r is the same as if the loan called for simple interest at rate r_{eff} per year.

The Doubling Rule

- If we put funds into an account that pays interest at rate r compounded annually, how many years does it take for the funds to double?
- The initial deposit of D will be worth $D(1+r)^n$ after n years.
- So we need to find the value of n such that $(1+r)^n = 2$.
- We can use the approximation

$$(1+r)^n = (1+\frac{nr}{n})^n \approx e^{nr},$$

  which is fairly precise provided that n is not too small.
- Therefore, we need $e^{nr} \approx 2$.
- We get $n \approx \frac{\log(2)}{r} = \frac{0.693}{r}$.
- So it will take $n \approx \frac{0.7}{r}$ years for the funds to double.

Example

- If $r = 0.01$, then it will take approximately 70 years for the funds to double.
- If $r = 0.02$, it will take about 35 years.
- If $r = 0.03$, it will take about $23\frac{1}{3}$ years.
- If $r = 0.05$, it will take about 14 years.
- If $r = 0.07$, it will take about 10 years.
- If $r = 0.10$, it will take about 7 years.
- Checking the preceding approximations, we note that:

$$\begin{array}{rclrcl} (1.01)^{70} & \approx & 2.007, & (1.05)^{14} & \approx & 1.980, \\ (1.02)^{35} & \approx & 2.000, & (1.07)^{10} & \approx & 1.967, \\ (1.03)^{23.33} & \approx & 1.993, & (1.10)^{7} & \approx & 1.949. \end{array}$$

Continuous Compounding

- Suppose now that we borrow the principal P for one year at a nominal interest rate of r per year, **compounded continuously**.
- We want to compute how much is owed at the end of the year.
- To answer this we must first decide on an appropriate definition of "continuous" compounding.
- Note that if the loan is compounded at n equal intervals in the year, then the amount owed at the end of the year is $P(1+\frac{r}{n})^n$.
- It is reasonable to suppose that continuous compounding refers to the limit of this process as n grows larger and larger.
- So the amount owed at time 1 is

$$P \lim_{n\to\infty} \left(1+\frac{r}{n}\right)^n = Pe^r.$$

Example

- If a bank offers interest at a nominal rate of 5% compounded continuously, what is the effective interest rate per year?

  The effective interest rate is

  $$r_{\text{eff}} = \frac{Pe^{0.05} - P}{P} = e^{0.05} - 1 \approx 0.05127.$$

  That is, the effective interest rate is 5.127% per year.

Continuous Compounding for Time t

- If the amount P is borrowed for t years at a nominal interest rate of r per year compounded continuously, then the amount owed at time t is Pe^{rt}.
- Note that, if interest is compounded n times during the year, then there are nt compoundings by time t.
- This gives a debt level of $P(1+\frac{r}{n})^{nt}$.
- Consequently, under continuous compounding the debt at time t is

$$P \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = P\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right)^{t} = Pe^{rt}.$$

- So continuous compounded interest at rate r per unit time can be interpreted as being a continuous compounding of a nominal interest rate of rt per (unit of time) t.

Subsection 2

Present Value Analysis

Present Value

- Suppose that one can both borrow and loan money at a nominal rate r per period that is compounded periodically.
- Under these conditions, we want to compute the present worth of a payment of v dollars that will be made at the end of period i.
- A bank loan of $v(1+r)^{-i}$ would require a payoff of

  $$v(1+r)^{-i}(1+r)^{i}=v$$

  at period i.
- So the **present value** of a payoff of v to be made at time period i is

  $$v(1+r)^{-i}.$$
- The concept of present value enables us to compare different income streams to see which is preferable.

Example

- Suppose we are to receive payments (in thousands of dollars) at the end of each of the next five years.

  Which of the following three payment sequences is preferable?

  A. 12, 14, 16, 18, 20;
  B. 16, 16, 15, 15, 15;
  C. 20, 16, 14, 12, 10.

  If the nominal interest rate is r compounded yearly, then the present value of the sequence of payments x_i, $i = 1, \ldots, 5$, is

  $$\sum_{i=1}^{5}(1+r)^{-i}x_i.$$

  The sequence having the largest present value is preferred.

  So the superior sequence of payments depends on the interest rate.

Dependence on Interest Rate

- If r is small, then the sequence A is best since its sum of payments is the highest.
- For a somewhat larger value of r, the sequence B would be best because - although the total of its payments (77) is less than that of A (80) - its earlier payments are larger than are those of A.
- For an even larger value of r, the sequence C, whose earlier payments are higher than those of either A or B, would be best.
- The table below gives the present values of these payment streams for three different values of r:

r	A	B	C
0.1	59.21	58.60	56.33
0.2	45.70	46.39	45.69
0.3	36.49	37.89	38.12

Comparing Sequences Any Time

- It should be noted that the payment sequences can be compared according to their values at any specified time.
- For instance, to compare them in terms of their time-5 values, we would determine which sequence of payments yields the largest value of
$$\sum_{i=1}^{5}(1+r)^{5-i}x_i = (1+r)^5\sum_{i=1}^{5}(1+r)^{-i}x_i.$$
- Consequently, we obtain the same preference ordering as a function of interest rate as before.

Equivalence Involving Present Value

- Let the given interest rate be r, compounded yearly.
- Any cash flow stream $\boldsymbol{a} = a_1, a_2, \ldots, a_n$ that returns you a_i dollars at the end of year i, $i = 1, \ldots, n$, can be replaced by:
  - Depositing
  $$\mathrm{PV}(\boldsymbol{a}) = \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_n}{(1+r)^n}$$
  in a bank at time 0;
  - Making the successive withdrawals $a_1, a_2, \ldots, a_n$.
- Withdrawing a_1 at the end of year 1 would leave on deposit
$$\begin{aligned}&(1+r)\left[\tfrac{a_1}{1+r} + \tfrac{a_2}{(1+r)^2} + \cdots + \tfrac{a_n}{(1+r)^n}\right] - a_1\\&= \tfrac{a_2}{(1+r)} + \cdots + \tfrac{a_n}{(1+r)^{n-1}}.\end{aligned}$$

Equivalence Involving Present Value (Cont'd)

- Thus, after withdrawing a_2 at the end of year 2 would leave

$$(1+r)\left[\frac{a_2}{1+r}+\cdots+\frac{a_n}{(1+r)^{n-1}}\right]-a_2$$
$$=\frac{a_3}{(1+r)}+\cdots+\frac{a_n}{(1+r)^{n-2}}.$$

- Continuing, withdrawing a_i at the end of year i, $i<n$, would leave on deposit

$$\frac{a_{i+1}}{(1+r)}+\cdots+\frac{a_n}{(1+r)^{n-i}}.$$

- Withdrawing a_{n-1}, would leave on deposit $\frac{a_n}{(1+r)}$.
- This is just enough to cover the next withdrawal of a_n at the end of the following year.

Equivalence Involving Present Value (Cont'd)

- In a similar manner, the cash flow sequence $a_1, a_2, \ldots, a_n$ can be transformed into the initial capital

$$\mathrm{PV}(\boldsymbol{a}) = \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_n}{(1+r)^n}$$

  by borrowing this amount from a bank and then using the cash flow to pay off this debt.
- Any cash flow sequence is equivalent to an initial reception of the present value of the cash flow sequence.
- So one cash flow sequence is preferable to another whenever the former has a larger present value than the latter.

Example: Purchasing a New Machine

- A company needs a certain type of machine for the next five years.
- They presently own such a machine, which is now worth \$6,000 but will lose \$2,000 in value in each of the next three years, after which it will be worthless and unuseable.
- The (beginning-of-the-year) value of its yearly operating cost is \$9,000, with this amount expected to increase by \$2,000 in each subsequent year that it is used.
- A new machine can be purchased at the beginning of any year for a fixed cost of \$22,000.
- The lifetime of a new machine is six years, and its value decreases by \$3,000 in each of its first two years of use and then by \$4,000 in each following year.
- The operating cost of a new machine is \$6,000 in its first year, with an increase of \$1,000 in each subsequent year.
- If the interest rate is 10%, when should a new machine be bought?

Example (Analysis of a Cash Flow Sequence)

- Suppose that the company will buy a new machine at the beginning of year 3.
  - Its year-1 cost is the \$9,000 operating cost of the old machine;
  - Its year-2 cost is the \$11,000 operating cost of this machine;
  - Its year-3 cost is the \$22,000 cost of a new machine, plus the \$6,000 operating cost of this machine, minus the \$2,000 obtained for the replaced machine;
  - Its year-4 cost is the \$7,000 operating cost;
  - Its year-5 cost is the \$8,000 operating cost;
  - Its year-6 cost is $-\$12,000$, the negative of the value of the 3-year-old machine that it no longer needs.

Thus, if the new machine is bought at the beginning of year 3, we have the six-year cash flow sequence (in units of \$1,000)

$$9, 11, 26, 7, 8, -12.$$

Example (Cash Flow Sequences)

- The company can purchase a new machine at the beginning of year 1, 2, 3, or 4, with similarly analyzed six-year cash flow sequences (in units of $1,000):
  - Buy at beginning of year 1: $22, 7, 8, 9, 10, -4$;
  - Buy at beginning of year 2: $9, 24, 7, 8, 9, -8$;
  - Buy at beginning of year 3: $9, 11, 26, 7, 8, -12$;
  - Buy at beginning of year 4: $9, 11, 13, 28, 7, -16$.

  To compute present values and compare the best option, we assume interest rate 10% ($r = 0.01$).

Example (Comparing Present Values)

- With the yearly interest rate $r = 0.10$, the present value of the first cost-flow sequence is

$$22 + \frac{7}{1.1} + \frac{8}{(1.1)^2} + \frac{9}{(1.1)^3} + \frac{10}{(1.1)^4} - \frac{4}{(1.1)^5} = 46.083.$$

The present values of the other cash flows are similarly determined, and the four present values are

$$46.083, \quad 43.794, \quad 43.760, \quad 45.627.$$

Therefore, the company should purchase a new machine two years from now.

Sum of a Geometric Sequence

- The following algebraic identity is often very useful.

$$1 + b + b^2 + \cdots + b^n = \frac{1 - b^{n+1}}{1 - b}.$$

Let

$$x = 1 + b + b^2 + \cdots + b^n.$$

Then

$$\begin{array}{rcl} x - 1 & = & b + b^2 + \cdots + b^n \\ & = & b(1 + b + \cdots + b^{n-1}) \\ & = & b(x - b^n). \end{array}$$

Sum of a Geometric Sequence (Cont'd)

- We got
$$x - 1 = b(x - b^n).$$
Now we get
$$\begin{aligned} x - 1 &= bx - b^{n+1} \\ x - bx &= 1 - b^{n+1} \\ x(1-b) &= 1 - b^{n+1} \\ x &= \frac{1 - b^{n+1}}{1 - b}. \end{aligned}$$
- Letting n go to infinity, we also see that, when $|b| < 1$,
$$1 + b + b^2 + \cdots = \frac{1}{1 - b}.$$

Example: Retirement

- An individual who plans to retire in 20 years has decided to put an amount A in the bank at the beginning of each of the next 240 months, after which she will withdraw \$1,000 at the beginning of each of the following 360 months.

  Assuming a nominal yearly interest rate of of 6% compounded monthly, how large does A need to be?

  Let $r = \frac{0.06}{12} = 0.005$ be the monthly interest rate.

  With $\beta = \frac{1}{1+r}$, the present value of all her deposits is

$$A + A\beta + A\beta^2 + \cdots + A\beta^{239} = A\frac{1-\beta^{240}}{1-\beta}.$$

Example (Cont'd)

- Similarly, if W is the amount withdrawn in the following 360 months, then the present value of all these withdrawals is

$$W\beta^{240} + W\beta^{241} + \cdots + W\beta^{599} = W\beta^{240}\frac{1-\beta^{360}}{1-\beta}.$$

Thus she will be able to fund all withdrawals (and have no money left in her account) if

$$A\frac{1-\beta^{240}}{1-\beta} = W\beta^{240}\frac{1-\beta^{360}}{1-\beta} \quad \text{or} \quad A = W\beta^{240}\frac{1-\beta^{360}}{1-\beta^{240}}.$$

With $W = 1,000$, and $\beta = \frac{1}{1.005}$, this gives $A = 360.99$.

Saving \$361 a month for 240 months will enable her to withdraw \$1,000 a month for the succeeding 360 months.

Example: Perpetuity

- A perpetuity entitles its holder to be paid the constant amount c at the end of each of an infinite sequence of years.

  That is, it pays its holder c at the end of year i, for $i = 1, 2, \ldots$.

  Suppose the interest rate is r, compounded yearly

  We find the present value of such a cash flow sequence.
- Note that such a cash flow can be replicated by:
  - Initially depositing the principle $\frac{c}{r}$;
  - Then withdrawing the interest earned (leaving the principal intact) at the end of each period.
- This intuition checks mathematically by

$$\begin{aligned} \mathrm{PV} &= \frac{c}{1+r} + \frac{c}{(1+r)^2} + \frac{c}{(1+r)^3} + \cdots \\ &= \frac{c}{1+r}\left[1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots\right] \\ &= \frac{c}{1+r} \cdot \frac{1}{1-\frac{1}{1+r}} = \frac{c}{r}. \end{aligned}$$

Example

- Suppose we want to borrow \$100,000 to purchase a house.

  The loan officer offers a \$100,000 loan, to be repaid in monthly installments over 15 years with an interest rate of 0.6% per month.

  The bank charges:
  - A loan initiation fee of \$600;
  - A house inspection fee of \$400;
  - 1 "point" (meaning that 1% of the nominal loan amount must be paid o the bank when the loan is received).

  We compute the effective annual interest rate of the loan.

Example (Cont'd)

- We first determine the monthly mortgage payment A.

  \$100,000 is to be repaid in 180 monthly payments at an interest rate of 0.6% per month.

  So we get, with $\alpha = \frac{1}{1.006}$,

$$A[\alpha + \alpha^2 + \cdots + \alpha^{180}] = 100,000.$$

  Therefore,

$$A = \frac{100,000(1-\alpha)}{\alpha(1-\alpha^{180})} = 910.05.$$

  So if actually receiving \$100,000 to be repaid in 180 monthly payments of \$910.05, then the effective monthly interest rate would be 0.6%.

Example (Cont'd)

- But the amount to be repaid in 180 monthly payments at an effective interest rate of r per month is actually

$$100,000 - 600 - 400 - 0.01 \cdot 100,000 = 98,000.$$

So, for $\beta = \frac{1}{1+r}$, we get

$$910.05[\beta + \beta^2 + \cdots + \beta^{180}] = 98,000.$$

Therefore,

$$\frac{\beta(1 - \beta^{180})}{1 - \beta} = 107.69.$$

Since $\beta = \frac{1}{1+r}$, i.e., $\frac{1-\beta}{\beta} = r$, we get

$$\frac{1 - (\frac{1}{1+r})^{180}}{r} = 107.69.$$

Solving numerically yields $r = .00627$. So the effective annual interest rate is $(1 + 0.00627)^{12} - 1 = 0.0779$.

Example (Mortgage Loan)

- Suppose that one takes a mortgage loan for the amount L that is to be paid back over n months with equal payments of A at the end of each month.

  The interest rate for the loan is r per month, compounded monthly.
  (a) We want to find the value of A in terms of L, n and r.
  (b) We want to compute the amount of loan principal that remains after payment has been made at the end of month j.
  (c) We want to compute the amount of the payment during month j that is for interest and the amount that is for principal reduction?

  This is important for several reasons.
  - Some contracts allow for the loan to be paid back early;
  - The interest part of the payment is tax-deductible.

Example (Part (a))

(a) The present value of the n monthly payments is

$$\begin{aligned} & \frac{A}{1+r} + \frac{A}{(1+r)^2} + \cdots + \frac{A}{(1+r)^n} \\ & = \frac{A}{1+r} \cdot \frac{1-(\frac{1}{1+r})^n}{1-\frac{1}{1+r}} \\ & = \frac{A}{r}[1-(1+r)^{-n}]. \end{aligned}$$

This must equal the loan amount L.

So we get

$$A = \frac{Lr}{1-(1+r)^{-n}}.$$

Letting $\alpha = 1 + r$, we can rewrite

$$A = \frac{L(\alpha-1)\alpha^n}{\alpha^n - 1}.$$

Example (Part (b))

(b) Let R_j denote the remaining amount of principal owed after the payment at the end of month j, $j = 0, \ldots, n$.

Suppose we owe R_j at the end of month j.

Then the amount owed immediately before the payment at the end of month $j+1$ is $(1+r)R_j$.

Because one then pays the amount A, it follows that

$$R_{j+1} = (1+r)R_j - A = \alpha R_j - A.$$

Noting that $R_0 = L$, we obtain:

$$R_1 = \alpha L - A;$$
$$R_2 = \alpha R_1 - A = \alpha(\alpha L - A) - A = \alpha^2 L - (1+\alpha)A;$$
$$R_3 = \alpha R_2 - A = \alpha(\alpha^2 L - (1+\alpha)A) - A = \alpha^3 L - (1+\alpha+\alpha^2)A.$$

Example (Part (b) Cont'd)

- In general, for $j = 0, \ldots, n$ we obtain

$$\begin{aligned} R_j &= \alpha^j L - A(1 + \alpha + \cdots + \alpha^{j-1}) \\ &= \alpha^j L - A\frac{\alpha^j - 1}{\alpha - 1} \\ &= \alpha^j L - \frac{L\alpha^n(\alpha^j - 1)}{\alpha^n - 1} \\ &= \frac{L(\alpha^{j+n} - \alpha^j) - L(\alpha^{j+n} - \alpha^n)}{\alpha^n - 1}. \\ &= \frac{L(\alpha^n - \alpha^j)}{\alpha^n - 1}. \end{aligned}$$

Example (Part (c))

(c) Let I_j and P_j denote the amounts of the payment at the end of month j that are for interest and for principal reduction, respectively.

Since R_{j-1} was owed at the end of the previous month, we have

$$I_j = rR_{j-1} = \frac{L(\alpha-1)(\alpha^n - \alpha^{j-1})}{\alpha^n - 1}.$$

Moreover,

$$P_j = A - I_j = \frac{L(\alpha-1)}{\alpha^n - 1}[\alpha^n - (\alpha^n - \alpha^{j-1})] = \frac{L(\alpha-1)\alpha^{j-1}}{\alpha^n - 1}.$$

As a check, note that $\sum_{j=1}^n P_j = L$.

Note also that the amount of principal repaid in successive months increases by the factor $\alpha = 1 + r$.

Comparing Two Cash Flows

- Consider two cash flow sequences,

$$b_1, b_2, \ldots, b_n \quad \text{and} \quad c_1, c_2, \ldots, c_n.$$

  Under what conditions is the present value of the first sequence at least as large as that of the second for every positive interest rate r?
- Clearly, $b_i \geq c_i$ $(i = 1, \ldots, n)$ is a sufficient condition.
- However, we can obtain weaker sufficient conditions.
- Let

$$B_i = \sum_{j=1}^{i} b_j \quad \text{and} \quad C_i = \sum_{j=1}^{i} c_j, \quad \text{for } i = 1, \ldots, n.$$

- It can be shown that the condition

$$B_i \geq C_i \quad \text{for each } i = 1, \ldots, n$$

  suffices.

Sufficient Condition

Proposition

If $B_n \geq C_n$ and if $\sum_{i=1}^{k} B_i \geq \sum_{i=1}^{k} C_i$, for each $k = 1, \ldots, n$, then

$$\sum_{i=1}^{n} b_i (1+r)^{-i} \geq \sum_{i=1}^{n} c_i (1+r)^{-i},$$

for every $r > 0$.

Sufficient Condition (Revisited)

- The proposition says that the cash flow sequence $b_1, \ldots, b_n$ will, for every positive interest rate r, have a larger present value than the cash flow sequence $c_1, \ldots, c_n$ if:
  (i) The total of the b cash flows is at least as large as the total of the c cash flows;
  (ii) For every $k = 1, \ldots, n$,

$$kb_1 + (k-1)b_2 + \cdots + b_k \geq kc_1 + (k-1)c_2 + \cdots + c_k.$$

Subsection 3

Rate of Return

Rate of Return

- Consider an investment that, for an initial payment of a ($a > 0$), returns the amount b after one period.
- The **rate of return** on this investment is defined to be the interest rate r that makes the present value of the return equal to the initial payment.
- That is, the rate of return is that value r such that $b = a(1 + r)$.
- This gives
$$r = \frac{b}{a} - 1.$$

  Example: A \$100 investment that returns \$150 after one year is said to have a yearly rate of return of
$$r = \frac{150}{100} - 1 = 0.50.$$

Rate of Return per Period

- Consider an investment that, for an initial payment of a ($a > 0$), yields a string of nonnegative returns $b_1, \ldots, b_n$.
- Here b_i is received at the end of period i, $i = 1, \ldots, n$, and $b_n > 0$.
- We define the **rate of return per period** of this investment to be the value of the interest rate, such that the present value of the cash flow sequence is equal to zero when values are compounded periodically at that interest rate.
- Formally, we first define the function P by

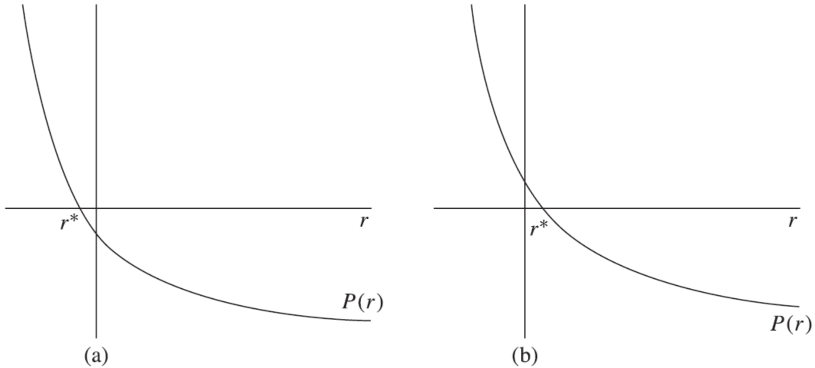
$$P(r) = -a + \sum_{i=1}^{n} b_i (1+r)^{-i}.$$

- Then the rate of return per period of the investment is that value $r^* > -1$ for which $P(r^*) = 0$.

Rate of Return Analysis

- We have
$$P(r) = -a + \sum_{i=1}^{n} b_i (1+r)^{-i}.$$
- We make the following observations.
  - $\lim_{r \to -1} P(r) = \infty$;
  - $\lim_{r \to \infty} P(r) = -a < 0$;
  - $P(r)$ is a strictly decreasing function of r when $r > -1$.
- Hence, there is a unique value r^* satisfying $P(r^*) = 0$.
- Now note that $P(0) = \sum_{i=1}^{n} b_i - a$.
- Therefore, r^* will be:
  - Positive if $\sum_{i=1}^{n} b_i > a$;
  - Negative if $\sum_{i=1}^{n} b_i < a$.

Rate of Return (Illustration)



$$P(r) = -a + \sum_{i \geq 1} b_i (1+r)^{-i}\text{: (a) } \sum_i b_i < a\text{; (b) } \sum_i b_i > a$$

Rate of Return Analysis (Cont'd)

- Because of the monotonicity of $P(r)$, it follows that the cash flow sequence will have:
  - A positive present value when the interest rate is less than r^*;
  - A negative present value when the interest rate is greater than r^*.
- When an investment's rate of return is r^* per period, we often say that the investment yields a $100r^*$ **percent rate of return per period**.

Example

- Find the rate of return from an investment that, for an initial payment of 100, yields returns of 60 at the end of each of the first two periods.

  The rate of return will be the solution to

  $$P(r) = -100 + \frac{60}{1+r} + \frac{60}{(1+r)^2} = 0.$$

  Let $x = \frac{1}{1+r}$.
  We get

  $$60x^2 + 60x - 100 = 0.$$

  So $x = \frac{-60 \pm \sqrt{60^2 + 4(60)(100)}}{120}$.
  Since $-1 < r$, $x = \frac{-60 + \sqrt{27{,}600}}{120} \approx 0.8844$.
  Hence, $r^* = \frac{1}{x} - 1 \approx \frac{1}{0.8844} - 1 \approx 1.131$.
  The rate of return is approximately 13.1% per period.

Other Situations

- Consider an investment cash flow sequence $c_0, c_1, \ldots, c_n$.
  - If $c_i \geq 0$, then the amount c_i is received by the investor at the end of period i;
  - If $c_i < 0$, then the amount $-c_i$ must be paid by the investor at the end of period i.
- Let

$$P(r) = \sum_{i=0}^{n} c_i (1+r)^{-i}$$

  be the present value of this cash flow when the interest rate is r per period.
- Then, in general, there will not necessarily be a unique solution of the equation $P(r) = 0$ in the region $r > -1$.
- As a result, the rate-of-return concept is unclear in the case of more general cash flows than the ones we considered.

Subsection 4

Continuously Varying Interest Rates

Instantaneous Interest Rate

- Suppose that interest is continuously compounded but with a rate that is changing in time.
- Let $r(s)$ denote the interest rate at time s.
- Thus, if you put x in a bank at time s, then the

$$\text{amount in your account at time } s+h \approx x(1+r(s)h) \quad (h \text{ small}).$$

- The quantity $r(s)$ is called the **spot** or the **instantaneous interest rate** at time s.

Rate of Change of Amount

- Suppose we deposit 1 at time 0.
- Let $D(t)$ be the amount in the account at time t.
- In order to determine $D(t)$ in terms of the interest rates $r(s)$, $0 \leq s \leq t$, note that, for h small,

$$D(s+h) \approx D(s)(1+r(s)h).$$

- Equivalently,

$$D(s+h) - D(s) \approx D(s)r(s)h.$$

- That is

$$\frac{D(s+h) - D(s)}{h} \approx D(s)r(s).$$

- This approximation becomes exact as h approaches 0.
- Taking the limit as $h \to 0$, we get

$$D'(s) = D(s)r(s).$$

Rate of Change of Amount

- We computed $\frac{D'(s)}{D(s)} = r(s)$.
- Therefore

$$\int_0^t \frac{D'(s)}{D(s)} ds = \int_0^t r(s) ds;$$

- Equivalently,

$$\log(D(t)) - \log(D(0)) = \int_0^t r(s) ds.$$

- Since $D(0) = 1$, we obtain from the preceding equation that

$$D(t) = \exp\left\{\int_0^t r(s) ds\right\}.$$

Present Value

- Now let $P(t)$ denote the present (i.e., time-0) value of the amount 1 that is to be received at time t.
  - $P(t)$ would be the cost of a bond that yields a return of 1 at time t;
  - It would equal e^{-rt} if the interest rate were always equal to r.
- A deposit of $\frac{1}{D(t)}$ at time 0 will be worth 1 at time t.
- So we get

$$P(t) = \frac{1}{D(t)} = \exp\left\{-\int_0^t r(s)ds\right\}.$$

The Yield Curve

- Let $\overline{r}(t)$ denote the average of the spot interest rates up to time t.
- That is,
$$\overline{r}(t) = \frac{1}{t}\int_0^t r(s)ds.$$
- The function $\overline{r}(t)$, $t \geq 0$, is called the **yield curve**.
- By the previous slides,
$$D(t) = e^{t\overline{r}(t)} \quad \text{and} \quad P(t) = e^{-t\overline{r}(t)}.$$

Example

- Find the yield curve and the present value function if

$$r(s) = \frac{1}{1+s}r_1 + \frac{s}{1+s}r_2.$$

Rewrite $r(s)$ as

$$r(s) = = \frac{r_2 + r_2 s + r_1 - r_2}{1+s} = r_2 + \frac{r_1 - r_2}{1+s}.$$

Thus, the yield curve is given by

$$\begin{aligned} \overline{r}(t) &= \tfrac{1}{t}\int_0^t \left(r_2 + \tfrac{r_1-r_2}{1+s}\right) ds \\ &= \tfrac{1}{t}\left[r_2 s + (r_1 - r_2)\log(1+s)\right]_0^t \\ &= \tfrac{1}{t} r_2 t + \tfrac{r_1-r_2}{t}\log(1+t) \\ &= r_2 + \tfrac{r_1-r_2}{t}\log(1+t). \end{aligned}$$

Example (Cont'd)

- We computed

$$\overline{r}(t) = r_2 + \frac{r_1 - r_2}{t} \log(1+t).$$

Consequently, the present value function is

$$\begin{aligned} P(t) &= e^{-t\overline{r}(t)} \\ &= e^{-t(r_2 + \frac{r_1 - r_2}{t} \log(1+t))} \\ &= e^{-tr_2} e^{(r_2 - r_1)\log(1+t)} \\ &= e^{-r_2 t} e^{\log((1+t)^{r_2 - r_1})} \\ &= e^{-r_2 t} (1+t)^{r_2 - r_1}. \end{aligned}$$