Introduction to Mathematical Finance

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 500

1. Pricing Contracts via Arbitrage
   - An Example in Options Pricing
   - Other Examples of Pricing via Arbitrage

Subsection 1

An Example in Options Pricing

Options Pricing

- Suppose that the nominal interest rate is r.
- We study a model for pricing an option to purchase a stock at a future time at a fixed price.
- Let the present price (in dollars) of the stock be 100 per share.
- Suppose after one time period its price will be either 200 or 50.
- Suppose that, for any y, at a cost of Cy we can purchase at time 0 the option to buy y shares of the stock at time 1 at a price of 150 per share.
    - Suppose we purchase this option.
        - If the stock rises to 200, we exercise the option at time 1 and realize a gain of $200 - 150 = 50$ for each of the y options purchased.
        - If the price of the stock at time 1 is 50, the option is worthless.
- In addition to the options, we may purchase x shares of the stock at time 0 at a cost of $100x$.
- Each share would be worth either 200 or 50 at time 1.

Range of Option Pricing

- We suppose that both x and y can be positive, negative or zero.
- I.e, we can either buy or sell both the stock and the option.
- E.g., suppose x is negative.
  - We would be selling $-x$ shares of stock, with an initial return of $-100x$.
  - We would be responsible for buying and returning $-x$ shares of the stock at time 1 at a cost of either 200 or 50 per share.
- When we sell a stock that we do not own, we say that we are **selling it short**.

Unit Cost of Option

- We are interested in determining the appropriate value of C, the unit cost of an option.
- We fix the one-period interest rate r.
- We show that, unless

$$C = \frac{1}{3}\left[100 - \frac{50}{1+r}\right],$$

there is a combination of purchases that will always result in a positive present value gain.

Unit Cost of Option (Cont'd)

- To show this, suppose that at time 0:
  - We purchase x units of stock;
  - We purchase y units of options.
- Here x and y (both of which can be either positive or negative) are to be determined.
- The cost of this transaction is

$$100x + Cy.$$

  - If this amount is positive, then it should be borrowed from a bank, to be repaid with interest at time 1;
  - if it is negative, then the amount received, $-(100x + Cy)$, should be put in the bank to be withdrawn at time 1.

Value of Time-1 Holdings

- The value of our holdings at time 1 depends on the price of the stock at that time.
- It is given by

$$\text{value} = \begin{cases} 200x + 50y, & \text{if the price is 200,} \\ 50x, & \text{if the price is 50.} \end{cases}$$

- This formula follows by noting the following.
  - If the stock's price at time 1 is 200, then:
    - The x shares of the stock are worth $200x$;
    - The y units of options to buy the stock at a share price of 150 are worth $(200 - 150)y$.
  - If the stock's price is 50, then:
    - The x shares are worth $50x$;
    - The y units of options are worthless.

Fixing Value of Time-1 Holdings

- Now, suppose we choose y so that the value of our holdings at time 1 is the same no matter what the price of the stock at that time.
- That is, we choose y so that $200x + 50y = 50x$ or $y = -3x$.
- Note that y has the opposite sign of x.
  - If $x > 0$:
    - x shares of the stock are purchased at time 0;
    - $3x$ units of stock options are sold at that time.
  - Similarly, if $x < 0$:
    - $-x$ shares of the stock are sold at time 0;
    - $-3x$ units of stock options are purchased at time 0.
- Thus, with $y = -3x$, the time-1 value of holdings equals $50x$ no matter what the value of the stock.

Gain

- Assume $y = -3x$.
  - If $100x + Cy > 0$, we took out a loan;
  - If $100x + Cy < 0$, we deposited a sum.
- In either case, after paying off our loan or withdrawing our money from the bank, we will have gained the amount

$$\begin{aligned} \text{gain} &= 50x - (100x + Cy)(1 + r) \\ &= 50x - (100x - 3xC)(1 + r) \\ &= (1 + r)x \left[3C - 100 + \frac{50}{1+r}\right]. \end{aligned}$$

Guaranteed Positive Gain

- We found gain amount

$$\text{gain} = (1+r)x\left[3C - 100 + \frac{50}{1+r}\right].$$

- Thus, if $3C = 100 - \frac{50}{1+r}$, then the gain is 0.
- On the other hand, suppose $3C \neq 100 - \frac{50}{1+r}$.
- Then we can guarantee a positive gain (no matter what the price of the stock at time 1) by letting:
  - x be positive when $3C > 100 - \frac{50}{1+r}$;
  - x be negative when $3C < 100 - \frac{50}{1+r}$.

Example: Arbitrage

- Suppose $\frac{1}{1+r} = 0.9$.
- Then the gain is 0 at cost per option

$$C = \frac{1}{3}[100 - 50 \cdot 0.9] \approx 18.33.$$

- Suppose the cost per option is $C = 20$.
- Then we:
  - Purchase one share of the stock;
  - Sell three units of options.
- The initial cost is $100 - 3(20) = 40$.
- We borrow this from the bank.
- The value of this holding at time 1 is 50 regardless of whether the stock price rises to 200 or falls to 50.
- We use $40(1 + r) = 44.44$ of this amount to pay our bank loan.
- This results in a guaranteed gain of 5.56.

Example: Arbitrage (Cont'd)

- Suppose the cost per option is $C = 15$.
- Then we:
  - Sell one share of the stock;
  - Buy three units of options.
- The initial gain is $100 - 3(15) = 55$.
- We deposit this in the bank.
- The value of this holding at time 1 is -50 regardless of whether the stock price rises to 200 or falls to 50.
- Our bank amount at time 1 is $55(1 + r) = 61.11$.
- This results in a guaranteed gain of 11.11.
- A sure-win betting scheme is called an **arbitrage**.
- For the numbers considered, the only option cost C that does not result in an arbitrage is $C = \frac{1}{3}(100 - 45) \approx 18.33$.

The Law of One Price

Proposition (The Law of One Price)

Consider two investments.

- The first costs the fixed amount C_1;
- The second costs the fixed amount C_2.

If the (present value) payoff from the first investment is always identical to that of the second investment, then either $C_1 = C_2$ or there is an arbitrage.

- The proof of the Law of One Price is straightforward.

  Suppose the costs of the investments are not equal.

  Then an arbitrage is obtained by:

  - Buying the cheaper investment;
  - Selling the more expensive one.

Example

- We apply the Law of One Price to our previous example.
- The payoff at time 1 from purchasing the option is

$$\text{payoff of option} = \begin{cases} 50, & \text{if the price is } 200, \\ 0, & \text{if the price is } 50. \end{cases}$$

- Consider now a second investment that calls for purchasing y shares of the security by:
  - Borrowing x from the bank - to be repaid (with interest) at time 1.
  - Investing $100y - x$ of our own funds.
- The initial cost of this investment is $100y - x$.
- The payoff at time 1 from this investment is

$$\text{payoff of investment} = \begin{cases} 200y - x(1+r), & \text{if the price is } 200, \\ 50y - x(1+r), & \text{if the price is } 50. \end{cases}$$

Example (Cont'd)

- We calculate x and y, so that the payoffs from this investment and the option would be identical.
- We must have
$$\begin{array}{rcl} 200y - x(1+r) & = & 50, \\ 50y - x(1+r) & = & 0. \end{array}$$
- Solving the system gives
$$y = \frac{1}{3}, \quad x = \frac{50}{3(1+r)}.$$
- The cost of the investment is
$$100y - x = \frac{1}{3}\left[100 - \frac{50}{1+r}\right].$$
- By the Law of One Price, either this is the cost of the option or there is an arbitrage.

Example (Cont'd)

- We specify the arbitrage (buy the cheaper investment and sell the more expensive one), when the cost of the option $C \neq \frac{1}{3}\left(100 - \frac{50}{1+r}\right)$.
- **Case 1** ($C < \frac{100-\frac{50}{1+r}}{3}$):
  In this case sell $\frac{1}{3}$ share. This yields $\frac{100}{3}$.
  Of this amount:
  - Use C to purchase an option;
  - Put the remainder ($> \frac{50}{3(1+r)}$) in the bank.

  There are two possible outcomes.
  - The price at time 1 is 200.
    The option is worth 50 and we have more than $\frac{50}{3}$ in the bank.
    So we have more than enough to meet the obligation of $\frac{200}{3}$.
  - The price at time 1 is 50.
    We have more than $\frac{50}{3}$ in the bank.
    This is more than enough to cover the obligation of $\frac{50}{3}$.

Example (Cont'd)

- **Case 2** ($C > \frac{100 - \frac{50}{1+r}}{3}$):

  In this case:

  - Sell an option;
  - Borrow $\frac{50}{3(1+r)}$ from the bank;
  - Use $\frac{100}{3}$ of the proceeds to purchase $\frac{1}{3}$ of a share.

  The amount left over is $C - \frac{100 - \frac{50}{1+r}}{3}$.

  - Suppose the price at time 1 is 200.
    - We use the $\frac{200}{3}$ from the $\frac{1}{3}$ share to pay $\frac{50}{3}$ to the bank;
    - We use 50 to pay the option buyer.
  - Suppose, next, that the price at time 1 is 50.
    Then the option sold is worthless.
    So we use the $\frac{50}{3}$ from the $\frac{1}{3}$ share to pay the bank.

  Remark: A global ongoing assumption is that there is always a market, i.e., any investment can always be bought or sold.

Subsection 2

Other Examples of Pricing via Arbitrage

Call Options and Styles

- The type of option considered in the preceding section is known as a **call option** because it gives one the option of calling for the stock at a specified price, known as the **exercise** or **strike price**.
- An **American style call option** allows the buyer to exercise the option at any time up to the expiration time.
- A **European style call option** can only be exercised at the expiration time.
- It appears that, because of its additional flexibility, the American style option would be worth more.
- However, it is never optimal to exercise a call option early.
- We will show the two style options have identical worths.

Exercise Time

Proposition

One should never exercise an American style call option before its expiration time t.

- Suppose that the present price of the stock is S.
  We own an option to buy one share of the stock at a fixed price K.
  This option expires after an additional time t.
  Exercising the option now, yields the amount $S - K$.
  Alternatively, we could sell the stock short and, then, purchase the stock at time t, in the least expensive of two ways:
  - Paying the market price at that time;
  - Exercising the option and paying K.

  The latter strategy involves:
  - An initial revenue of S;
  - A subsequent expenditure of the minimum of the market price and the exercise price K after an additional time t.

  This is clearly preferable to receiving S and immediately paying out K.

Put Options and Styles

- In addition to call options there are also **put options** on stocks.
- These give their owners the option of putting a stock up for sale at a specified price.
- An **American style put option** allows the owner to put the stock up for sale - that is, to exercise the option - at any time up to the expiration time of the option.
- A **European style put option** can only be exercised at its expiration time.
- Contrary to the situation with call options, it may be advantageous to exercise a put option before its expiration time.
- So the American style put option may be worth more than the European.

Put-Call Option Parity Formula

- The absence of arbitrage implies a relationship between:
  - The price of a European put option having exercise price K and expiration time t;
  - The price of a call option on that stock that also has exercise price K and expiration time t.

Proposition (Put-Call Option Parity Formula)

Let C be the price of a call option that enables its holder to buy one share of a stock at an exercise price K at time t.
Let P be the price of a European put option that enables its holder to sell one share of the stock for the amount K at time t.
Let S be the price of the stock at time 0.
Then, assuming that interest is continuously discounted at a nominal rate r, either

$$S + P - C = Ke^{-rt}$$

or there is an arbitrage opportunity.

Proof (Case 1)

- Suppose, first, that $S + P - C < Ke^{-rt}$.

  We can effect a sure win by initially:
  - Buying one share of the stock;
  - Buying one put option;
  - Selling one call option.

  The payout $S + P - C$ is borrowed from a bank to be repaid at time t.

  Let us now consider the value of our holdings at time t.

  The value depends on $S(t)$, the stock's market price at time t.
  - If $S(t) \leq K$, then the call option we sold is worthless.
    We can exercise our put option to sell the stock for the amount K.
  - If $S(t) > K$ then our put option is worthless.
    The call option we sold forces selling our stock for the price K.

  In either case we will realize the amount K at time t.

  Since $K > e^{rt}(S + P - C)$, we can pay off our bank loan and realize a positive profit in all cases.

Proof (Case 2)

- Suppose, next, that $S+P-C>Ke^{-rt}$.

  Then we can make a sure profit by reversing the preceding procedure.
  - We sell one share of stock;
  - We sell one put option;
  - We buy one call option.

  We have initial revenue $S+P-C$, which we deposit in the bank.

  The value of our holdings at time t depends on $S(t)$, the stock's market price at time t.
  - If $S(t)\leq K$, then the call option we bought is worthless. The put option sold forces up to buy a stock for the amount K.
  - If $S(t)>K$ then the put option is worthless. The call option we bought allows up to buy a stock for K.

  In either case we will spend the amount K at time t.

  Since $K<(S+P-C)e^{rt}$, we have enough in the bank to pay our obligations and realize a positive profit.

Example (Forward Contracts)

- Let S be the present market price of a specified stock.
- In a **forwards agreement**, one agrees at time 0 to pay the amount F at time t for one share of the stock that will be delivered at the time of payment.
- That is, one contracts a price for the stock, which is to be delivered and paid for at time t.
- Suppose interest is continuously discounted at the nominal interest rate r.
- We show, using an arbitrage argument, that, in order for there to be no arbitrage opportunity, we must have

$$F = Se^{rt}.$$

Example (Forward Contracts Case I)

- Suppose first that $F < Se^{rt}$.

  In this case:
  - We sell a stock at time 0.
  - We put the sale proceeds S into a bond that matures at time t.
  - We buy a forwards contract for delivery of one share of the stock at time t.

  At time t we will receive Se^{rt} from the bond.

  We pay F to obtain one share of the stock.

  We end up with a positive profit of $Se^{rt} - F$.

Example (Forward Contracts Case II)

- Suppose, next, that $F > Se^{rt}$.

  Then we do the following.
  - We sell a forwards contract;
  - We borrow S to purchase the stock.

  At time t, we will receive F for our stock.

  Since, $F > Se^{rt}$, we repay the loan amount Se^{rt}.

  We are guaranteed a profit of $F - Se^{rt}$.

Remark (Using Law of One Price)

- Another way to see that $F = Se^{rt}$ in the preceding example is to use the law of one price.
- Consider the following investments, both of which result in owning the security at time t:
  (1) Put Fe^{-rt} in the bank and purchase a forward contract.
  (2) Buy the stock.

  By the Law of One Price, either

$$Fe^{-rt} = S$$

  or there is an arbitrage.

Commodity Markets

- When one purchases a share of a stock in the stock market, one is purchasing a share of ownership in the entity that issues the stock.
- On the other hand, the commodity market deals with more concrete objects:
  - Agricultural items like oats, corn or wheat;
  - Energy products like crude oil and natural gas;
  - Metals such as gold, silver or platinum;
  - Animal parts such as hogs, pork-bellies and beef;

    ⋮
- Almost all of the activity on the commodities market is involved with contracts for future purchases and sales of the commodity.

Futures Contracts

- You could purchase a contract to buy natural gas in 90 days for a price that is specified today.

  (Such a **futures contract** differs from a forwards contract in that, although one pays in full when delivery is taken for both, in futures contracts one settles up on a daily basis depending on the change of the price of the futures contract on the commodity exchange.)
- You could also write a futures contract that obligates you to sell gas at a specified price at a specified time.
- Most people who play the commodities market never have actual contact with the commodity.
- Rather, people who buy a futures contract most often sell that contract before the delivery date.

Futures Contracts versus Forwards Contracts

- The relationship given in the preceding example does not hold for futures contracts in the commodity market.
  - Suppose, first, $F > Se^{rt}$.
    The plan called for purchasing the commodity (say, crude oil) and selling it back at time t.
    In this case, will will incur additional costs related to storing and insuring the commodity.
  - Suppose, next, $F < Se^{rt}$.
    The plan called for selling the commodity for today's price.
    This requires that we be able to deliver it immediately.

Forward Contracts and Currency Exchanges

- On January 7, 2024, a web site gave the following listing for the price of a euro (€):
  - Today: 1.09;
  - 90-day forward: 1.08.
- In other words, you can purchase €1 today at the price of \$1.09.
- In addition, you can sign a contract to purchase €1 in 90 days at a price, to be paid on delivery, of \$1.08.
- Why are these prices different?
- One might suppose that the difference is caused by the market's expectation of the worth in 90 days of the euro relative to the U.S. dollar.
- However, the entire price differential is due to the different interest rates in Europe and in the United States.

Forward Contracts and Currency Exchanges (Cont'd)

- Suppose the interest in both systems is continuously compounded:
  - At nominal yearly rate r_u in the United States;
  - At nominal yearly rate r_e in Europe.
- Let S denote the present price of €1.
- Let F be the price for a forwards contract to be delivered at time t.
- In the case of the example:
  - $S = 1.09$;
  - $F = 1.08$;
  - $t = \frac{90}{365}$.

Forward Contracts and Currency Exchanges (Cont'd)

- In order for there not to be an arbitrage opportunity, we must have

$$F = Se^{(r_u - r_e)t}.$$

Consider two ways to obtain €1 at time t.

(1) Put $Fe^{-r_u t}$ in a U.S. bank.
Buy a forward contract to purchase €1 at time t.

(2) Purchase $e^{-r_e t}$ euros;
Put them in a European bank.

Note that:

- The first investment costs $Fe^{-r_u t}$;
- The second investment costs $Se^{-r_e t}$;
- Both investments yield €1 at time t.

Therefore, by the Law of One Price, either

$$Fe^{-r_u t} = Se^{-r_e t}$$

or there is an arbitrage.

Forward Contracts and Currency Exchanges (Case 1)

- Now suppose $Fe^{-r_u t} < Se^{-r_e t}$.

  We obtain an arbitrage.
  - Borrow €1 from a European bank;
  - Sell it for S U.S. dollars;
  - Put that amount in a U.S. bank.
  - Buy a forward contract to purchase $e^{r_e t}$ euros at time t.

  At time t, we will have $Se^{r_u t}$ dollars.
  - We use $Fe^{r_e t}$ to pay the forward contract for $e^{r_e t}$ euros.
  - We give these euros to the European bank to pay off our loan.

  Since $Se^{r_u t} > Fe^{r_e t}$, we have a positive amount remaining.

Forward Contracts and Currency Exchanges (Case 2)

- Next, suppose $Fe^{-r_u t} > Se^{-r_e t}$.

  We get an arbitrage.
  - Borrow $Se^{-r_e t}$ dollars from a U.S. bank;
  - Use them to purchase $e^{-r_e t}$ euros;
  - Deposit the purchased euros in a European bank;
  - Sell a forward contract for the purchase of €1 at time t.

  At time t:
  - Take out your €1 from the European bank;
  - Give it to the buyer of the forward contract, who will pay F for it.

  The amount we owe the U.S. bank is $Se^{-r_e t}e^{r_u t}$.

  Since $Se^{-r_e t}e^{r_u t} < F$, we have an arbitrage.

Generalized Law of One Price

Proposition (The Generalized Law of One Price)

Consider two investments.

- The first costs the fixed amount C_1;
- The second costs the fixed amount C_2.

If $C_1 < C_2$ and the (present value) payoff from the first investment is always at least as large as that from the second investment, then there is an arbitrage.

- We obtain an arbitrage by simultaneously:
  - Buying investment 1;
  - Selling investment 2.

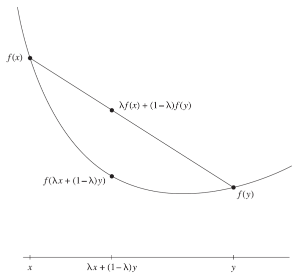
Convex Functions

Definition

A function $f(x)$ is said to be **convex** if, for for all x, y and $0 < \lambda < 1$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

- Consider $\lambda x + (1-\lambda)y$ as in the figure.
- $f(\lambda x + (1-\lambda)y)$ is the corresponding value on the curve.
- $\lambda f(x) + (1-\lambda)f(y)$ is a point on the straight line between $f(x)$ and $f(y)$.



- So convexity states that the straight line segment connecting two points on the curve $f(x)$ always lies above (or on) the curve.

Cost of a Call Option

Proposition

Let $C(K, t)$ be the cost of a call option on a specified security that has strike price K and expiration time t.

(a) For fixed expiration time t, $C(K, t)$ is a convex and nonincreasing function of K.

(b) For $s > 0$, $C(K, t) - C(K + s, t) \leq se^{-rt}$.

- Let $S(t)$ be the price of the security at time t.

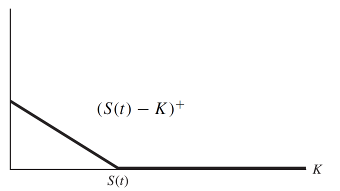
  Then the payoff at time t from a (K, t) call option is

$$\text{payoff of option} = \begin{cases} S(t) - K, & \text{if } S(t) \geq K, \\ 0, & \text{if } S(t) < K. \end{cases}$$

  That is, payoff of option $= (S(t) - K)^+$, where x^+ is the **positive part** of x (equals x, if $x \geq 0$, and 0, if $x < 0$).

Cost of a Call Option (Part (a))

- For fixed $S(t)$, a plot of the payoff function $(S(t)-K)^+$ indicates that it is a convex function of K.



- We show that $C(K,t)$ is a convex function of K.

  Suppose that $K = \lambda K_1 + (1-\lambda)K_2$, for $0 < \lambda < 1$.

  Now consider two investments.

  (1) Purchase a (K,t) call option;
  (2) Purchase λ (K_1,t) call options and $1-\lambda$ (K_2,t) call options.

  The payoffs at time t are:

  - From Investment (1), $(S(t)-K)^+$;
  - From Investment (2), $\lambda(S(t)-K_1)^+ + (1-\lambda)(S(t)-K_2)^+$.

  By the convexity of $(S(t)-K)^+$, the payoff from Investment (2) is at least as large as that from Investment (1).

Cost of a Call Option (Part (a) Cont'd)

- We reasoned that he payoff from Investment (2) is at least as large as that from Investment (1).

  By the Generalized Law of One Price, one of the following must hold.
  - The cost of Investment (2) is at least as large as that of Investment (1);
  - There is an arbitrage.

  That is, either

  $$C(K,t) \leq \lambda C(K_1,t) + (1-\lambda)C(K_2,t)$$

  or there is an arbitrage.

  Hence, convexity is established.

  We show, next that $C(K,t)$ is nonincreasing in K.

  Suppose, to the contrary, for some $h > 0$,

  $$C(K,t) < C(K+h,t).$$

  In general, $(S(t)-(K+h))^+ \leq (S(t)-K)^+$.

  Thus, by the Generalized Law of One Price, there is an arbitrage.

Cost of a Call Option (Part (b))

- Suppose that
$$C(K,t) > C(K+s,t) + se^{-rt}.$$
Then, we can obtain an arbitrage.
  - Sell a call with strike price K and exercise time t;
  - Buy a call with strike price $K+s$ and exercise time t;
  - Deposit the remaining amount
$$C(K,t) - C(K+s,t) \geq se^{-rt}$$
in the bank.

The payoff of the call with strike price K can exceed that of the one with price $K+s$ by at most s.

So this combination of buying one call and selling the other always yields a positive profit.

Remark

- Part (b) of the preceding proposition is equivalent to

$$\frac{\partial}{\partial K} C(K,t) \geq -e^{-rt}.$$

Part (b) implies

$$C(K+s,t) - C(K,t) \geq -se^{-rt}, \quad \text{for } s > 0.$$

Dividing by s and letting s go to 0 yields the result.

Suppose, conversely, that the inequality holds.

Then

$$\int_K^{K+s} \frac{\partial}{\partial x} C(x,t) dx \geq \int_K^{K+s} -e^{-rt} dx.$$

This shows that $C(K+s,t) - C(K,t) \geq -se^{-rt}$.

The Option Portfolio Property

- An **option on an index** is a weighted sum of the prices of a collection of specified securities.
- The **option portfolio property**: The option on an index will never be more expensive than the costs of a corresponding collection of options on the individual securities.
- We prove this by using the Generalized Law of One Price.

Market Value of a Portfolio

- Consider a collection of n securities.
- Suppose that
$$S_j(y), \quad j = 1, \ldots, n,$$
is the price of security j at a future time y.
- For fixed positive constants w_j, let
$$I(y) = \sum_{j=1}^{n} w_j S_j(y).$$
- That is, $I(y)$ is the market value at time y of a portfolio of the securities, where the portfolio consists of w_j shares of security j.

Call Options

- A (K_j, t) **call option** on security j refers to a call option having:
  - Strike price K_j;
  - Expiration time t.
- Suppose

$$C_j, \quad j = 1, \ldots, n,$$

  are the costs of these options.
- Denote by C the cost of a call option on the index I that has:
  - Strike price $\sum_{j=1}^{n} w_j K_j$;
  - Expiration time t.

The Option Portfolio Property

- The payoff of the call option on the index is always less than or equal to the sum of the payoffs from buying w_j (K_j, t) call options on security j, for $j = 1, \ldots, n$.

$$\begin{aligned}
&\text{index option payoff at time } t \\
&= (I(t) - \textstyle\sum_{j=1}^n w_j K_j)^+ \\
&= (\textstyle\sum_{j=1}^n w_j S_j(t) - \sum_{j=1}^n w_j K_j)^+ \\
&= (\textstyle\sum_{j=1}^n w_j (S_j(t) - K_j))^+ \\
&\leq (\textstyle\sum_{j=1}^n (w_j (S_j(t) - K_j))^+)^+ \\
&= (\textstyle\sum_{j=1}^n w_j (S_j(t) - K_j)^+)^+ \\
&= \textstyle\sum_{j=1}^n w_j (S_j(t) - K_j)^+ \\
&= \textstyle\sum_{j=1}^n w_j \cdot [\text{payoff from } (K_j, t) \text{ call option}].
\end{aligned}$$

By the Law of One Price, $C \leq \sum_{j=1}^n w_j C_j$ or there is an arbitrage.