Introduction to Mathematical Finance

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Subsection 1

The Arbitrage Theorem

Experiments, Wagers and Returns

- Consider an experiment with set of possible outcomes $\{1, 2, \ldots, m\}$.
- Suppose that n wagers concerning this experiment are available.
- If the amount x is bet on Wager i, then $xr_i(j)$ is received, if the outcome of the experiment is j, for $j = 1, \ldots, m$.
- In other words, $r_i(\cdot)$ is the **return function** for a unit bet on wager i.
- The amount bet on a wager can be positive, negative or zero.

Betting Strategies and Returns

- A **betting strategy** is a vector

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n),$$

  with the interpretation that:
  - x_1 is bet on Wager 1;
  - x_2 is bet on Wager 2;

    $\vdots$
  - x_n is bet on Wager n.
- If the outcome of the experiment is j, then the return from the betting strategy $\boldsymbol{x}$ is given by

$$\text{return from } \boldsymbol{x} = \sum_{i=1}^{n} x_i r_i(j).$$

Introducing the Arbitrage Theorem

- The **arbitrage theorem** asserts that one of the following must hold.
  - There exists a probability vector
    $$\boldsymbol{p} = (p_1, p_2, \ldots, p_m)$$
    on the set of possible outcomes of the experiment, under which the expected return of each wager is equal to zero;
  - There exists a betting strategy that yields a positive win for each outcome of the experiment.

The Arbitrage Theorem

Theorem (The Arbitrage Theorem)

Exactly one of the following is true.

(a) There is a probability vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_m)$ for which

$$\sum_{j=1}^{m} p_j r_i(j) = 0, \quad \text{for all } i = 1, \ldots, n.$$

(b) There is a betting strategy $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ for which

$$\sum_{i=1}^{n} x_i r_i(j) > 0, \quad \text{for all } j = 1, \ldots, m.$$

Rephrasing the Arbitrage Theorem

- If X is the outcome of the experiment, then the arbitrage theorem states that one of the following holds.
  - There is a set of probabilities $(p_1, p_2, \ldots, p_m)$ such that, if
  $$P\{X = j\} = p_j, \quad \text{for all } j = 1, \ldots, m,$$
  then
  $$E[r_i(X)] = 0 \quad \text{for all } i = 1, \ldots, n;$$
  - There is a betting strategy that leads to a sure win.
- In other words, one of the following holds.
  - There is a probability vector on the outcomes of the experiment that results in all bets being fair;
  - There is a betting scheme that guarantees a win.

Risk-Neutral Probabilities

Definition

A **risk-neutral probability** on the set of outcomes of an experiment is a probability distribution on the outcomes of the experiment that results in all bets being fair.

Betting on an Outcome

- Consider a situation in which the only type of wager allowed is one that:
  - Chooses one of the outcomes i, $i = 1, \ldots, m$;
  - Bets that i is the outcome of the experiment.
- The return from such a bet is often quoted in terms of odds.
- Suppose the odds against outcome i are o_i (expressed as "o_i to 1").
- Then a one-unit bet will return:
  - o_i, if i is the outcome of the experiment;
  - -1, if i is not the outcome.
- That is, a one-unit bet on i will either win o_i or lose 1.
- The return function for such a bet is given by

$$r_i(j) = \begin{cases} o_i, & \text{if } j = i, \\ -1, & \text{if } j \neq i. \end{cases}$$

Betting on an Outcome (Cont'd)

- Suppose that the odds $o_1, o_2, \ldots, o_m$ are quoted.
- In order for there not to be a sure win, there must be a probability vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_m)$, such that, for each i ($i = 1, \ldots, m$),

$$0 = E_{\boldsymbol{p}}[r_i(X)] = o_i p_i - (1 - p_i).$$

- That is, we must have $p_i = \frac{1}{1+o_i}$.
- But the p_i must sum to 1.
- So the condition for there not to be an arbitrage is

$$\sum_{i=1}^{m} \frac{1}{1+o_i} = 1.$$

- That is, if $\sum_{i=1}^{m} \frac{1}{1+o_i} \neq 1$, then a sure win is possible.

Example

- Suppose there are three possible outcomes and the quoted odds are shown below.

Outcome	Odds
1	1
2	2
3	3

  That is, we have:
  - Odds against Outcome 1 are 1 to 1;
  - Odds against Outcome 2 are 2 to 1;
  - Odds against Outcome 3 are 3 to 1.

  We verify the condition that ensures that a sure win is possible.

$$\sum_{i=1}^{m} \frac{1}{1+o_i} \neq 1$$
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} \neq 1$$
$$\frac{13}{12} \neq 1.$$

Example (Cont'd)

- One possibility is to:
  - Bet -1 on outcome 1 (we either win 1 if the outcome is not 1 or we lose 1 if the outcome is 1);
  - Bet -0.7 on outcome 2 (we either win 0.7 if the outcome is not 2 or we lose 1.4 if it is 2);
  - Bet -0.5 on outcome 3 (we either win 0.5 if the outcome is not 3 or we lose 1.5 if it is 3).

  If the experiment results in:
  - Outcome 1, we win $-1 + 0.7 + 0.5 = 0.2$;
  - Outcome 2, we win $1 - 1.4 + 0.5 = 0.1$;
  - Outcome 3, we win $1 + 0.7 - 1.5 = 0.2$.

  Hence, in all cases we win a positive amount.

Example: Option Pricing (Revisited)

- We reconsider the option pricing example where the initial price of a stock is 100 and the price after one period is assumed to be either 200 or 50.

  At a cost of C per share, we can purchase at time 0 the option to buy the stock at time 1 for the price of 150.

  We want to find the value of C for which no sure win is possible.
- In the context of this section, the outcome of the experiment is the value of the stock at time 1, i.e., there are two possible outcomes.
- There are also two different wagers:
  - Buy (or sell) the stock;
  - Buy (or sell) the option.
- By the arbitrage theorem, there will be no sure win if there are probabilities $(p, 1-p)$ on the outcomes that make the expected present value return equal to zero for both wagers.

Example: Option Pricing (Cont'd)

- The present value return from purchasing one share of the stock is

$$\text{return} = \begin{cases} 200\frac{1}{1+r} - 100, & \text{if the price is 200 at time 1,} \\ 50\frac{1}{1+r} - 100, & \text{if the price is 50 at time 1.} \end{cases}$$

- If p is the probability that the price is 200 at time 1, then

$$\begin{aligned} E[\text{return}] &= p[\tfrac{200}{1+r} - 100] + (1-p)[\tfrac{50}{1+r} - 100] \\ &= p\tfrac{150}{1+r} + \tfrac{50}{1+r} - 100. \end{aligned}$$

- Setting this equal to zero yields $p = \frac{1+2r}{3}$.
- Therefore, the only probability vector $(p, 1-p)$ that results in a zero expected return for the wager of purchasing the stock has $p = \frac{1+2r}{3}$.

Example: Option Pricing (Cont'd)

- The present value return from purchasing one option is

$$\text{return} = \begin{cases} 50\frac{1}{1+r} - C, & \text{if the price is 200 at time 1,} \\ -C, & \text{if the price is 50 at time 1.} \end{cases}$$

- When $p = \frac{1+2r}{3}$, the expected return of purchasing one option is

$$E[\text{return}] = \frac{1+2r}{3}\frac{50}{1+r} - C.$$

- By the Arbitrage Theorem, the only value of C for which there will not be a sure win is $C = \frac{1+2r}{3}\frac{50}{1+r}$.
- We have

$$C = \frac{50+100r}{3(1+r)} = \frac{1}{3}\left(\frac{100r+50}{1+r}\right) = \frac{1}{3}\left(100 - \frac{50}{1+r}\right).$$

- This is in accord with the result of a previous section.

Subsection 2

The Multiperiod Binomial Model

Stocks Over Multiple Periods

- Consider a stock option scenario in which:
  - There are n periods;
  - The nominal interest rate is r per period.
- Let $S(0)$ be the initial price of the stock,.
- For $i = 1, \ldots, n$ let $S(i)$ be its price at i time periods later.
- Suppose $S(i)$ is either $uS(i-1)$ or $dS(i-1)$, with $d < 1 + r < u$.
- That is, going from one time period to the next, the price either goes up by the factor u or down by the factor d.
- Furthermore, suppose that at time 0 an option may be purchased that enables buying the stock after n periods for the amount K.
- In addition, the stock may be purchased and sold anytime within these n time periods.

Outcomes of the Experiment

- Let X_i equal 1, if the stock's price goes up by the factor u from period $i-1$ to i, and 0, if the price goes down by the factor d.

$$X_i = \begin{cases} 1, & \text{if } S(i) = uS(i-1), \\ 0, & \text{if } S(i) = dS(i-1). \end{cases}$$

- The outcome of the experiment can now be regarded as the value of the vector $(X_1, X_2, \ldots, X_n)$.
- By the Arbitrage Theorem, in order for there not to be an arbitrage opportunity, there must be probabilities on these outcomes that make all bets fair.
- That is, there must be a set of probabilities

$$P\{X_1 = x_1, \ldots, X_n = x_n\}, \quad x_i = 0, 1, \ i = 1, \ldots, n,$$

that make all bets fair.

Bets

- Now consider the following type of bet:
  - First choose a value of i ($i = 1, \ldots, n$) and a vector $(x_1, \ldots, x_{i-1})$ of zeros and ones;
  - Then observe the first $i - 1$ changes.
    - If $X_j = x_j$ for each $j = 1, \ldots, i - 1$, immediately buy one unit of stock and then sell it back the next period.
- If the stock is purchased, then its cost at time $i - 1$ is $S(i - 1)$;
- The time-$(i - 1)$ value of the amount obtained when sold at time i is:
  - $\frac{1}{1+r} uS(i - 1)$ if the stock goes up;
  - $\frac{1}{1+r} dS(i - 1)$ if it goes down.
- Let $\alpha = P\{X_1 = x_1, \ldots, X_{i-1} = x_{i-1}\}$ denote the probability that the stock is purchased.
- Let $p = P\{X_i = 1 | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}\}$ denote the probability that a purchased stock goes up the next period.

Expected Gain

- Then the expected gain on this bet (in time-$(i-1)$ units) is

$$\alpha\left[p\frac{1}{1+r}uS(i-1)+(1-p)\frac{1}{1+r}dS(i-1)-S(i-1)\right].$$

- The expected gain on this bet will be zero, provided that

$$\frac{pu}{1+r}+\frac{(1-p)d}{1+r}=1 \quad \text{or} \quad p=\frac{1+r-d}{u-d}.$$

- In other words, the only probability vector that results in an expected gain of zero for this type of bet has

$$P\{X_i=1|X_1=x_1,\ldots,X_{i-1}=x_{i-1}\}=\frac{1+r-d}{u-d}.$$

- Since $x_1,\ldots,x_n$ are arbitrary, the only probability vector on the set of outcomes that results in all these bets being fair is the one that takes $X_1,\ldots,X_n$ to be independent random variables with $P\{X_i=1\}=p=1-P\{X_i=0\}$, $i=1,\ldots,n$, where $p=\frac{1+r-d}{u-d}$.

Arbitrage

- It can be shown that, with these probabilities, any bet on buying stock will have zero expected gain.
- By the Arbitrage Theorem one of the following must hold.
  - The cost of the option must be equal to the expectation of the present (i.e., time-0) value of owning it using the preceding probabilities
  - There is an arbitrage opportunity.
- So, to determine the no-arbitrage cost, we assume that the X_i are independent 0-or-1 random variables whose common probability p of being equal to 1 given by

$$p = \frac{1 + r - d}{u - d}.$$

- If Y is their sum, Y is the number of the X_i equal to 1.
- Thus Y is a binomial random variable with parameters n and p.

No-Arbitrage Cost of Option

- In going from period to period, the stock's price is its old price multiplied either by u or by d.
- At time n, the price:
  - Would have gone up Y times;
  - Would have gone down $n - Y$ times.
- So the stock's price after n periods is

$$S(n) = u^Y d^{n-Y} S(0).$$

- The value of owning the option after n periods have elapsed is

$$(S(n) - K)^+.$$

- Recall this is defined to equal either $S(n) - K$, if this quantity is nonnegative, or zero, if it is negative.

No-Arbitrage Cost of Option (Cont'd)

- The present (time-0) value of owning the option is

$$\frac{1}{(1+r)^n}(S(n)-K)^+.$$

- So the expectation of the present value of owning the option is

$$\frac{1}{(1+r)^n}E[(S(n)-K)^+]=\frac{1}{(1+r)^n}E[(S(0)u^Y d^{n-Y}-K)^+].$$

- Thus, the only option cost C that does not result in an arbitrage is

$$C=\frac{1}{(1+r)^n}E[(S(0)u^Y d^{n-Y}-K)^+].$$

Subsection 3

Proof of the Arbitrage Theorem

Primary and Dual Linear Programs

- We first present the Duality Theorem of linear programming.
- Suppose that, for given constants c_i, b_j and a_{ij} ($i = 1, \ldots, n$, $j = 1, \ldots, m$), we want to choose values $x_1, \ldots, x_n$ that will

$$\text{maximize } \sum_{i=1}^{n} c_i x_i \text{ subject to } \sum_{i=1}^{n} a_{ij} x_i \leq b_j,\ j = 1, 2, \ldots, m.$$

- This problem is called a **primal linear program**.
- The **dual** of the preceding linear program is to choose values $y_1, \ldots, y_m$ that

$$\text{minimize} \sum_{j=1}^{m} b_j y_j \text{ subject to } \sum_{j=1}^{m} a_{ij} y_j = c_i,\ i = 1, \ldots, n,$$

$$y_j \geq 0,\ j = 1, \ldots, m.$$

Duality Theorem of Linear Programming

- A linear program is said to be **feasible** if there are values for the variables ($x_1, \ldots, x_n$ in the primal linear program or $y_1, \ldots, y_m$ in the dual) that satisfy the constraints.
- The key theoretical result of linear programming is the **Duality Theorem**, which we state without proof.

Proposition (Duality Theorem of Linear Programming)

If a primal and its dual linear program are both feasible, then:

- They both have optimal solutions;
- The maximal value of the primal is equal to the minimal value of the dual.

If either problem is infeasible, then the other does not have an optimal solution.

The Arbitrage Setting Revisited

- A consequence of the Duality Theorem is the Arbitrage Theorem.
- Recall that the arbitrage theorem refers to a situation in which there are n wagers with payoffs that are determined by the result of an experiment having possible outcomes $1, 2, \ldots, n$.
- If we bet Wager i at level x, then we win the amount $xr_i(j)$ if the outcome of the experiment is j.
- A betting strategy is a vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, where each x_i can be positive, negative or zero.
- The interpretation of a betting strategy is that we simultaneously bet Wager i at level x_i, for all $i = 1, \ldots, n$.
- If the outcome of the experiment is j, then our winnings from the betting strategy $\boldsymbol{x}$ are $\sum_{i=1}^{n} x_i r_i(j)$.

The Arbitrage Theorem

Proposition (Arbitrage Theorem)

Exactly one of the following is true:

(i) There exists a probability vector $\boldsymbol{p} = (p_1, \ldots, p_m)$, for which

$$\sum_{j=1}^{m} p_j r_i(j) = 0, \quad \text{for all } i = 1, \ldots, n;$$

(ii) There exists a betting strategy $\boldsymbol{x} = (x_1, \ldots, x_n)$, such that

$$\sum_{i=1}^{n} x_i r_i(j) > 0, \quad \text{for all } j = 1, \ldots, m.$$

That is, one of the following holds:

- There exists a probability vector under which all wagers have expected gain equal to zero;
- There is a betting strategy that always results in a positive win.

Proof

- Let x_{n+1} denote an amount that the gambler can be sure of winning.
- Consider the problem of maximizing this amount.
- If the gambler uses the betting strategy $(x_1, \ldots, x_n)$ then she will win $\sum_{i=1}^n x_i r_i(j)$ if the outcome of the experiment is j.
- Hence, she wants to choose her betting strategy $(x_1, \ldots, x_n)$ and x_{n+1} so as to

$$\text{maximize } x_{n+1} \text{ subject to } \sum_{i=1}^{n} x_i r_i(j) \geq x_{n+1},\ j = 1, \ldots, m.$$

- Set $a_{ij} = -r_i(j)$, $i = 1, \ldots, n$, $a_{n+1,j} = 1$.
- Then we can rewrite the preceding as follows:

$$\text{maximize } x_{n+1} \text{ subject to } \sum_{i=1}^{n+1} a_{ij} x_i \leq 0,\ j = 1, \ldots, m.$$

Proof (Cont'd)

- The preceding linear program has $c_1 = c_2 = \cdots = c_n = 0$, $c_{n+1} = 1$, and upper-bound constraint values all equal to zero (i.e., all $b_j = 0$).
- Consequently, its dual program is to choose variables $y_1, \ldots, y_m$ so as to

$$\text{minimize } 0 \text{ subject to } \sum_{j=1}^{m} a_{ij} y_j = 0, \ i = 1, \ldots, n,$$

$$\sum_{j=1}^{m} a_{n+1,j} y_j = 1, \ y_j \geq 0, \ j = 1, \ldots, m.$$

- Using $a_{ij} = -r_i(j)$, $i = 1, \ldots, n$, $a_{n+1,j} = 1$, we get the dual program

$$\text{minimize } 0 \text{ subject to } \sum_{j=1}^{m} r_i(j) y_j = 0, \ i = 1, \ldots, n,$$

$$\sum_{j=1}^{m} y_j = 1, \ y_j \geq 0, \ j = 1, \ldots, m.$$

Proof (Cont'd)

- The dual program is feasible, and its minimal value is zero, if and only if there is a probability vector $(y_1, \ldots, y_m)$ under which all wagers have expected return 0.
- The primal problem is feasible because $x_i = 0$, $i = 1, \ldots, n+1$, satisfies its constraints.
- By the Duality Theorem:
  - If the dual problem is also feasible, then the optimal value of the primal is zero. Hence, no sure win is possible.
  - If the dual is infeasible, then there is no optimal solution of the primal. This implies that zero is not the optimal solution. Thus, there is a betting scheme whose minimal return is positive.
- The reason there is no primal optimal solution when the dual is infeasible is because the primal is unbounded in this case.