Introduction to Mathematical Finance

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Subsection 1

The Black-Scholes Formula

The Set Up

- Consider a call option having:
  - Strike price K;
  - Expiration time t.
- That is, the option allows one to purchase a single unit of an underlying security at time t for the price K.
- Let the nominal interest rate be r, compounded continuously.
- Suppose the price of the security follows a geometric Brownian motion, with:
  - Drift parameter μ;
  - Volatility parameter σ.
- Under these assumptions, we find the unique cost of the option that does not give rise to an arbitrage.

The Price Fluctuation

- Let $S(y)$ denote the price of the security at time y.
- By hypothesis, $\{S(y), 0 \leq y \leq t\}$ follows a geometric Brownian motion with volatility parameter σ and drift parameter μ.
- So the n-stage approximation of this model supposes that, every $\frac{t}{n}$ time units, the price changes.
- Its new value is equal to its old value multiplied:
  - By the factor $u = e^{\sigma\sqrt{t/n}}$ with probability $\frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\frac{t}{n}}\right)$;
  - By the factor $d = e^{-\sigma\sqrt{t/n}}$ with probability $\frac{1}{2}\left(1 - \frac{\mu}{\sigma}\sqrt{\frac{t}{n}}\right)$.
- So the n-stage approximation model is an n-stage binomial model in which the price at each time interval $\frac{t}{n}$ changes in one of two ways:
  - Goes up by a multiplicative factor u;
  - Goes down by a multiplicative factor d.

Probability of Fair Bets

- Let
$$X_i = \begin{cases} 1, & \text{if } S\left(i\frac{t}{n}\right) = uS\left((i-1)\frac{t}{n}\right), \\ 0, & \text{if } S\left(i\frac{t}{n}\right) = dS\left((i-1)\frac{t}{n}\right). \end{cases}$$
- By previous results, the only probability law on $X_1, \ldots, X_n$ that makes all security buying bets fair in the n-stage approximation model is the one that takes the X_i to be independent with
$$\begin{aligned} p := P\{X_i = 1\} &= \frac{1 + r\frac{t}{n} - d}{u - d} \\ &= \frac{1 - e^{-\sigma\sqrt{t/n}} + r\frac{t}{n}}{e^{\sigma\sqrt{t/n}} - e^{-\sigma\sqrt{t/n}}}. \end{aligned}$$

Rewriting Using Taylor Expansions

- We obtained

$$p = \frac{1 - e^{-\sigma\sqrt{t/n}} + r\frac{t}{n}}{e^{\sigma\sqrt{t/n}} - e^{-\sigma\sqrt{t/n}}}.$$

- Recall the Taylor series expansion about 0 of the function e^x

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots .$$

- Using the first three terms, we get

$$\begin{aligned} e^{-\sigma\sqrt{t/n}} &\approx 1 - \sigma\sqrt{\frac{t}{n}} + \sigma^2\frac{t}{2n}, \\ e^{\sigma\sqrt{t/n}} &\approx 1 + \sigma\sqrt{\frac{t}{n}} + \sigma^2\frac{t}{2n}. \end{aligned}$$

Rewriting Using Taylor Expansions (Cont'd)

- Setting $e^{-\sigma\sqrt{t/n}} \approx 1 - \sigma\sqrt{\frac{t}{n}} + \sigma^2\frac{t}{2n}$ and $e^{\sigma\sqrt{t/n}} \approx 1 + \sigma\sqrt{\frac{t}{n}} + \sigma^2\frac{t}{2n}$ in

$$p = \frac{1 - e^{-\sigma\sqrt{t/n}} + r\frac{t}{n}}{e^{\sigma\sqrt{t/n}} - e^{-\sigma\sqrt{t/n}}}$$

gives

$$\begin{aligned} p &\approx \frac{\sigma\sqrt{\frac{t}{n}} - \sigma^2\frac{t}{2n} + r\frac{t}{n}}{2\sigma\sqrt{\frac{t}{n}}} \\ &= \frac{1}{2} + \frac{r\sqrt{\frac{t}{n}}}{2\sigma} - \frac{\sigma\sqrt{\frac{t}{n}}}{4} \\ &= \frac{1}{2}\left(1 + \frac{r - \frac{\sigma^2}{2}}{\sigma}\sqrt{\frac{t}{n}}\right). \end{aligned}$$

Risk Neutrality versus Arbitrage

- The unique risk-neutral probabilities on the n-stage approximation model result from supposing that, in each period, the price changes in one of two ways:
  - Goes up by the factor $e^{\sigma\sqrt{t/n}}$ with probability p;
  - Goes down by the factor $e^{-\sigma\sqrt{t/n}}$ with probability $1-p$.
- From previous work, it follows that as $n \to \infty$ this risk-neutral probability law converges to geometric Brownian motion with drift coefficient $r - \frac{\sigma^2}{2}$ and volatility parameter σ.
- So is reasonable to suppose (and can be rigorously proven) that this risk-neutral geometric Brownian motion is the only probability law on the evolution of prices over time that makes all security buying bets fair.

Risk Neutrality versus Arbitrage (Cont'd)

- We have just argued that if the underlying price of a security follows a geometric Brownian motion with volatility parameter σ, then the only probability law on the sequence of prices that results in all security buying bets being fair is that of a geometric Brownian motion with drift parameter $r - \frac{\sigma^2}{2}$ and volatility parameter σ.
- Consequently, by the Arbitrage Theorem, one of the following holds:
  - The options are priced to be fair bets according to the risk-neutral geometric Brownian motion probability law;
  - There will be an arbitrage.

The Black-Scholes Option Pricing Formula

- Suppose $S(t)$ is a risk-neutral geometric Brownian motion.
- Then $\frac{S(t)}{S(0)}$ is a lognormal random variable with:
  - Mean parameter $(r - \frac{\sigma^2}{2})t$;
  - Variance parameter $\sigma^2 t$.
- Hence, the unique no-arbitrage cost C of a call option to purchase the security at time t for the specified price K, is

$$C = e^{-rt}E[(S(t) - K)^+] = e^{-rt}E[(S(0)e^W - K)^+],$$

where W is a normal random variable with mean $(r - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$.

The Black-Scholes Option Pricing Formula (Cont'd)

- We have
$$C = e^{-rt}E[(S(0)e^W - K)^+],$$
where W is a normal random variable with mean $(r - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$.
- The right side can be explicitly evaluated to give the following expression, known as the **Black-Scholes option pricing formula**.
$$C = S(0)\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}),$$
where
$$\omega = \frac{rt + \sigma^2\frac{t}{2} - \log\left(\frac{K}{S(0)}\right)}{\sigma\sqrt{t}}$$
and $\Phi(x)$ is the standard normal distribution function.

Example

- We make the following assumptions:
  - A security is presently selling for a price of 30;
  - The nominal interest rate is 8% (unit of time being one year);
  - The security's volatility is 0.20.

  We want to compute the no-arbitrage cost of a call option that expires in three months and has a strike price of 34.

  We first identify the value of the parameters.
  - $S(0) = 30$;
  - $r = 0.08$;
  - $\sigma = 0.20$;
  - $t = 0.25$;
  - $K = 34$.

Example (Cont'd)

- The parameters are

$$t = 0.25, \quad r = 0.08, \quad \sigma = 0.20, \quad K = 34, \quad S(0) = 30.$$

We apply the formula to find ω

$$\begin{aligned} \omega &= \frac{rt+\sigma^2 \frac{t}{2} - \log\left(\frac{K}{S(0)}\right)}{\sigma\sqrt{t}} \\ &= \frac{0.02+0.005-\log\frac{34}{30}}{(0.2)(0.5)} \\ &\approx -1.0016. \end{aligned}$$

Therefore,

$$\begin{aligned} C &= S(0)\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}) \\ &= 30\Phi(-1.0016) - 34e^{-0.02}\Phi(-1.1016) \\ &= 30(0.15827) - 34(0.9802)(0.13532) \approx 0.2383. \end{aligned}$$

The appropriate price of the option is thus 24 cents.

Remarks

1. Another way to derive the no-arbitrage option cost C is to:
   - Consider the unique no-arbitrage cost of an option in the n-period approximation model;
   - Let n go to infinity.
2. Let $C(s, t, K)$ be the no-arbitrage cost of an option having strike price K and exercise time t when the initial price of the security is s.

   That is, $C(s, t, K)$ is the C of the Black-Scholes, with $S(0) = s$.

   Suppose the price of the security at time y $(0 < y < t)$ is $S(y) = s_y$.

   The unique no-arbitrage cost of the option at time y is $C(s_y, t - y, K)$.

   This is because at time y:
   - The option will expire after an additional time $t - y$;
   - It has the same exercise price K;
   - For the next $t - y$ units of time the security will follow a geometric Brownian motion with initial value s_y.

Remarks (Cont'd)

3. Recall from our study of pricing via arbitrage that, for no-arbitrage,

$$S + P - C = Ke^{-rt},$$

where

- S be the price of the stock at time 0;
- P is the price of a European put option for selling one share of the stock for the amount K at time t;
- C is the price of a call option for buying one share of a stock at an exercise price K at time t;
- r is the nominal rate for continuous discounting.

So the no-arbitrage cost $P(s,t,K)$ of a European put option with initial price s, strike price K, and exercise time t is given by

$$P(s,t,K) = C(s,t,K) + Ke^{-rt} - s.$$

Subsection 2

Properties of the Black-Scholes Option Cost

Arbitrage Option Cost Revisited

- The no-arbitrage option cost $C = C(s, t, K, \sigma, r)$ is a function of five variables:
  - The security's initial price s;
  - The expiration time t of the option;
  - The strike price K;
  - The security's volatility parameter σ;
  - The interest rate r.
- To see what happens to the cost as a function of each of these variables, we use the equation

$$C(s, t, K, \sigma, r) = e^{-rt} E[(se^{W} - K)^{+}],$$

where W is a normal random variable with mean $(r - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$.

Properties of the Cost Function (1)

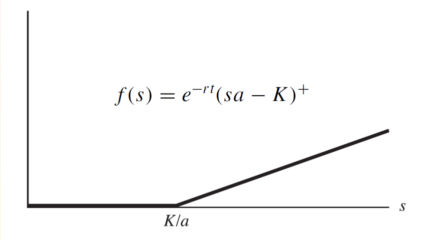
1. C is an increasing, convex function of s.

   This means that if the other four variables remain the same, then the no-arbitrage cost of the option is:
   - An increasing function of the security's initial price;
   - A convex function of the security's initial price.

   For any positive constant a, the function

   $$e^{-rt}(sa-K)^+$$

   is an increasing, convex function of s.

   But the probability distribution of W does not depend on s.

   So $e^{-rt}(se^W - K)^+$ is, for all W, increasing and convex in s.

   Thus, so is its expected value.

Properties of the Cost Function (2-3)

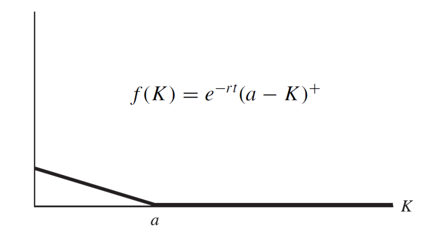
2. C is a decreasing, convex function of K.

   This follows from the fact that

   $$e^{-rt}(se^W - K)^+$$

   is, for all W, decreasing and convex in K.

   Thus, so is its expectation.



3. C is increasing in t.

   It is immediate that the option cost would be increasing in t if the option were an American call option (any additional time to exercise could not hurt, since one could always elect not to use it).

   The value of a European call option is the same as that of an American call option.

   So we obtain the result for both options.

Properties of the Cost Function (4)

4. C is increasing in σ.

   The result seems at first sight to be quite intuitive.

   This is because an option holder:

   - Will greatly benefit from very large prices at the exercise time;
   - Will not incur any additional loss for any additional price decrease below the exercise price.

   However, it is more subtle than it appears.

   Since $E\left[\log \frac{S(t)}{S(0)}\right] = \left(r - \frac{\sigma^2}{2}\right) t$, an increase in σ results not only in an increase in the variance of the logarithm of the final price under the risk-neutral valuation but also in a decrease in the mean.

   Nevertheless, the result is true and will be shown mathematically later.

Properties of the Cost Function (5)

5. C is increasing in r.

   We can express W, a normal random variable with mean $(r - \frac{\sigma^2}{2})t$ and variance $t\sigma^2$, as

   $$W = rt - \frac{\sigma^2 t}{2} + \sigma\sqrt{t}Z,$$

   where Z is a standard normal random variable with mean 0 and variance 1.

   Hence, from the cost equation we have that

   $$C = E\left[\left(se^{-\frac{\sigma^2 t}{2} + \sigma\sqrt{t}Z} - Ke^{-rt}\right)^+\right].$$

   The result now follows because $\left(se^{-\frac{\sigma^2 t}{2} + \sigma\sqrt{t}Z} - Ke^{-rt}\right)^+$, and, thus, its expected value, is increasing in r.

Rate of Change With Respect to Price

- The rate of change in the value of the call option as a function of a change in the price of the underlying security is described by the quantity **delta**, denoted as Δ.
- Formally, if $C(s, t, K, \sigma, r)$ is the Black-Scholes cost valuation of the option, then Δ is its partial derivative with respect to s,

$$\Delta = \frac{\partial}{\partial s} C(s, t, K, \sigma, r).$$

- We will show later that

$$\Delta = \Phi(\omega),$$

where, $\omega = \frac{rt + \frac{\sigma^2 t}{2} - \log\left(\frac{K}{S(0)}\right)}{\sigma\sqrt{t}}$.

Using Rate of Change for Investment

- Delta can be used to construct investment portfolios that hedge against risk.
- For instance, suppose that an investor feels that a call option is underpriced and consequently buys the call.
- To protect himself against a decrease in its price, he can simultaneously sell a certain number of shares of the security.
    - Suppose the price of the security decreases by the small amount h.
    - Then the worth of the option will decrease by the amount $h\Delta$.
    - So the investor would be covered if he sold Δ shares of the security.
- Therefore, a reasonable hedge might be to sell Δ shares of the security for each option purchased.
- This will be made precise by the delta hedging arbitrage strategy.
- This strategy can, in theory, be used to construct an arbitrage if a call option is not priced according to the Black-Scholes formula.

Subsection 3

The Delta Hedging Arbitrage Strategy

The Set Up

- Consider a security whose initial price is s.
- Suppose that, after each time period, its price changes in one of two ways:
  - Goes up by the multiple u;
  - Goes down by the multiple d.
- We determine the amount of money x that we must have at time 0 in order to meet the following payment at time 1:
  - a, if the price of the stock is us at time 1;
  - b, if the price of the stock is ds at time 1.
- Suppose we purchase y shares of the stock and:
  - Put the remaining $x - ys$ in the bank, if $x - ys \geq 0$;
  - Borrow $ys - x$ from the bank, if $x - ys < 0$.

Initial Cost

- Suppose that:
  - $S(1)$ is the price of the security at time 1;
  - r is the interest rate per period.
- Then the return at time 1 is given by

$$\text{return at time } 1 = \begin{cases} yus + (x - ys)(1 + r), & \text{if } S(1) = us, \\ yds + (x - ys)(1 + r), & \text{if } S(1) = ds. \end{cases}$$

- We choose x and y, such that

$$\begin{aligned} yus + (x - ys)(1 + r) &= a, \\ yds + (x - ys)(1 + r) &= b. \end{aligned}$$

- Then, after taking our money out of the bank (or meeting our loan payment), we will have the desired amount.
- Subtracting, we get $y = \frac{a-b}{s(u-d)}$.

Initial Cost (Cont'd)

- We obtained $y = \frac{a-b}{s(u-d)}$.
- Substituting into the first equation yields

$$\frac{a-b}{u-d}[u-(1+r)] + x(1+r) = a.$$

- So we have

$$\begin{aligned} x &= \frac{1}{1+r}\left(a\left[1-\frac{u-(1+r)}{u-d}\right] + b\frac{u-1-r}{u-d}\right) \\ &= \frac{1}{1+r}\left(a\frac{1+r-d}{u-d} + b\frac{u-1-r}{u-d}\right) \\ &= p\frac{a}{1+r} + (1-p)\frac{b}{1+r}, \end{aligned}$$

  where $p = \frac{1+r-d}{u-d}$.
- The amount of money needed at time 0 equals the expected present value, under the risk-neutral probabilities, of the payoff at time 1.
- The investment strategy calls for purchasing of $y = \frac{a-b}{s(u-d)}$ shares of the security and putting the remainder in the bank.

The Case at Time 2 (Step 1)

- Consider the problem of determining how much money is needed at time 0 to meet a payoff at time 2 of $x_{i,2}$ if the price of the security at time 2 is $u^i d^{2-i} s$ ($i = 0, 1, 2$).
- We first determine, for each possible price of the security at time 1, the amount that is needed at time 1 to meet the payment at time 2.
- If the price at time 1 is us, then the amount needed at time 2 is:
  - $x_{2,2}$, if the price at time 2 is $u^2 s$;
  - $x_{1,2}$, if the price at time 2 is uds.
- Thus, it follows from our preceding analysis that, if the price at time 1 is us, then we would, at time 1, need the amount

$$x_{1,1} = p\frac{x_{2,2}}{1+r} + (1-p)\frac{x_{1,2}}{1+r}.$$

- Moreover the strategy is to purchase $y_{1,1} = \frac{x_{2,2}-x_{1,2}}{us(u-d)}$ shares of the security and put the remainder in the bank.

The Case at Time 2 (Step 1 Cont'd)

- Similarly, if the price at time 1 is ds, then to meet the final payment at time 2 we would, at time 1, need the amount

$$x_{0,1} = p\frac{x_{1,2}}{1+r} + (1-p)\frac{x_{0,2}}{1+r}.$$

- Moreover, the strategy is to purchase

$$y_{0,1} = \frac{x_{1,2} - x_{0,2}}{ds(u-d)}$$

shares of the security and put the remainder in the bank.

The Case at Time 2 (Step 2)

- At time 0 we need to have enough to invest so as to be able to have:
  - $x_{1,1}$ at time 1, if the price of the security is us at time 1;
  - $x_{0,1}$ at time 1, if the price of the security is ds at time 1.
- Consequently, at time 0 we need the amount

$$\begin{array}{rcl} x_{0,0} & = & p\frac{x_{1,1}}{1+r} + (1-p)\frac{x_{0,1}}{1+r} \\ & = & p^2\frac{x_{2,2}}{(1+r)^2} + 2p(1-p)\frac{x_{1,2}}{(1+r)^2} + (1-p)^2\frac{x_{0,2}}{(1+r)^2}. \end{array}$$

- Once again, the amount needed is the expected present value, under the risk-neutral probabilities, of the final payoff.
- The strategy is to:
  - Purchase $y_{0,0} = \frac{x_{1,1}-x_{0,1}}{s(u-d)}$ shares of the security;
  - Put the remainder in the bank.

The n-Period Case

- The preceding is easily generalized to an n-period problem, where the payoff at the end of period n is $x_{i,n}$ if the price at that time is $u^i d^{n-i} s$.
- The amount $x_{i,j}$ needed at time j, given that the price of the security at that time is $u^i d^{j-i} s$, is equal to the conditional expected time-j value of the final payoff, where the expected value is computed under the assumption that the successive changes in price are governed by the risk-neutral probabilities.
- That is, the successive changes are independent, with each new price equal to the previous period's price multiplied either by the factor u with probability p or by the factor d with probability $1 - p$.

The n-Period Case (No-Arbitrage)

- If the payoff results from paying the holder of a call option that has strike price K and expiration time n, then the payoff at time n is

$$x_{i,n} = (u^i d^{n-i} s - K)^+, \quad i = 0, \ldots, n,$$

  when the price of the security at time n is $u^i d^{n-i} s$.
- Our investment strategy replicates the payoff from this option.
- By the Law of One Price (as well as from the Arbitrage Theorem), $x_{0,0}$, the initial amount needed, is equal to the unique no-arbitrage cost of the option.
- Moreover, $x_{i,j}$, the amount needed at time j, when the price at that time is $su^i d^{j-i}$, is the unique no-arbitrage cost of the option at that time and price.

The n-Period Case (Arbitrage I)

- Suppose C, the cost of the option at time 0, is larger than $x_{0,0}$. Then we may effect an arbitrage.
  - Sell the option;
  - Use $x_{0,0}$ from this sale to meet the option payoff at time n;
  - Walk away with a positive profit of $C - x_{0,0}$.

The n-Period Case (Arbitrage II)

- Suppose that $C < x_{0,0}$.
- By reversing the procedure (changing buying into selling, and vice versa) we can transform an initial debt of $x_{0,0}$ into a time-n debt of $x_{i,n}$, when the price at time n is $su^i d^{n-i}$.
- So we can also make an arbitrage.
  - Borrow the amount $x_{0,0}$;
  - Use C of this amount to buy the option;
  - Use the investment procedure to transform the initial debt into a time-n debt whose amount is exactly that of the return from the option.
- In either case we can:
  - Gain $|C - x_{0,0}|$ at time 0;
  - Follow an investment strategy that guarantees we have no additional losses or gains by hedging all future risks.

The Case of Geometrical Brownian Motion

- Suppose a security follows a geometric Brownian motion with volatility σ.
- Let a call option for the security have:
  - Strike price K;
  - Expiration time t.
- We determine the hedging strategy for the call option.
- We first consider a finite-period approximation.
- In each h time units the price of the security changes in one of two ways.
  - Increases by the factor $e^{\sigma\sqrt{h}}$;
  - Decreases by the factor $e^{-\sigma\sqrt{h}}$.
- Suppose the present price of the stock is s.

Amount Needed in Next Period

- Let $C(s,t)$ be the no-arbitrage cost of the call option.
- The notation suppresses the dependence of C on K, r and σ.
- The price after an additional time h is either $se^{\sigma\sqrt{h}}$ or $se^{-\sigma\sqrt{h}}$.
- So the amount we will need in the next period to utilize the hedging strategy is:
  - $C(se^{\sigma\sqrt{h}}, t-h)$ if the price is $se^{\sigma\sqrt{h}}$;
  - $C(se^{\sigma\sqrt{h}}, t-h)$ if the price is $se^{-\sigma\sqrt{h}}$.
- When the price of the security is s and time t remains before the option expires, the hedging strategy calls for owning

$$\frac{C(se^{\sigma\sqrt{h}}, t-h) - C(se^{-\sigma\sqrt{h}}, t-h)}{se^{\sigma\sqrt{h}} - se^{-\sigma\sqrt{h}}}$$

shares of the security.

Number of Shares to be Owned

- To determine the number of shares that should be owned, we need to let h go to zero.
- Thus, we need to determine

$$\begin{aligned}
&\lim_{h\to 0}\frac{C(se^{\sigma\sqrt{h}},t-h)-C(se^{-\sigma\sqrt{h}},t-h)}{se^{\sigma\sqrt{h}}-se^{-\sigma\sqrt{h}}}\\
&\quad=\lim_{a\to 0}\frac{C(se^{\sigma a},t-a^2)-C(se^{-\sigma a},t-a^2)}{se^{\sigma a}-se^{-\sigma a}}\\
&\quad\overset{\text{L'Hôpital}}{=}\lim_{a\to 0}\frac{s\sigma e^{\sigma a}\frac{\partial}{\partial y}C(y,t)|_{y=se^{\sigma a}}+s\sigma e^{-\sigma a}\frac{\partial}{\partial y}C(y,t)|_{y=se^{-\sigma a}}}{s\sigma e^{\sigma a}+s\sigma e^{\sigma a}}\\
&\quad=\frac{\partial}{\partial y}C(y,t)|_{y=s}\\
&\quad=\frac{\partial}{\partial s}C(s,t).
\end{aligned}$$

Number of Shares to be Owned (Cont'd)

- Therefore, the return from a call option having strike price K and exercise time T can be replicated by an investment strategy that:
  - Requires an investment capital of $C(S(0), T, K)$;
  - Calls for owning exactly $\frac{\partial}{\partial s}C(s, t, K)$ shares of the security, when its current price is s and time t remains before the option expires;
  - The absolute value of the remaining capital at that time being:
    - In the bank, if the remaining capital is positive;
    - Borrowed, if the remaining capital is negative.

Attaining an Arbitrage

- Suppose the market price of the (K, T) call option is greater than $C(S(0), T, K)$;

  Then an arbitrage can be made.
  - Sell the option;
  - Use $C(S(0), T, K)$ from this sale along with the preceding strategy to replicate the return from the option.
- Suppose the market cost C is less than $C(S(0), T, K)$.

  An arbitrage is obtained by doing the reverse.
  - Borrow $C(S(0), T, K)$;
  - Use C of this amount to buy a (K, T) call option;
  - Maintain a short position of $\frac{\partial}{\partial s} C(s, t, K)$ shares of the security when its current price is s and time t remains before the option expires.

  The invested money from these short positions, along with your call option, will cover your loan of $C(S(0), T, K)$ and also pay off your final short position.

Subsection 4

Some Derivations

The Set Up

- Consider a security that:
  - Has initial price s;
  - Follows a geometric Brownian motion with volatility parameter σ.
- Consider a call option for the security, with:
  - Strike price K;
  - Expiration time t.
- Let r be the interest rate.
- Denote by

$$C(s, t, K, \sigma, r) = E\left[e^{-rt}(S(t) - K)^+\right]$$

the risk-neutral cost of the security.

The Goals and Some Notation

- We wish to derive:
  - The Black-Scholes option pricing formula;
  - The partial derivatives of C.
- We use the fact that, under the risk-neutral probabilities, $S(t)$ can be expressed as

$$S(t) = s \exp\left\{\left(r - \frac{\sigma^2}{2}\right) t + \sigma\sqrt{t} Z\right\},$$

  where Z is a standard normal random variable.
- Let I be the indicator random variable for the event that the option finishes in the money:

$$I = \begin{cases} 1, & \text{if } S(t) > K, \\ 0, & \text{if } S(t) \leq K. \end{cases}$$

Expression for I

Lemma

We have

$$I = \begin{cases} 1, & \text{if } Z > \sigma\sqrt{t} - \omega, \\ 0, & \text{otherwise,} \end{cases}$$

where $\omega = \dfrac{rt + \frac{\sigma^2 t}{2} - \log\frac{K}{s}}{\sigma\sqrt{t}}$.

- We have

$$\begin{aligned} S(t) > K \quad & \Leftrightarrow \quad \exp\{(r - \tfrac{\sigma^2}{2})t + \sigma\sqrt{t}Z\} > \tfrac{K}{s} \\ & \Leftrightarrow \quad Z > \tfrac{\log\frac{K}{s} - (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \\ & \Leftrightarrow \quad Z > \sigma\sqrt{t} - \omega. \end{aligned}$$

Expression for $E(I)$

Lemma

We have

$$E[I] = P\{S(t) > K\} = \Phi(\omega - \sigma\sqrt{t}),$$

where Φ is the standard normal distribution function.

- It follows from its definition that

$$\begin{aligned} E[I] &= P\{S(t) > K\} \\ &= P\{Z > \sigma\sqrt{t} - \omega\} \quad \text{(from preceding lemma)} \\ &= P\{Z < \omega - \sigma\sqrt{t}\} \\ &= \Phi(\omega - \sigma\sqrt{t}). \end{aligned}$$

Expression for $E[IS(t)]$

Lemma

We have

$$e^{-rt}E[IS(t)] = s\Phi(\omega).$$

- Using the formula for $S(t)$ and the expression for I, with $c = \sigma\sqrt{t} - \omega$,

$$\begin{aligned}
E[IS(t)] &= \int_c^\infty s\exp\left\{(r-\frac{\sigma^2}{2})t + \sigma\sqrt{t}x\right\}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}}s\exp\{rt\}\int_c^\infty \exp\left\{-\frac{x^2-2\sigma\sqrt{t}x+\sigma^2 t}{2}\right\}dx \\
&= \frac{1}{\sqrt{2\pi}}se^{rt}\int_c^\infty \exp\left\{-\frac{(x-\sigma\sqrt{t})^2}{2}\right\}dx \\
&= se^{rt}\frac{1}{\sqrt{2\pi}}\int_{-\omega}^\infty e^{-y^2/2}dy \quad \text{(by letting } y = x - \sigma\sqrt{t}) \\
&= se^{rt}P\{Z > -\omega\} \\
&= se^{rt}\Phi(\omega).
\end{aligned}$$

The Black-Scholes Pricing Formula

Theorem (The Black-Scholes Pricing Formula)

We have

$$C(s,t,K,\sigma,r) = s\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}).$$

- We obtain

$$\begin{aligned} C(s,t,K,\sigma,r) &= e^{-rt}E[(S(t)-K)^+] \\ &= e^{-rt}E[I(S(t)-K)] \\ &= e^{-rt}E[IS(t)] - Ke^{-rt}E[I] \\ &= s\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}) \end{aligned}$$

(by the preceding lemmas).

The Black-Scholes Call Option Formula Revisited

- Let Z be a normal random variable with mean 0 and variance 1.
- Let $W = (r - \frac{\sigma^2}{2})t + \sigma\sqrt{t}Z$.
- Thus, W is normal with mean $(r - \frac{\sigma^2}{2})t$ and variance $t\sigma^2$.
- The Black-Scholes call option formula can be written as

$$C = C(s, t, K, \sigma, r) = e^{-rt}E[(S(t) - K)^+] = E[e^{-rt}I(se^W - K)],$$

where $I = \begin{cases} 1, & \text{if } se^W > K \\ 0, & \text{if } se^W \leq K \end{cases}$ is the indicator of $se^W > K$.

- We have

$$e^{-rt}I(se^W - K) = \begin{cases} e^{-rt}(se^W - K), & \text{if } se^W > K, \\ 0, & \text{if } se^W \leq K. \end{cases}$$

Partial Derivatives of C

- The preceding is, for given Z, a differentiable function of the parameters s, t, K, σ, r.
- So for x equal to any one of these variables,

$$\frac{\partial}{\partial x} e^{-rt} I(se^W - K) = \begin{cases} \frac{\partial}{\partial x} e^{-rt}(se^W - K), & \text{if } se^W > K, \\ 0, & \text{if } se^W \leq K. \end{cases}$$

- That is, $\frac{\partial}{\partial x} e^{-rt} I(se^W - K) = I \frac{\partial}{\partial x} e^{-rt}(se^W - K)$.
- Using that the partial derivative and the expectation operation can be interchanged, the preceding gives that

$$\begin{aligned} \frac{\partial C}{\partial x} &= \frac{\partial}{\partial x} E[e^{-rt} I(se^W - K)] \\ &= E[\frac{\partial}{\partial x} e^{-rt} I(se^W - K)] \\ &= E[I \frac{\partial}{\partial x} e^{-rt}(se^W - K)]. \end{aligned}$$

Partial Derivative of C With Respect to K

Proposition

We have

$$\frac{\partial C}{\partial K} = -e^{-rt}\Phi(\omega - \sigma\sqrt{t}).$$

- Because $S(t)$ does not depend on K,

$$\frac{\partial}{\partial K}e^{-rt}(S(t) - K) = -e^{-rt}.$$

Using the equation obtained in the preceding slide, this gives

$$\frac{\partial C}{\partial K} = E[-Ie^{-rt}] = -e^{-rt}E[I] = -e^{-rt}\Phi(\omega - \sigma\sqrt{t}),$$

the final equality by a previous lemma.

Partial Derivative of C With Respect to s

Proposition

We have

$$\frac{\partial C}{\partial s} = \Phi(\omega).$$

- Using the representation $S(t) = s \exp\{(r - \frac{\sigma^2}{2})t + \sigma\sqrt{t}Z\}$, we see that

$$\frac{\partial}{\partial s} e^{-rt}(S(t) - K) = e^{-rt}\frac{\partial S(t)}{\partial s} = \frac{S(t)}{s}e^{-rt}.$$

Hence, using $\frac{\partial C}{\partial s} = E[I \frac{\partial}{\partial s} e^{-rt}(se^W - K)]$,

$$\frac{\partial C}{\partial s} = \frac{e^{-rt}}{s}E[IS(t)] = \Phi(\omega),$$

the final equality using a previous lemma.

Partial Derivative of C With Respect to r

Proposition

We have

$$\frac{\partial C}{\partial r} = Kte^{-rt}\Phi(\omega - \sigma\sqrt{t}).$$

- We have

$$\begin{aligned}\frac{\partial}{\partial r}[e^{-rt}(S(t)-K)] &= -te^{-rt}(S(t)-K) + e^{-rt}\frac{\partial S(t)}{\partial r} \\ &= -te^{-rt}(S(t)-K) + e^{-rt}tS(t) \\ &= Kte^{-rt}.\end{aligned}$$

Using $\frac{\partial C}{\partial r} = E[I\frac{\partial}{\partial r}e^{-rt}(se^W - K)]$ and the expression for $E[I]$,

$$\frac{\partial C}{\partial r} = Kte^{-rt}E[I] = Kte^{-rt}\Phi(\omega - \sigma\sqrt{t}).$$

Auxiliary Lemma

Lemma

With $S(t) = s\exp\{(r - \frac{\sigma^2}{2})t + \sigma\sqrt{t}Z\}$,

$$e^{-rt}E[IS(t)Z] = s(\Phi'(\omega) + \sigma\sqrt{t}\Phi(\omega)).$$

- With $c = \sigma\sqrt{t} - \omega$, it follows from a previous lemma that

$$\begin{aligned}
E[IZS(t)] &= \int_c^\infty xs\exp\{(r - \frac{\sigma^2}{2})t + \sigma\sqrt{t}x\}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}}s\exp\{rt\}\int_c^\infty x\exp\{-\frac{x^2 - 2\sigma\sqrt{t}x + \sigma^2 t}{2}\}dx \\
&= \frac{1}{\sqrt{2\pi}}se^{rt}\int_c^\infty x\exp\{-\frac{(x - \sigma\sqrt{t})^2}{2}\}dx \\
&= \frac{1}{\sqrt{2\pi}}se^{rt}\int_{-\omega}^\infty (y + \sigma\sqrt{t})e^{-y^2/2}dy \quad (y = x - \sigma\sqrt{t}) \\
&= se^{rt}[\int_{-\omega}^\infty \frac{1}{\sqrt{2\pi}}ye^{-y^2/2}dy + \sigma\sqrt{t}\frac{1}{\sqrt{2\pi}}\int_{-\omega}^\infty e^{-y^2/2}dy] \\
&= se^{rt}[\frac{1}{\sqrt{2\pi}}e^{-\omega^2/2} + \sigma\sqrt{t}\Phi(\omega)].
\end{aligned}$$

Partial Derivative of C With Respect to σ

Proposition

We have
$$\frac{\partial C}{\partial \sigma} = s\sqrt{t}\Phi'(\omega).$$

- The equation $S(t) = s\exp\{(r - \frac{\sigma^2}{2})t + \sigma\sqrt{t}Z\}$ yields
$$\frac{\partial}{\partial \sigma}[e^{-rt}(S(t) - K)] = e^{-rt}S(t)(-t\sigma + \sqrt{t}Z).$$
Hence, by $\frac{\partial C}{\partial \sigma} = E[I\frac{\partial}{\partial \sigma}e^{-rt}(se^W - K)]$, we get
$$\begin{aligned}
\frac{\partial C}{\partial \sigma} &= E[e^{-rt}IS(t)(-t\sigma + \sqrt{t}Z)] \\
&= -t\sigma e^{-rt}E[IS(t)] + \sqrt{t}e^{-rt}E[IS(t)Z] \\
&= -t\sigma s\Phi(\omega) + s\sqrt{t}(\Phi'(\omega) + \sigma\sqrt{t}\Phi(\omega)) \\
&\quad \text{(by previous lemmas)} \\
&= s\sqrt{t}\Phi'(\Omega).
\end{aligned}$$

Partial Derivative of C With Respect to t

Proposition

We have

$$\frac{\partial C}{\partial t} = \frac{\sigma}{2\sqrt{t}} s\Phi'(\omega) + Kre^{-rt}\Phi(\omega - \sigma\sqrt{t}).$$

- We get

$$\begin{aligned} \tfrac{\partial}{\partial t}[e^{-rt}(S(t)-K)] &= e^{-rt}\tfrac{\partial S(t)}{\partial t} - re^{-rt}S(t) + Kre^{-rt} \\ &= e^{-rt}S(t)(r - \tfrac{\sigma^2}{2} + \tfrac{\sigma}{2\sqrt{t}}Z) \\ &\quad - re^{-rt}S(t) + Kre^{-rt} \\ &= e^{-rt}S(t)(-\tfrac{\sigma^2}{2} + \tfrac{\sigma}{2\sqrt{t}}Z) + Kre^{-rt}. \end{aligned}$$

Partial Derivative of C With Respect to t (Cont'd)

- Therefore, using $\frac{\partial C}{\partial t} = E[I \frac{\partial}{\partial t} e^{-rt}(se^W - K)]$,

$$\begin{aligned} \frac{\partial C}{\partial t} &= -e^{-rt}E[IS(t)]\frac{\sigma^2}{2} + e^{-rt}E[IZS(t)]\frac{\sigma}{2\sqrt{t}} + Kre^{-rt}E[I] \\ &= -s\Phi(\omega)\frac{\sigma^2}{2} + \frac{\sigma}{2\sqrt{t}}s(\Phi'(\omega) + \sigma\sqrt{t}\Phi(\omega)) \\ &\qquad + Kre^{-rt}\Phi(\omega - \sigma\sqrt{t}) \\ &= \frac{\sigma}{2\sqrt{t}}s\Phi'(\omega) + Kre^{-rt}\Phi(\omega - \sigma\sqrt{t}). \end{aligned}$$

Terminology: Terms used for the partial derivatives:

- **delta** for $\frac{\partial C}{\partial s}$;
- **rho** for $\frac{\partial C}{\partial r}$;
- **vega** for $\frac{\partial C}{\partial \sigma}$;
- **theta** for $\frac{\partial C}{\partial t}$.

Monotonicity and Convexity

Corollary

$C(s, t, K, \sigma, r)$ is:

(a) Decreasing and convex in K;

(b) Increasing and convex in s;

(c) Increasing, but neither convex nor concave, in r, σ and t.

(a) From a previous proposition $\frac{\partial C}{\partial K} = -e^{-rt}\Phi(\omega - \sigma\sqrt{t}) < 0$.

Moreover, recalling that $\omega = \frac{rt + \frac{\sigma^2 t}{2} - \log\frac{K}{s}}{\sigma\sqrt{t}}$,

$$\begin{aligned} \frac{\partial^2 C}{\partial K^2} &= -e^{-rt}\Phi'(\omega - \sigma\sqrt{t})\frac{\partial\omega}{\partial K} \\ &= e^{-rt}\Phi'(\omega - \sigma\sqrt{t})\frac{1}{K\sigma\sqrt{t}} > 0. \end{aligned}$$

So C is decreasing and convex in K.

Monotonicity and Convexity (Cont'd)

(b) By a previous proposition, we get $\frac{\partial C}{\partial s} = \Phi(\omega) > 0$.

Moreover, taking again into account $\omega = \frac{rt + \frac{\sigma^2 t}{2} - \log\frac{K}{s}}{\sigma\sqrt{t}}$,

$$\frac{\partial^2 C}{\partial s^2} = \Phi'(\omega)\frac{\partial \omega}{\partial s} = \Phi'(\omega)\frac{1}{s\sigma\sqrt{t}} > 0.$$

(c) By previous propositions, we have:

- $\frac{\partial C}{\partial r} = Kte^{-rt}\Phi(\omega - \sigma\sqrt{t}) > 0$;
- $\frac{\partial C}{\partial \sigma} = s\sqrt{t}\Phi'(\omega) > 0$;
- $\frac{\partial C}{\partial t} = \frac{\sigma}{2\sqrt{t}}s\Phi'(\omega) + Kre^{-rt}\Phi(\omega - \sigma\sqrt{t}) > 0$.

Each of the second derivatives can be shown to be sometimes positive and sometimes negative. So C is neither convex nor concave in r, σ or t.

Remarks

- The results that $C(s,t,K,\sigma,r)$ is decreasing and convex in K and increasing in t would be true no matter what model we assumed for the price evolution of the security.
- The results that $C(s,t,K,\sigma,r)$ is increasing and convex in s, increasing in r and increasing in σ depend on the assumption that the price evolution follows a geometric Brownian motion with volatility parameter σ.
- The second partial derivative of C with respect to s, whose value is given by

$$\frac{\partial^2 C}{\partial s^2} = \Phi'(\omega)\frac{1}{s\sigma\sqrt{t}}$$

  is called **gamma**.

Subsection 5

European Put Options

No-Arbitrage Cost of European Options

- The put call option parity formula, in conjunction with the Black-Scholes equation, yields the unique no arbitrage cost of a European (K, t) put option:

$$P(s,t,K,r,\sigma) = C(s,t,K,r,\sigma) + Ke^{-rt} - s.$$

- This formula is useful for computational purposes.
- To determine monotonicity and convexity properties of $P = P(s,t,K,r,\sigma)$ we use the fact that $P(s,t,K,r,\sigma)$ must equal the expected return from the put under the risk neutral geometric Brownian motion process.
- Consequently, with Z being a standard normal random variable,

$$\begin{aligned} P(s,t,K,r,\sigma) &= e^{-rt}E[(K - se^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{t}Z})^+] \\ &= E[(Ke^{-rt} - se^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}Z})^+]. \end{aligned}$$

Properties of P

- Consider Z fixed.
- The function $(Ke^{-rt} - se^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}Z})^+$ is decreasing and convex in s.
  For $b > 0$, $(a - bs)^+$ is decreasing and convex in s.
- The function $(Ke^{-rt} - se^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}Z})^+$ is decreasing and convex in r.
  For $a > 0$, $(ae^{-rt} - b)^+$ is decreasing and convex in r.
- The function $(Ke^{-rt} - se^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}Z})^+$ is increasing and convex in K.
  For $a > 0$, $(aK - b)^+$ is increasing and convex in K.

Properties of P (Cont'd)

- Because the preceding properties remain true when we take expectations, we infer the following:
  - $P(s, t, K, r, \sigma)$ is decreasing and convex in s;
  - $P(s, t, K, r, \sigma)$ is decreasing and convex in r;
  - $P(s, t, K, r, \sigma)$ is increasing and convex in K.
- Because $C(s, t, K, r, \sigma)$ is increasing in σ, it follows
  - $P(s, t, K, r, \sigma)$ is increasing in σ.
- $P(s, t, K, r, \sigma)$ is not necessarily increasing or decreasing in t.
- The partial derivatives of $P(s, t, K, r, \sigma)$ can be obtained by using the corresponding partial derivatives of $C(s, t, K, r, \sigma)$.