Introduction to Mathematical Finance

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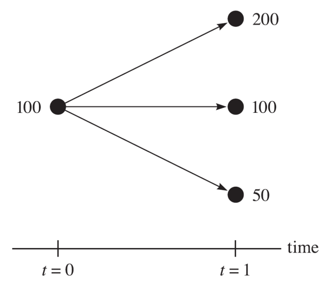
Subsection 1

Limitations of Arbitrage Pricing

Example

- Let the initial price of a security be 100.
- Suppose that the price at time 1 can be any of the values 50, 200, and 100.
- That is, we also allow for the possibility that the price of the stock at time 1 is unchanged from its initial price.



- We want to price an option to purchase the stock at time 1 for the fixed price of 150.

Example (Cont'd)

- For simplicity, let the interest rate r equal zero.
- The Arbitrage Theorem states that there will be no guaranteed win if there are nonnegative numbers p_{50}, p_{100}, p_{200}, such that:
  (a) Their sum equals 1;
  (b) The expected gains if one purchases either the stock or the option are zero, when p_i is the probability that the stock's price at time 1 is i $(i = 50, 100, 200)$.
- Let G_s denote the gain at time 1 from buying one share of the stock.
- Let $S(1)$ be the price of that stock at time 1.
- Then

$$G_s = \begin{cases} 100, & \text{if } S(1) = 200, \\ 0, & \text{if } S(1) = 100, \\ -50, & \text{if } S(1) = 50. \end{cases}$$

- Hence, $E[G_s] = 100p_{200} - 50p_{50}$.

Example (Cont'd)

- Let c be the cost of the option.
- Then the gain from purchasing one option is

$$G_o = \begin{cases} 50 - c, & \text{if } S(1) = 200, \\ -c, & \text{if } S(1) = 100 \text{ or } S(1) = 50. \end{cases}$$

- Therefore,

$$E[G_o] = (50 - c)p_{200} - c(p_{50} + p_{100}) = 50p_{200} - c.$$

- Equating both $E[G_s]$ and $E[G_o]$ to zero shows that the conditions for the absence of arbitrage are that there exist probabilities and a cost c such that

$$p_{200} = \frac{1}{2}p_{50} \quad \text{and} \quad c = 50p_{200}.$$

Example (Conclusion)

- We found that the conditions for the absence of arbitrage are that there exist probabilities and a cost c such that

$$p_{200} = \frac{1}{2} p_{50} \quad \text{and} \quad c = 50 p_{200}.$$

- The first implies that $p_{200} \leq \frac{1}{3}$.
- So, for any value of c satisfying

$$0 \leq c \leq \frac{50}{3},$$

  we can find probabilities that make both buying the stock and buying the option fair bets.
- By the Arbitrage Theorem, for any option cost in the interval $[0, \frac{50}{3}]$, no arbitrage is possible.

Subsection 2

Valuing Investments by Expected Utility

Choice Between Two Possible Investments

- Suppose we must choose one of two possible investments, each of which can result in any of n consequences, denoted $C_1, \ldots, C_n$.
- We make the following assumptions:
  - If the first investment is chosen, then consequence C_i will result with probability p_i, $i = 1, \ldots, n$, with $\sum_{i=1}^{n} p_i = 1$;
  - If the second investment is chosen, then consequence C_i will result with probability q_i, $i = 1, \ldots, n$, with $\sum_{i=1}^{n} q_i = 1$.
- We first assign numerical values to the different consequences.
- Identify the least and the most desirable consequence, say c and C.
  - Give the consequence c the value 0;
  - Give the consequence C the value 1.

Utility of Consequences

- Now consider any of the other $n-2$ consequences, say C_i.
- We want to assign a value to this consequence.
- Imagine that we are given the choice between:
  - Receiving C_i;
  - Taking part in a random experiment that earns one of the following:
    - Consequence C with probability u;
    - Consequence c with probability $1-u$.
- Clearly the choice will depend on the value of u.
  - If $u=1$ then the experiment is certain to result in consequence C.
    C is the most desirable consequence.
    So we clearly prefer the experiment to receiving C_i.
  - If $u=0$ then the experiment will result in c.
    c is the least desirable consequence.
    So in this case we prefer the consequence C_i to the experiment.

Utility of Consequences (Cont'd)

- As u decreases from 1 down to 0, it seems reasonable that the choice will at some point switch from the experiment to receiving C_i.
  At that critical point, we will be indifferent between the two choices.
  We adopt that indifference probability u as the value of the consequence C_i.
- So the value of C_i is that probability u such that we are indifferent between:
  - Receiving the consequence C_i;
  - Taking part in the experiment that returns consequence C with probability u or consequence c with probability $1 - u$.
- We call this indifference probability the **utility** of the consequence C_i, denoted by $u(C_i)$.

Evaluation of the Investments

- Consider the first investment, which results in consequence C_i with probability p_i, $i = 1, \ldots, n$.
- We can think of the result of this investment as being determined by a two-stage experiment.
  - In the first stage, one of the values $1, \ldots, n$ is chosen according to the probabilities $p_1, \ldots, p_n$;
  - If value i is chosen, we receive consequence C_i.
- Now C_i is equivalent to obtaining consequence C with probability $u(C_i)$ or consequence c with probability $1 - u(C_i)$.
- So the result of the two-stage experiment is equivalent to an experiment in which:
  - Either consequence C or consequence c is obtained;
  - Consequence C is obtained with probability

$$\sum_{i=1}^{n} p_i u(C_i).$$

Comparison of the Investments

- Similarly, the result of choosing the second investment is equivalent to taking part in an experiment in which:
  - Either consequence C or consequence c is obtained;
  - Consequence C being obtained with probability

$$\sum_{i=1}^{n} q_i u(C_i).$$

- We know that C is preferable to c.
- So the first investment is preferable to the second if

$$\sum_{i=1}^{n} p_i u(C_i) > \sum_{i=1}^{n} q_i u(C_i).$$

- In other words:
  - The value of an investment can be measured by the expected value of the utility of its consequence;
  - The investment with the largest expected utility is most preferable.

Utility Functions and Comparison

- In many investments, the consequences correspond to the investor receiving a certain amount of money.
- In this case, we let the dollar amount represent the consequence.
- Thus, $u(x)$ is the investor's utility of receiving the amount x.
- We call $u(x)$ a **utility function**.
- Suppose an investor must choose between two investments.
  - The first investment returns an amount X;
  - The second investment returns an amount Y;
  - The investor has utility function u.
- Then the investor should choose:
  - The first investment, if

$$E[u(X)] > E[u(Y)];$$

  - The second investment, if the inequality is reversed.
- Often, the possible monetary returns form an infinite set.
- So we may drop the requirement that $u(x)$ be between 0 and 1.

Nondecreasing and Concave Utility Functions

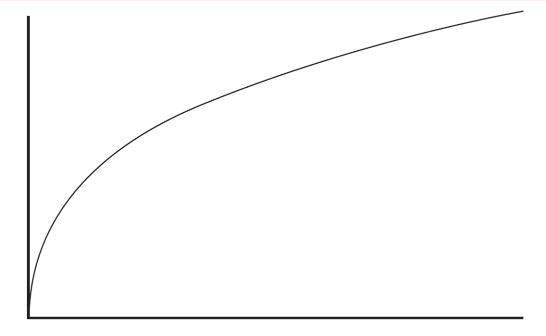
- An investor's utility function is specific to that investor.
- However, a general property usually assumed of utility functions is that $u(x)$ is a nondecreasing function of x.
- Another common (but not universal) property is that, if an investor expects to receive x, then the extra utility gained, if they are given an additional amount Δ, is nonincreasing in x.
- That is, for fixed $\Delta > 0$, their utility function satisfies

$$u(x + \Delta) - u(x) \text{ is nonincreasing in } x.$$

- A utility function that satisfies this condition is called **concave**.
- The condition of concavity is equivalent to $u''(x) \leq 0$.
- That is, a function is concave if and only if its second derivative is nonpositive.

Concave Utility Functions (Illustration)

- A function is concave if and only if its second derivative is nonpositive.



- The curve of a concave function has the property that the line segment connecting any two of its points always lies below the curve.

Risk-Averse Investors

- An investor with a concave utility function is said to be **risk-averse**.
- An explanation of this terminology follows.
- **Jensen's Inequality** states that if u is a concave function then, for any random variable X,

$$E[u(X)] \leq u(E[X]).$$

- This inequality, interpreted in terms of the return X from an investment, says that:

  Any investor with a concave utility function would prefer the certain return of $E[X]$ to receiving a random return X having this mean.

- This explains the term risk-averse.

Jensen's Inequality

Theorem (Jensen's Inequality)

If U is concave, then

$$E[U(X)] \leq U(E[X]).$$

- The Taylor series formula with remainder of $U(x)$ expanded about $\mu = E[X]$ gives, for some value of τ between x and μ, that

$$U(x) = U(\mu) + U'(\mu)(x-\mu) + \frac{U''(\tau)}{2}(x-\mu)^2.$$

But U being concave implies that $U'' \leq 0$, showing that

$$U(x) \leq U(\mu) + U'(\mu)(x-\mu).$$

Taking expectations of both sides, we get

$$E[U(X)] \leq U(\mu) + U'(\mu)E[X-\mu] = U(\mu).$$

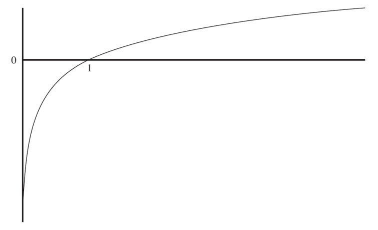
Risk-Neutral Investors

- An investor with a linear utility function
$$u(x) = a + bx, \quad b > 0,$$
is said to be **risk-neutral** or **risk-indifferent**.
- For such a utility function,
$$E[u(X)] = a + bE[X].$$
- So a risk-neutral investor values an investment only through its expected return.

Log Utility Function

- A commonly assumed utility function is the log utility function

$$u(x) = \log(x).$$



- Because $\log(x)$ is a concave function, an investor with a log utility function is risk-averse.
- This is a particularly important utility function.
- It can be mathematically proven in a variety of situations that an investor faced with an infinite sequence of investments can maximize his long-term rate of return by:
  - Adopting a log utility function;
  - Maximizing the expected utility in each period.

Maximizing Returns Under Log Utility Function

- Suppose that the result of each investment is to multiply the investor's wealth by a random amount X.
  - Let W_0 be the investor's initial wealth;
  - Let W_n be the investor's wealth after the nth investment;
  - Let X_n be the nth multiplication factor.
- Then we have

$$W_n = X_n W_{n-1}, \quad n \geq 1.$$

- Moreover,

$$\begin{aligned} W_n &= X_n W_{n-1} \\ &= X_n X_{n-1} W_{n-2} \\ &= X_n X_{n-1} X_{n-2} W_{n-3} \\ &= \cdots \\ &= X_n X_{n-1} \cdots X_1 W_0. \end{aligned}$$

Maximizing Returns (Cont'd)

- We calculated
$$W_n = X_n X_{n-1} \cdots X_1 W_0.$$
- Let R_n denote the rate of return (per investment) from the n investments.
- Then
$$\frac{W_n}{(1+R_n)^n} = W_0 \quad \text{or} \quad (1+R_n)^n = \frac{W_n}{W_0} = X_1 \cdots X_n.$$
- Taking logarithms yields that
$$\log(1+R_n) = \frac{\sum_{i=1}^n \log(X_i)}{n}.$$

Maximizing Returns (Cont'd)

- Suppose X_i are independent and identically distributed.
- By the Strong Law of Large Numbers, the average of the values $\log(X_i)$, $i = 1, \ldots, n$, converges to $E[\log(X_i)]$ as $n \to \infty$.
- Consequently,

$$\log(1 + R_n) \to E[\log(X)], \text{ as } n \to \infty.$$

- So the long-run rate of return is maximized by choosing the investment that yields the largest value of $E[\log(X)]$.
- Moreover, because $W_n = W_0 X_1 \cdots X_n$, it follows that

$$\log(W_n) = \log(W_0) + \sum_{i=1}^{n} \log(X_i).$$

- Hence,

$$E[\log(W_n)] = \log(W_0) + nE[\log(X)].$$

- This shows that maximizing $E[\log(X)]$ is equivalent to maximizing the expectation of the log of the final wealth.

Example

- Suppose an investor has capital x.
- He can invest any amount y, between 0 and x.
- In that case, one of the following occurs.
  - y is won with probability p;
  - y is lost with probability $1-p$.
- Suppose $p > \frac{1}{2}$ and the investor has a log utility function.
- We want to calculate how much should be invested.
- Suppose the amount αx is invested, where $0 \leq \alpha \leq 1$.
- Then the investor's final fortune X, will be:
  - $x + \alpha x$, with probability p;
  - $x - \alpha x$, with probability $1-p$.
- Hence, the expected utility of his final fortune is

$$\begin{aligned} & p \log((1+\alpha)x) + (1-p)\log((1-\alpha)x) \\ = & \ p\log(1+\alpha) + p\log(x) + (1-p)\log(1-\alpha) + (1-p)\log(x) \\ = & \ \log(x) + p\log(1+\alpha) + (1-p)\log(1-\alpha). \end{aligned}$$

Example (Cont'd)

- To find the optimal value of α, we differentiate

$$p \log(1+\alpha) + (1-p)\log(1-\alpha).$$

- We obtain

$$\frac{d}{d\alpha}(p \log(1+\alpha) + (1-p)\log(1-\alpha)) = \frac{p}{1+\alpha} - \frac{1-p}{1-\alpha}.$$

- Setting this equal to zero yields

$$p - \alpha p = 1 - p + \alpha - \alpha p \;\Rightarrow\; \alpha = 2p - 1.$$

- The investor should always invest

$$100(2p-1)\%$$

  of his present fortune.
  - If $p = 0.6$, the investor should invest 20% of his fortune;
  - If $p = 0.7$, he should invest 40% of his fortune.

Example

- We modify the preceding example.
  - The investment αx must be paid immediately;
  - The payoff of $2\alpha x$ (if it occurs) takes place after one period;
  - The amount not invested earns interest at a rate of r per period.
- We want to calculate how much should be invested.
- Suppose an investor invests αx and puts $(1-\alpha)x$ in the bank.
- After one period, she will have:
  - $(1+r)(1-\alpha)x$ in the bank;
  - Either $2\alpha x$, with probability p, or 0, with probability $1-p$.
- Hence, the expected value of the utility of her fortune is

$$\begin{aligned} & p\log((1+r)(1-\alpha)x+2\alpha x)+(1-p)\log((1+r)(1-\alpha)x) \\ = {} & \log(x)+p\log(1+r+\alpha-\alpha r) \\ & \quad +(1-p)\log(1+r)+(1-p)\log(1-\alpha). \end{aligned}$$

Example (Cont'd)

- Hence, once again the optimal fraction of one's fortune to invest does not depend on the amount of that fortune.
- Differentiating the previous equation yields

$$\frac{d}{d\alpha}(\text{expected utility}) = \frac{p(1-r)}{1+r+\alpha-\alpha r} - \frac{1-p}{1-\alpha}.$$

- Setting equal to zero and solving yields the optimal value of α

$$\begin{aligned} p(1-r)(1-\alpha) - (1-p)(1+r+\alpha-\alpha r) &= 0 \\ p(1-r) - (1-p)(1+r) &= \alpha[p(1-r) + (1-p)(1-r)] \\ \alpha = \frac{p(1-r) - (1-p)(1+r)}{1-r} &= \frac{2p-1-r}{1-r}. \end{aligned}$$

Example (Cont'd)

- We found that the optimal value of α is

$$\alpha = \frac{2p - 1 - r}{1 - r}.$$

- Let $p = 0.6$ and $r = 0.05$.
- The expected rate of return on the investment is 20%, whereas the bank pays only 5%.
- Still, the optimal fraction of money to be invested is

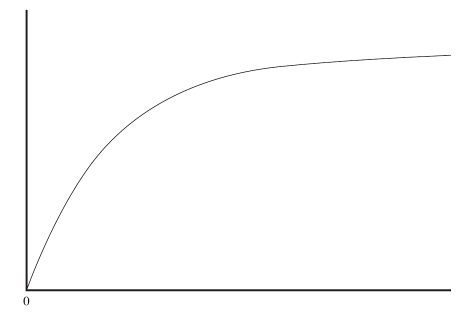
$$\alpha = \frac{2 \cdot 0.6 - 1 - 0.05}{1 - 0.05} = \frac{0.15}{0.95} \approx 0.158.$$

- That is, the investor should invest approximately 15.8% of his capital and put the remainder in the bank.

The Exponential Utility Function

- Another commonly used utility function is the **exponential utility function**

$$u(x) = 1 - e^{-bx}, \quad b > 0.$$



- The exponential is also a risk-averse utility function.

Subsection 3

The Portfolio Selection Problem

Portfolio

- Suppose one has the positive amount w to be invested among n different securities.
- Suppose the amount a is invested in security i $(i = 1, \ldots, n)$.
- Then, after one period, that investment returns aX_i, where X_i is a nonnegative random variable.
- So if R_i is the the rate of return from investment i, then

$$a = \frac{aX_i}{1 + R_i} \quad \text{or} \quad R_i = X_i - 1.$$

- Suppose w_i is invested in each security $i = 1, \ldots, n$.
- Then the end-of-period wealth is

$$W = \sum_{i=1}^{n} w_i X_i.$$

- The vector $w_1, \ldots, w_n$ is called a **portfolio**.

Maximizing the Expected Utility

- Let U be the investor's utility function for the end-of-period wealth.
- The problem of determining the portfolio that maximizes the expected utility of one's end-of-period wealth can be expressed mathematically as follows:

$$\text{choose } w_1, \ldots, w_n \text{ satisfying } w_i \geq 0, i = 1, \ldots, n, \\ \text{and } \sum_{i=1}^n w_i = w, \text{ to maximize } E[U(W)].$$

- We assume that the end-of-period wealth W is a normal random variable.
  - This is a reasonable approximation if many securities are not too highly correlated.
  - It is exactly true if the X_i, $i = 1, \ldots, n$, have a multivariate normal distribution.

Using the Exponential Utility Function

- Suppose now that the investor has an exponential utility function

$$U(x) = 1 - e^{-bx}, \quad b > 0.$$

- So the utility function is concave.
- If Z is a normal random variable, then e^Z is lognormal and has expected value

$$E[e^Z] = \exp\left\{E[Z] + \frac{\text{Var}(Z)}{2}\right\}.$$

- Hence, as $-bW$ is normal with mean $-bE[W]$ and variance $b^2\text{Var}(W)$, it follows that

$$E[U(W)] = 1 - E[e^{-bW}] = 1 - \exp\left\{-bE[W] + \frac{b^2\text{Var}(W)}{2}\right\}.$$

- Therefore, the investor's expected utility will be maximized by choosing a portfolio that maximizes $E[W] - \frac{b\text{Var}(W)}{2}$.

Comparing Portfolios

- Suppose two portfolios give rise to random end-of-period wealths W_1 and W_2.
- If W_1 has a larger mean and a smaller variance than does W_2, then the first portfolio results in a larger expected utility than does the second.

$$E[W_1] \geq E[W_2] \text{ \& } \mathrm{Var}(W_1) \leq \mathrm{Var}(W_2) \text{ imply } E[U(W_1)] \geq E[U(W_2)].$$

- In fact, provided that all end-of-period fortunes are normal random variables, this implication remains valid even when the utility function is not exponential, as long as it is nondecreasing and concave.
- Consequently, if one investment portfolio offers a risk-averse investor an expected return that is at least as large as that offered by a second investment portfolio and with a variance that is no greater than that of the second portfolio, then the investor prefers the first portfolio.

Expectation and Variance of Wealth

- We now compute, for a given portfolio, the mean and variance of W.
- Let $R_i = X_i - 1$ be security i's rate of return.
- Let $r_i = E[R_i]$ and $v_i^2 = \text{Var}(R_i)$.
- We have

$$W = \sum_{i=1}^{n} w_i(1+R_i) = w + \sum_{i=1}^{n} w_i R_i.$$

- Hence, we obtain

$$\begin{aligned} E[W] &= w + \textstyle\sum_{i=1}^n E[w_i R_i] = w + \sum_{i=1}^n w_i r_i; \\ \text{Var}(W) &= \text{Var}(\textstyle\sum_{i=1}^n w_i R_i) \\ &= \textstyle\sum_{i=1}^n \text{Var}(w_i R_i) + \sum_{i=1}^n \sum_{j\neq i} \text{Cov}(w_i R_i, w_j R_j) \\ &= \textstyle\sum_{i=1}^n w_i^2 v_i^2 + \sum_{i=1}^n \sum_{j\neq i} w_i w_j c(i,j), \end{aligned}$$

where $c(i,j) = \text{Cov}(R_i, R_j)$.

Example (Multivariate Normal Distribution)

Definition

Let $Z_1, \ldots, Z_m$ be independent standard normal random variables. If for some constants μ_i, $i = 1, \ldots, n$ and a_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$,

$$\begin{array}{rcl} X_1 & = & \mu_1 + a_{11}Z_1 + a_{12}Z_2 + \cdots + a_{1m}Z_m \\ X_2 & = & \mu_2 + a_{21}Z_1 + a_{22}Z_2 + \cdots + a_{2m}Z_m \\ & \vdots & \\ X_i & = & \mu_i + a_{i1}Z_1 + a_{i2}Z_2 + \cdots + a_{im}Z_m \\ & \vdots & \\ X_n & = & \mu_n + a_{n1}Z_1 + a_{n2}Z_2 + \cdots + a_{nm}Z_m \end{array}$$

we say that $(X_1, \ldots, X_n)$ has a **multivariate normal distribution**.

Because any linear combination $\sum_{i=1}^n w_i X_i$ is also a linear combination of the independent normal random variables $Z_1, \ldots, Z_m$, it follows that $\sum_{i=1}^n w_i X_i$ is a normal random variable.

Example

- We are investing a fortune of 100 in two securities.
- Let our utility function be

$$U(x) = 1 - e^{-0.005x}.$$

- The rates of return have the following expected values and standard deviations:
  - $r_1 = 0.15$ and $v_1 = 0.20$;
  - $r_2 = 0.18$ and $v_2 = 0.25$.
- The correlation between the rates of return is $\rho = -0.4$.
- We want to calculate the optimal portfolio.

Example (Cont'd)

- Suppose $w_1 = y$ and $w_2 = 100 - y$.
- We know $E[W] = w + \sum_{i=1}^{2} w_i r_i$.
- So we obtain

$$E[W] = 100 + 0.15y + 0.18(100 - y) = 118 - 0.03y.$$

- We also have

$$c(1,2) = \rho v_1 v_2 = -0.02.$$

- We know $\text{Var}(W) = \sum_{i=1}^{2} w_i^2 v_i^2 + 2 w_1 w_2 c(1,2)$.
- So we get

$$\begin{aligned} \text{Var}(W) &= y^2(0.04) + (100 - y)^2(0.0625) - 2y(100 - y)(0.02) \\ &= 0.1425y^2 - 16.5y + 625. \end{aligned}$$

Example (Cont'd)

- We must choose y to maximize

$$\begin{aligned} E[W] - \frac{b\mathrm{Var}(W)}{2} &= 118 - 0.03y - \frac{0.005(0.1425y^2 - 16.5y + 625)}{2} \\ &= 0.01125y - \frac{0.0007125y^2}{2}. \end{aligned}$$

- Using calculus, we get $y = \frac{0.01125}{0.0007125} = 15.789$.
- So we must invest:
  - 15.789 in investment 1
  - 84.211 in investment 2.
- The value $y = 15.789$ gives:
  - $E[W] = 18 - 0.03 \cdot 15.789 = 117.526$;
  - $\mathrm{Var}(W) = 0.1425 \cdot (15.789)^2 - 16.5 \cdot 15.789 + 625 = 400.006$.
- So the maximal expected utility is

$$1 - \exp\left\{-0.005\left(117.526 + \frac{0.005(400.006)}{2}\right)\right\} = 0.4416.$$

Example

- Suppose only two securities are under consideration, both with normally distributed returns that have same expected rate of return.
- Every portfolio will yield the same expected value.
- The best portfolio for any concave utility function is the one whose end-of-period wealth has minimal variance.
- Suppose αw is invested in security 1 and $(1-\alpha)w$ is invested in security 2.
- With $c = c(1,2)$ we have

$$\begin{aligned} \mathrm{Var}(W) &= \alpha^2 w^2 v_1^2 + (1-\alpha)^2 w^2 v_2^2 + 2\alpha(1-\alpha) w^2 c \\ &= w^2[\alpha^2 v_1^2 + (1-\alpha)^2 v_2^2 + 2c\alpha(1-\alpha)]. \end{aligned}$$

- Thus, the optimal portfolio is obtained by choosing the value of α that minimizes

$$\alpha^2 v_1^2 + (1-\alpha)^2 v_2^2 + 2c\alpha(1-\alpha).$$

Example (Cont'd)

- We want to minimize

$$\alpha^2 v_1^2 + (1-\alpha)^2 v_2^2 + 2c\alpha(1-\alpha).$$

- Differentiating this quantity and setting the derivative equal to zero yields

$$2\alpha v_1^2 - 2(1-\alpha)v_2^2 + 2c - 4c\alpha = 0.$$

- Solving for α gives the optimal fraction to invest in security 1:

$$\alpha = \frac{v_2^2 - c}{v_1^2 + v_2^2 - 2c}.$$

Example (Special Case)

- If the rates of returns are independent, then $c = 0$.
- So the optimal fraction to invest in security 1 is

$$\alpha = \frac{v_2^2}{v_1^2 + v_2^2} = \frac{\frac{1}{v_1^2}}{\frac{1}{v_1^2} + \frac{1}{v_2^2}}.$$

- In this case, the optimal percentage of capital to invest in a security is determined by a weighted average, where the weight given to a security is inversely proportional to the variance of its rate of return.
- This result also remains true when there are n securities whose rates of return are uncorrelated and have equal means.
- Under these conditions, the optimal fraction of one's capital to invest in security i is

$$\frac{\frac{1}{v_i^2}}{\sum_{j=1}^{n} \frac{1}{v_j^2}}.$$

Estimating Covariances

- In order to create good portfolios, we must first use historical data to estimate, for all i and j, the values of

$$r_i = E[R_i], \quad v_i^2 = \text{Var}(R_i) \quad \text{and} \quad c(i,j) = \text{Cov}(R_i, R_j).$$

- Suppose we have historical data that covers m periods.
- Let $r_{i,k}$ and $r_{j,k}$ denote (respectively) the rates of return of security i and of security j for period k, $k = 1, \ldots, m$.
- Then we take:
  - $\overline{r}_i = \frac{\sum_{k=1}^m r_{i,k}}{m}$;
  - $\overline{v}_i^2 = \frac{\sum_{k=1}^m (r_{i,k} - \overline{r}_i)^2}{m-1}$;
  - $\overline{c}(i,j) = \frac{\sum_{k=1}^m (r_{i,k} - \overline{r}_i)(r_{j,k} - \overline{r}_j)}{m-1}$.

Subsection 4

Value at Risk and Conditional Value at Risk

Value at Risk Criterion

- Suppose an investment:
  - Calls for an initial payment of c;
  - Returns X after one period.
- Let G denote the present value gain from the investment.

$$G = \frac{X}{1+r} - c.$$

- The **value at risk** (**VAR**) of the investment is the value v, such that there is only a 1-percent chance that the loss from the investment will be greater than v.
- Because $-G$ is the loss, the value at risk is the value v such that

$$P\{-G > v\} = 0.01.$$

- The **VAR criterion** for choosing among different investments selects the investment having the smallest VAR.

Example

- Suppose that the gain G from an investment is a normal random variable with mean μ and standard deviation σ.
- Then $-G$ is normal with mean $-\mu$ and standard deviation σ.
- So the VAR of this investment is the value of v, such that

$$\begin{aligned} 0.01 &= P\{-G > v\} \\ &= P\left\{\frac{-G+\mu}{\sigma} > \frac{v+\mu}{\sigma}\right\} \\ &= P\left\{Z > \frac{v+\mu}{\sigma}\right\}, \end{aligned}$$

  where Z is a standard normal random variable.
- From the table we see that $P\{Z > 2.33\} = 0.01$.
- Therefore, $\frac{v+\mu}{\sigma} = 2.33$, which gives $\text{VAR} = -\mu + 2.33\sigma$.
- So, among investments whose gains are normally distributed, the VAR criterion selects the one having the largest value of $\mu - 2.33\sigma$.

Conditional Value at Risk Criterion

- The VAR gives a value that has only a 1-percent chance of being exceeded by the loss from an investment.
- The VAR criterion chooses the investment having the smallest VAR.
- An alternative proposal considers the conditional expected loss, given that it exceeds the VAR.
- In other words, we consider the amount lost, given that the 1-percent event occurs and there is a large loss.
- This a quantity larger than the VAR.
- The conditional expected loss, given that it exceeds the VAR, is called the **conditional value at risk** or **CVAR**.
- The **CVAR criterion** selects the investment having the smallest CVAR.

The Conditional Expectation Formula

- For a standard normal random variable Z,

$$E[Z|Z>a]=\frac{1}{\sqrt{2\pi}P\{Z\geq a\}}e^{-a^2/2}.$$

- The conditional density of Z, given that $Z>a$, is

$$f_{Z|Z>a}(x)=\frac{1}{\sqrt{2\pi}}\frac{e^{-x^2/2}}{P(Z>a)},\quad x>a.$$

- This gives

$$\begin{aligned}E[Z|Z>a] &= \frac{1}{\sqrt{2\pi}P(Z>a)}\int_a^\infty xe^{-x^2/2}dx\\ &= \frac{1}{\sqrt{2\pi}P(Z>a)}e^{-a^2/2}.\end{aligned}$$

Example

- Suppose the gain G from an investment is a normal random variable with mean μ and standard deviation σ.
- Then the CVAR is given by

$$\begin{aligned}
\text{CVAR} &= E[-G \mid -G > VAR] \\
&= E[-G \mid -G > -\mu + 2.33\sigma] \\
&= E\left[-G \Big| \frac{-G+\mu}{\sigma} > 2.33\right] \\
&= E\left[\sigma\left(\frac{-G+\mu}{\sigma}\right) - \mu \Big| \frac{-G+\mu}{\sigma} > 2.33\right] \\
&= \sigma E\left[\frac{-G+\mu}{\sigma} \Big| \frac{-G+\mu}{\sigma} > 2.33\right] - \mu \\
&= \sigma E[Z \mid Z > 2.33] - \mu,
\end{aligned}$$

where Z is a standard normal random variable.

Example (Cont'd)

- We computed
$$\text{CVAR} = \sigma E[Z|Z > 2.33] - \mu.$$
- We showed that, for a standard normal random variable Z,
$$E[Z|Z > a] = \frac{1}{\sqrt{2\pi} P\{Z \geq a\}} e^{-a^2/2}.$$
- Hence we obtain that
$$\begin{aligned} \text{CVAR} &= \sigma \frac{1}{\sqrt{2\pi} P\{Z \geq 2.33\}} e^{-(2.33)^2/2} - \mu \\ &= \sigma \frac{100}{\sqrt{2\pi}} \exp\left\{-\frac{(2.33)^2}{2}\right\} - \mu \\ &= 2.64\sigma - \mu. \end{aligned}$$
- So, the CVAR, which attempts to maximize $\mu - 2.64\sigma$, gives a little more weight to the variance than does the VAR.

Subsection 5

The Capital Assets Pricing Model

The Capital Assets Pricing Model

- Let R_i be the one-period rate of return of a specified security i.
- Let R_m be the one-period rate of return of the entire market (as measured, say, by the Standard and Poor's index of 500 stocks).
- The **Capital Assets Pricing Model** (**CAPM**) relates R_i to R_m.
- Let r_f be the risk-free interest rate (usually taken to be the current rate of a U.S. Treasury bill).
- The model assumes that, for some constant β_i,

$$R_i = r_f + \beta_i(R_m - r_f) + e_i,$$

where e_i is a normal random variable with mean 0 that is assumed to be independent of R_m.

Expected Rates of Return

- Let r_i be the expected value of R_i.
- Let r_m be the expected value of R_m.
- The CAPM model (which treats r_f as a constant) implies that

$$r_i = r_f + \beta_i(r_m - r_f).$$

- Equivalently, we have

$$r_i - r_f = \beta_i(r_m - r_f).$$

- That is, the difference between the expected rate of return of the security and the risk-free interest rate is assumed to equal β_i times the difference between the expected rate of return of the market and the risk-free interest rate.
- The quantity β_i is known as the **beta** of security i.

Covariance of Specific and Market Return

- Recall that:
  - Covariance is linear;
  - The covariance of a random variable and a constant is 0.
- As a result, we get

$$\begin{aligned}
\mathrm{Cov}(R_i, R_m) &= \mathrm{Cov}(r_f + \beta_i(R_m - r_f) + e_i, R_m) \\
&= \beta_i \mathrm{Cov}(R_m, R_m) + \mathrm{Cov}(e_i, R_m) \\
&= \beta_i \mathrm{Var}(R_m). \quad (e_i \text{ and } R_m \text{ independent}).
\end{aligned}$$

- Therefore, letting $v_m^2 = \mathrm{Var}(R_m)$, we see that

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_m)}{v_m^2}.$$

Example

- Suppose that:
  - The current risk-free interest rate is 6%;
  - The expected value of the market rate of return is 0.10;
  - The standard deviation of the market rate of return is 0.20;
  - The covariance of the rate of return of a given stock and the market's rate of return is 0.05.
- We compute the expected rate of return of the stock based on CAPM.
- We have

$$\beta = \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)} = \frac{0.05}{(0.20)^2} = 1.25.$$

It follows that

$$r_i = r_f + \beta_i(r_m - r_f) = 0.06 + 1.25(0.10 - 0.06) = 0.11.$$

That is, the stock's expected rate of return is 11%.

Systematic and Specific Risks

- We are still in the framework of CAPM.
- Again, let $v_i^2 = \mathrm{Var}(R_i)$ and $v_m^2 = \mathrm{Var}(R_m)$.
- Recall that R_m and e_i are independent.
- Then we have

$$\begin{aligned} v_i^2 &= \mathrm{Var}(R_i) \\ &= \mathrm{Var}(r_f + \beta_i(R_m - r_f) + e_i) \\ &= \beta_i^2 v_m^2 + \mathrm{Var}(e_i). \end{aligned}$$

- Think of the variance of a rate of return as the risk of a security.
- Then the equation states that the risk of a security is the sum of:
  - The **systematic risk** $\beta_i^2 v_m^2$, due to the combination of the security's beta and the inherent risk in the market;
  - The **specific risk** $\mathrm{Var}(e_i)$, due to the specific stock being considered.

Subsection 6

Rates of Return: Single-Period and Geometric Brownian Motion

One Period Returns

- Let $S_i(t)$ be the price of security i at time t ($t \geq 0$).
- Assume these prices follow a geometric Brownian motion with:
  - Drift parameter μ_i;
  - Volatility parameter σ_i.
- Let R_i be the one-period rate of return for security i.
- Then we have
$$\frac{S_i(1)}{1+R_i} = S_i(0).$$
- Equivalently,
$$R_i = \frac{S_i(1)}{S_i(0)} - 1.$$

Expectation and Variance of Return

- Now $\frac{S_i(1)}{S_i(0)}$ has the same probability distribution as e^X, where X is a normal random variable with mean μ_i and variance σ_i^2.
- So we get

$$r_i = E[R_i] = E\left[\frac{S_i(1)}{S_i(0)}\right] - 1 = E[e^X] - 1 = \exp\left\{\mu_i + \frac{\sigma_i^2}{2}\right\} - 1.$$

- Also,

$$\begin{aligned} v_i^2 = \mathrm{Var}(R_i) &= \mathrm{Var}\left(\frac{S_i(1)}{S_i(0)}\right) \\ &= \mathrm{Var}(e^X) \\ &= E[e^{2X}] - (E[e^X])^2 \\ &= \exp\{2\mu_i + 2\sigma_i^2\} - \left(\exp\left\{\mu_i + \frac{\sigma_i^2}{2}\right\}\right)^2 \\ &= \exp\{2\mu_i + 2\sigma_i^2\} - \exp\{2\mu_i + \sigma_i^2\}. \end{aligned}$$

Expected Value and Variance of One-Period Yield

- The average spot rate of return by time t, $\overline{R}_i(t)$, satisfies
$$\frac{S_i(t)}{S_i(0)} = e^{t\overline{R}_i(t)}.$$
- This implies that
$$\overline{R}_i(t) = \frac{1}{t}\log\left(\frac{S_i(t)}{S_i(0)}\right).$$
- Now $\log\left(\frac{S_i(t)}{S_i(0)}\right)$ is a normal random variable with:
  - Mean $\mu_i t$;
  - Variance $t\sigma_i^2$.
- So $\overline{R}_i(t)$ is a normal random variable with
$$E[\overline{R}_i(t)] = \mu_i, \quad \mathrm{Var}(\overline{R}_i(t)) = \frac{\sigma_i^2}{t}.$$
- The expected value and variance of the one-period yield function for geometric Brownian motion are its parameters μ_i and σ_i^2.