Finite Model Theory

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1. The Ehrenfeucht-Fraïssé Method
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Subsection 1

Elementary Classes

Expressivity of First-Order Logic in the Finite

Proposition

Every finite structure can be characterized in first-order logic up to isomorphism. I.e., for every finite structure $\mathcal{A}$, there is a sentence $\varphi_{\mathcal{A}}$ of first-order logic, such that for all structures $\mathcal{B}$, we have

$$\mathcal{B} \vDash \varphi_{\mathcal{A}} \quad \text{iff} \quad \mathcal{A} \cong \mathcal{B}.$$

- Suppose $\mathcal{A} = \{a_1, \ldots, a_n\}$. Set $\overline{a} = a_1 \ldots a_n$. Let

$$\begin{aligned} \Theta_n \quad := \quad & \{\psi : \psi \text{ has the form } Rx_1 \ldots x_k, x = y \text{ or } c = x, \\ & \text{and variables among } v_1, \ldots, v_n\}. \end{aligned}$$

Finally set

$$\begin{aligned} \varphi_{\mathcal{A}} \quad := \quad \exists v_1 \cdots \exists v_n (& \bigwedge \{\psi : \psi \in \Theta_n, \mathcal{A} \vDash \psi[\overline{a}]\} \\ & \wedge \bigwedge \{\neg\psi : \psi \in \Theta_n, \mathcal{A} \vDash \neg\psi[\overline{a}]\} \\ & \wedge \forall v_{n+1} (v_{n+1} = v_1 \vee \cdots \vee v_{n+1} = v_n)). \end{aligned}$$

Generalized Axiomatizability

Corollary

Let K be a class of finite structures. Then there is a set Φ of first-order sentences such that $K = \text{Mod}(\Phi)$, i.e., K is the class of finite models of Φ.

- Let K be a class of finite structures. For each n, there is only a finite number of pairwise nonisomorphic structures of cardinality n. Let $\{\mathcal{A}_1, \ldots, \mathcal{A}_k\}$ be a maximal subset of K of pairwise nonisomorphic structures of cardinality n. Set

$$\psi_n := (\varphi_{=n} \to (\varphi_{\mathcal{A}_1} \vee \cdots \vee \varphi_{\mathcal{A}_k})),$$

where $\varphi_{=n}$ is a first-order sentence expressing "there are exactly n elements". Then $K = \text{Mod}(\{\psi_n : n \geq 1\})$.

Elementary Classes and m-Equivalence

Definition

Let K be a class of finite structures. K is called **axiomatizable in first-order logic** or **elementary**, if there is a sentence φ of first-order logic such that $K = \text{Mod}(\varphi)$.

- Some remarks:
  - In the literature, instead of axiomatizable one often uses the term *finitely axiomatizable*.
  - In (general) model theory a class of arbitrary structures K is called *elementary* if, for some φ, K is the class of arbitrary models of φ. And given classes K_0 and K with $K_0 \supseteq K$, it is said that K is *elementary relative to* K_0 if, for some φ, K is the class of models of φ in K_0. In this terminology our notion of elementary corresponds to elementary relative to the class of finite structures.
- For structures $\mathcal{A}$ and $\mathcal{B}$ and $m \in \mathbb{N}$, we write $\mathcal{A} \equiv_m \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are m-**equivalent**, if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same first-order sentences of quantifier rank $\leq m$.

Necessary Condition for Axiomatizability

Theorem

Let K be a class of finite structures. Suppose that for every m, there are finite structures $\mathcal{A}$ and $\mathcal{B}$, such that $\mathcal{A} \in K$, $\mathcal{B} \notin K$, and $\mathcal{A} \equiv_m \mathcal{B}$. Then K is not axiomatizable in first-order logic.

- Let φ be any first-order sentence.

  Set $m := \text{qr}(\varphi)$.

  By hypothesis, there are $\mathcal{A}$, $\mathcal{B}$, such that $\mathcal{A} \in K$, $\mathcal{B} \notin K$, and $\mathcal{A} \equiv_m \mathcal{B}$.

  Hence, $K \neq \text{Mod}(\varphi)$.

Subsection 2

Ehrenfeucht's Theorem

Partial Isomorphisms

Definition

Assume $\mathcal{A}$ and $\mathcal{B}$ are structures. Let p be a map with $\text{dom}(p) \subseteq A$ and $\text{ran}(p) \subseteq B$, where $\text{dom}(p)$ and $\text{ran}(p)$ denote the domain and the range of p, respectively. Then p is a **partial isomorphism** from $\mathcal{A}$ to $\mathcal{B}$ if:

- p is injective;
- for every $c \in \tau$, $c^{\mathcal{A}} \in \text{dom}(p)$ and $p(c^{\mathcal{A}}) = c^{\mathcal{B}}$;
- for every n-ary $R \in \tau$, and all $a_1, \ldots, a_n \in \text{dom}(p)$,

$$R^{\mathcal{A}} a_1 \ldots a_n \quad \text{iff} \quad R^{\mathcal{B}} p(a_1) \ldots p(a_n).$$

We write $\text{Part}(\mathcal{A}, \mathcal{B})$ for the set of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

- We identify a map p with its graph $\{(a, p(a)) : a \in \text{dom}(p)\}$.
- Then $p \subseteq q$ means that q is an extension of p.

Remarks

(a) The empty map, $p = \varnothing$, is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ just in case the vocabulary contains no constants.

(b) If $p \neq \varnothing$ is a map with $\text{dom}(p) \subseteq A$ and $\text{ran}(p) \subseteq B$, then p is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ iff
- $\text{dom}(p)$ contains $c^{\mathcal{A}}$, for all constants $c \in \tau$, and
- $p : \text{dom}(p)^{\mathcal{A}} \cong \text{ran}(p)^{\mathcal{B}}$, where $\text{dom}(p)^{\mathcal{A}}$ and $\text{ran}(p)^{\mathcal{B}}$ denote the substructures of $\mathcal{A}$ and $\mathcal{B}$ with universes $\text{dom}(p)$ and $\text{ran}(p)$, respectively.

Remarks (Cont'd)

(c) For $a = a_1 \dots a_s \in A$ and $b = b_1 \dots b_s \in B$, the following statements are equivalent:

(i) The clauses $p(a_i) = b_i$, for $i = 1, \dots, s$ and $p(c^{\mathcal{A}}) = c^{\mathcal{B}}$, for c in τ, define a map, which is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ (henceforth denoted by $\overline{a} \mapsto \overline{b}$, a notation that suppresses the constants).

(ii) For all quantifier-free $\varphi(v_1, \dots, v_s)$, $\mathcal{A} \vDash \varphi[\overline{a}]$ iff $\mathcal{B} \vDash \varphi[\overline{b}]$.

(iii) For all atomic $\varphi(v_1, \dots, v_s)$, $\mathcal{A} \vDash \varphi[\overline{a}]$ iff $\mathcal{B} \vDash \varphi[\overline{b}]$.

- Note that for an arbitrary structure $\mathcal{D}$ and $\overline{d}$ in D,
  - $d_i = d_j$ iff $\mathcal{D} \vDash v_i = v_j[\overline{d}]$;
  - $c^{\mathcal{D}} = d_j$ iff $\mathcal{D} \vDash c = v_j[\overline{d}]$;
  - $R^{\mathcal{D}} c^{\mathcal{D}} d_i d_j$ iff $\mathcal{D} \vDash Rcv_iv_j[\overline{d}]$, $c, R \in \tau, R$ ternary.

Using such equivalences, we can show that (i) and (iii) are equivalent.

Clearly, (ii) implies (iii).

(ii) follows from (iii), since every quantifier-free formula is a boolean combination of atomic formulas.

Partial Isomorphisms and Quantifiers

- In general, a partial isomorphism does not preserve the validity of formulas with quantifiers.

  Example: Let $\tau = \{<\}$, $\mathcal{A} = (\{0,1,2\},<)$, $\mathcal{B} = (\{0,1,2,3\},<)$, where in both cases $<$ denotes the natural ordering.

  $p_0 := 02 \mapsto 01$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

  Note that
  $$\mathcal{A} \vDash \exists v_3(v_1 < v_3 \wedge v_3 < v_2)[0,2]$$
  but
  $$\mathcal{B} \nvDash \exists v_3(v_1 < v_3 \wedge v_3 < v_2)[p_0(0), p_0(2)].$$

  Since $\mathcal{A} \vDash (v_1 < v_3 \wedge v_3 < v_2)[0,2,1]$, we see that, for any $p \in \text{Part}(\mathcal{A},\mathcal{B})$, with $\text{dom}(p) = \{0,2\}$, the validity of $\mathcal{B} \vDash \exists v_3(v_1 < v_3 \wedge v_3 < v_2)[p(0), p(2)]$ is equivalent to the existence of some $q \in \text{Part}(\mathcal{A},\mathcal{B})$ which extends p and has 1 in its domain.

Ehrenfeucht Games: The Play

- Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures, $\overline{a} \in A^s$, $\overline{b} \in B^s$, and $m \in \mathbb{N}$.
- The **Ehrenfeucht game** $G_m(A, \overline{a}, B, \overline{b})$ is played by two players called the **spoiler** and the **duplicator**.
- Each player has to make m moves in the course of a play and the players take turns.
- In his i-th move the spoiler first selects a structure, $\mathcal{A}$ or $\mathcal{B}$, and an element in this structure.
- In his i-th move:
  - If the spoiler chooses e_i in $\mathcal{A}$, then the duplicator must choose an element f_i in $\mathcal{B}$;
  - If the spoiler chooses f_i in $\mathcal{B}$, then the duplicator must choose an element e_i in $\mathcal{A}$.

Ehrenfeucht Games: Winners and Losers

- At the end of the game $G_m(\mathcal{A},\overline{a},\mathcal{B},\overline{b})$ elements $e_1,\ldots,e_m$ in $\mathcal{A}$ and $f_1,\ldots,f_m$ in $\mathcal{B}$ have been chosen.

	$\mathcal{A},\overline{a}$	$\mathcal{B},\overline{b}$
Move 1	e_1	f_1
Move 2	e_2	f_2
$\vdots$	$\vdots$	$\vdots$
Move m	e_m	f_m

- The duplicator wins iff $\overline{a}\ \overline{e} \mapsto \overline{b}\ \overline{f} \in \text{Part}(\mathcal{A},\mathcal{B})$.

  In case $m = 0$, we just require that $\overline{a} \mapsto \overline{b} \in \text{Part}(\mathcal{A},\mathcal{B})$.
- Otherwise, the spoiler wins.

  Equivalently, the spoiler wins if, after some $i \leq m$, $\overline{a}e_1\ldots e_i \mapsto \overline{b}f_1\ldots f_i$ is not a partial isomorphism.

Ehrenfeucht Games: Winning Strategies

- We say that a player, the spoiler or the duplicator, has a **winning strategy** in $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$, or shortly, that he **wins** $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$, if it is possible for him to win each play whatever choices are made by the opponent.
- If $s = 0$ (and hence $\overline{a}$ and $\overline{b}$ are empty), we denote the game by $G_m(\mathcal{A}, \mathcal{B})$.

Duplicator Win

Lemma

(a) If $\mathcal{A} \cong \mathcal{B}$ then the duplicator wins $G_m(\mathcal{A}, \mathcal{B})$.

(b) If the duplicator wins $G_{m+1}(\mathcal{A}, \mathcal{B})$ and $\|A\| \le m$, then $\mathcal{A} \cong \mathcal{B}$.

(a) Suppose $\pi : \mathcal{A} \cong \mathcal{B}$.

A winning strategy for the duplicator consists in always choosing the image or preimage under π of the spoiler's selection:

- If the spoiler chooses $a \in A$, then the duplicator chooses $\pi(a)$;
- If the spoiler chooses $b \in B$, then the duplicator answers with $\pi^{-1}(b)$.

Duplicator Win (Cont'd)

(b) Suppose that the duplicator wins $G_{m+1}(A, B)$ and $\|A\| \leq m$.

Assume $A = \{a_1, \ldots, a_m\}$.

Consider a play where the spoiler, in his first m moves, chooses $a_1, \ldots, a_m$.

Let $b_1, \ldots, b_m$ be the responses of the duplicator according to his winning strategy.

Then $p : \overline{a} \mapsto \overline{b} \in \text{Part}(\mathcal{A}, \mathcal{B})$ with $\text{dom}(p) = A$.

Claim: p is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Suppose, to the contrary, that $\text{ran}(p) \neq B$.

Then the spoiler, in the last move, chooses some $b \in B \backslash \text{ran}(p)$.

Now there is no answer for the duplicator.

So the spoiler wins, a contradiction.

Properties of Ehrenfeucht Games

- The following lemma collects some facts about the Ehrenfeucht game.

Lemma

Let $\mathcal{A}$ and $\mathcal{B}$ be structures, $\overline{a} \in A^s$, $\overline{b} \in B^s$, and $m \geq 0$.

(a) The duplicator wins $G_0(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$ iff $\overline{a} \mapsto \overline{b}$ is a partial isomorphism.

(b) For $m > 0$ the following are equivalent:

  (i) The duplicator wins $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.

  (ii) For all $a \in A$, there is $b \in B$, such that the duplicator wins the game $G_{m-1}(\mathcal{A}, \overline{a}a, \mathcal{B}, \overline{b}b)$ and for all $b \in B$, there is $a \in A$, such that the duplicator wins $G_{m-1}(\mathcal{A}, \overline{a}a, \mathcal{B}, \overline{b}b)$.

(c) If the duplicator wins $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$ and $m' < m$, then the duplicator wins $G_{m'}(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.

- The proofs are immediate from the definition.

Introducing the Hintikka Formula

- Let $\mathcal{A}$ be given.
- For $\overline{a} = a_1 \dots a_s \in A$ and $m \geq 0$, we introduce a formula

$$\varphi^m_{\overline{a}}(v_1, \dots, v_s)$$

  that describes the game theoretic properties of $\overline{a}$ in any game $G_m(\mathcal{A}, \overline{a}, \dots)$.
- More precisely, we want to define $\varphi^m_{\overline{a}}$ in such a way that for any $\mathcal{B}$ and $\overline{b} = b_1 \dots b_s \in B$,

$$\mathcal{B} \vDash \varphi^m_{\overline{a}}[\overline{b}] \quad \text{iff} \quad \text{the duplicator wins } G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b}).$$

- If the structure $\mathcal{A}$ is not clear from the context, we use the notation $\varphi^m_{\mathcal{A},\overline{a}}$ for $\varphi^m_{\overline{a}}$.
- We also allow $s = 0$, the case of the empty sequence $\varnothing$ of elements in A, and write $\varphi^m_{\mathcal{A}}$ for the sentence $\varphi^m_{\mathcal{A},\varnothing}$.

The Hintikka Formula

Definition

Let $\overline{v}$ be $v_1, \ldots, v_s$. Set

$$\varphi^0_{\overline{a}}(\overline{v}) := \bigwedge\{\varphi(\overline{v}) : \varphi \text{ atomic or negated atomic, } \mathcal{A} \vDash \varphi[\overline{a}]\}.$$

For $m > 0$, set

$$\varphi^m_{\overline{a}}(\overline{v}) := \bigwedge_{a \in A} \exists v_{s+1} \varphi^{m-1}_{\overline{a}a}(\overline{v}, v_{s+1}) \wedge \forall v_{s+1} \bigvee_{a \in A} \varphi^{m-1}_{\overline{a}a}(\overline{v}, v_{s+1}).$$

- $v^0_{\overline{a}}$ describes the isomorphism type of the substructure generated by $\overline{a}$ in $\mathcal{A}$.
- For $m > 0$ the formula $\varphi^m_{\overline{a}}$ tells us to which isomorphism types the tuple $\overline{a}$ can be extended in m steps adding one element in each step.
- $\varphi^m_{\overline{a}}$ is called the m-**isomorphism type** (or m-**Hintikka formula**) of $\overline{a}$ in $\mathcal{A}$.

Properties of the Hintikka Formula

- Since $\{\varphi(v_1,\ldots,v_s) : \varphi$ atomic or negated atomic$\}$ is finite, a simple induction on m shows

Lemma

For $s, m \geq 0$, the set $\{\varphi^m_{\mathcal{A},\overline{a}} : \mathcal{A}$ a structure and $\overline{a} \in A^s\}$ is finite.

- In particular, the conjunctions and disjunctions in the above definition are finite.

Lemma

(a) $\mathrm{qr}(\varphi^m_{\overline{a}}) = m$;

(b) $\mathcal{A} \models \varphi^m_{\overline{a}}[\overline{a}]$;

(c) For any $\mathcal{B}$ and $\overline{b}$ in B, $\mathcal{B} \models \varphi^0_{\overline{a}}[\overline{b}]$ iff $\overline{a} \mapsto \overline{b} \in \mathrm{Part}(\mathcal{A},\mathcal{B})$.

- The proofs of (a) and (b) are straightforward.

  (c) holds by Part (c) of previous remarks.

Ehrenfeucht's Theorem

Theorem (Ehrenfeucht's Theorem)

Given $\mathcal{A}$, $\mathcal{B}$, $\overline{a} \in A^s$, $\overline{b} \in B^s$, and $m \geq 0$, the following are equivalent:

(i) The duplicator wins $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$;

(ii) $\mathcal{B} \models \varphi_{\overline{a}}^m[\overline{b}]$;

(iii) $\overline{a}$ and $\overline{b}$ satisfy the same formulas of quantifier rank $\leq m$, that is, if $\varphi(x_1, \ldots, x_s)$ is of quantifier rank $\leq m$, then $\mathcal{A} \models \varphi[\overline{a}]$ iff $\mathcal{B} \models \varphi[\overline{b}]$.

(iii)$\Rightarrow$(ii): We have $\text{qr}(\varphi_{\overline{a}}^m) = m$ and $\mathcal{A} \models \varphi_{\overline{a}}^m[\overline{a}]$.

It follows, using (iii), that $\mathcal{B} \models \varphi_{\overline{a}}^m[\overline{b}]$.

(i)$\Leftrightarrow$(ii): This is done by induction on m.

Assume $m = 0$.

The duplicator wins $G_0(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$ iff $\overline{a} \mapsto \overline{b} \in \text{Part}(\mathcal{A}, \mathcal{B})$ iff $\mathcal{B} \models \varphi_{\overline{a}}^0[\overline{b}]$.

Ehrenfeucht's Theorem (Cont'd)

- Assume, next, $m > 0$.

  The duplicator wins $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$ iff
  for all $a \in A$, there is $b \in B$, such that the duplicator wins $G_{m-1}(\mathcal{A}, \overline{a}a, \mathcal{B}, \overline{b}b)$, and
  for all $b \in B$, there is $a \in A$, such that the duplicator wins $G_{m-1}(\mathcal{A}, \overline{a}a, \mathcal{B}, \overline{b}b)$ iff
  for all $a \in A$, there is $b \in B$, with $\mathcal{B} \vDash \varphi^{m-1}_{\overline{a}a}[\overline{b}b]$ and
  for all $b \in B$, there is $a \in A$, with $\mathcal{B} \vDash \varphi^{m-1}_{\overline{a}a}[\overline{b}b]$ iff
  $\mathcal{B} \vDash \bigwedge_{a \in A} \exists v_{s+1} \varphi^{m-1}_{\overline{a}a}(\overline{v}, v_{s+1}) \wedge \forall v_{s+1} \bigvee_{a \in A} \varphi^{m-1}_{\overline{a}a}(\overline{v}, v_{s+1})[\overline{b}]$ iff
  $\mathcal{B} \vDash \varphi^m_{\overline{a}}[\overline{b}]$.

Ehrenfeucht's Theorem (Cont'd)

- (i)⇒(iii): The proof proceeds by induction on m.

  The case $m = 0$ is handled as above.

  Let $m > 0$ and suppose that the duplicator wins $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.

  Clearly, the set of formulas $\varphi(x_1, \ldots, x_s)$ satisfying the equivalence in (iii) contains the atomic formulas and is closed under $\neg$ and $\vee$.

  Suppose that $\varphi(\overline{x}) = \exists y \psi$ and $\text{qr}(\varphi) \leq m$.

  Since $y \notin \text{free}(\varphi)$, we can assume that y is distinct from $\overline{x}$. Hence, $\psi = \psi(\overline{x}, y)$.

  Assume, for instance, $\mathcal{A} \vDash \varphi[\overline{a}]$.

  Then there is $a \in A$, such that $\mathcal{A} \vDash \psi[\overline{a}a]$.

  By (i), the duplicator wins $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.

  So there is $b \in B$, such that the duplicator wins $G_{m-1}(\mathcal{A}, \overline{a}a, \mathcal{B}, \overline{b}b)$.

  Since $\text{qr}(\psi) \leq m - 1$, the induction hypothesis yields $\mathcal{B} \vDash \psi[\overline{b}, b]$.

  Hence $\mathcal{B} \vDash \varphi[\overline{b}]$.

The Case $s = 0$

Corollary

For structures $\mathcal{A}, \mathcal{B}$ and $m \geq 0$ the following are equivalent:

(i) The duplicator wins $G_m(\mathcal{A}, \mathcal{B})$.

(ii) $\mathcal{B} \models \varphi_{\mathcal{A}}^m$.

(iii) $\mathcal{A} \equiv_m \mathcal{B}$.

- Moreover, by Part (b) of a previous lemma, we get

Corollary

Let $\mathcal{A}$ be a structure with $\|A\| \leq m$. Then for all $\mathcal{B}$,

$$\mathcal{B} \models \varphi_{\mathcal{A}}^{m+1} \quad \text{iff} \quad \mathcal{A} \cong \mathcal{B}.$$

Satisfiability of a Formula

- The next result shows that the formulas $\varphi^m_{\overline{a}}$ give a clear picture of the expressive power of first-order logic.

Theorem

Let $\varphi(v_1, \ldots, v_s)$ be a formula of quantifier rank $\leq m$. Then

$$\vDash \varphi \ \leftrightarrow \ \bigvee\{\varphi^m_{\mathcal{A},\overline{a}} : \mathcal{A} \text{ a structure, } \overline{a} \in A, \text{ and } \mathcal{A} \vDash \varphi[\overline{a}]\}.$$

- Suppose first that $\mathcal{B} \vDash \varphi[\overline{b}]$. Then the formula $\varphi^m_{\mathcal{B},\overline{b}}$ is a member of the disjunction on the right side of the equivalence. So the latter is satisfied by $\overline{b}$.

  Conversely, suppose $\mathcal{B} \vDash \bigvee\{\varphi^m_{\mathcal{A},\overline{a}}\}[\overline{b}]$.

  Then, for some $\mathcal{A}$ and $\overline{a}$, such that $\mathcal{A} \vDash \varphi[\overline{a}]$, we have $\mathcal{B} \vDash \varphi^m_{\mathcal{A},\overline{a}}[\overline{b}]$.

  By Ehrenfeucht's Theorem, $\overline{a}$ and $\overline{b}$ satisfy the same formulas of quantifier rank $\leq m$. Therefore, $\mathcal{B} \vDash \varphi[\overline{b}]$.

Characterization of Axiomatizable Classes

Theorem

For a class K of finite structures the following are equivalent:

(i) K is not axiomatizable in first-order logic.

(ii) For each m there are finite structures $\mathcal{A}$ and $\mathcal{B}$ such that

$$\mathcal{A} \in K,\ \mathcal{B} \notin K \text{ and } \mathcal{A} \equiv_m \mathcal{B}.$$

(ii) ⇒ (i) was proven previously.

(i) ⇒ (ii) Suppose that (ii) does not hold.

For some m and all finite $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \in K$ and $\mathcal{A} \equiv_m \mathcal{B}$ imply $\mathcal{B} \in K$.

Then $K = \mathrm{Mod}(\bigvee\{\varphi^m_{\mathcal{A}} : \mathcal{A} \in K\})$.

Thus K is axiomatizable.

Subsection 3

Examples and Fraïssé's Theorem

The Back and Forth Properties

- Given structures $\mathcal{A}, \mathcal{B}$ and $m \in \mathbb{N}$, let

$$W_m(\mathcal{A},\mathcal{B}) := \{\overline{a} \mapsto \overline{b} : s \geq 0, \overline{a} \in A^s, \overline{b} \in B^s, \text{ the duplicator wins } G_m(\mathcal{A},\overline{a},\mathcal{B},\overline{b})\}$$

be the set of winning positions for the duplicator.

Definition

Structures $\mathcal{A}$ and $\mathcal{B}$ are said to be m-**isomorphic**, written $\mathcal{A} \cong_m \mathcal{B}$, if there is a sequence $(I_j)_{j \leq m}$ with the following properties:

(a) Every I_j is a nonempty set of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

(b) (**Forth Property**) For every $j < m$, $p \in I_{j+1}$, and $a \in A$, there is $q \in I_j$, such that $q \supseteq p$ and $a \in \text{dom}(q)$.

(c) (**Back Property**) For every $j < m$, $p \in I_{j+1}$, and $b \in B$, there is $q \in I_j$, such that $q \supseteq p$ and $b \in \text{ran}(g)$.

If $(I_j)_{j \leq m}$ has the properties (a), (b), and (c), we write $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are m-**isomorphic via** $(I_j)_{j \leq m}$.

Property of m-Isomorphism

Proposition

Suppose $(I_j)_{j \le m} : \mathcal{A} \cong_m \mathcal{B}$. Then $(\widetilde{I_j})_{j \le m} : \mathcal{A} \cong_m \mathcal{B}$ with

$$\widetilde{I_j} := \{q \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) : q \subseteq p \text{ for some } p \in I_j\}.$$

In particular, $\varnothing \mapsto \varnothing \in \widetilde{I_j}$, for all $j \le m$. Moreover, $\widetilde{W_j(\mathcal{A}, \mathcal{B})} = W_j(\mathcal{A}, \mathcal{B})$.

- We first show that $(\widetilde{I_j})_{j \le m} : \mathcal{A} \cong_m \mathcal{B}$.
  - (a) Note that $I_j \ne \varnothing$ and $I_j \subseteq \widetilde{I_j}$. So $\widetilde{I_j} \ne \varnothing$.
    If $\widetilde{p} \in \widetilde{I_j}$, then, by definition, it is a partial isomorphism.
  - (b) Suppose $j < m$, $\widetilde{p} \subseteq \widetilde{I}_{j+1}$ and $a \in A$.
    Then, there is $p \in I_{j+1}$, with $\widetilde{p} \subseteq p$.
    Since $(I_j)_{j \le m} : \mathcal{A} \cong_m \mathcal{B}$, there exists $q \in I_j$, such that $p \subseteq q$ and $a \in \mathrm{dom}(q)$.
    Then, by definition, $\widetilde{q} = q \in \widetilde{I_j}$, such that $\widetilde{p} \subseteq p \subseteq \widetilde{q}$ and $a \in \mathrm{dom}(\widetilde{q})$.
  - (c) Similar to (b).

Property of m-Isomorphism (Cont'd)

- We next show that $\widetilde{W_j(\mathcal{A},\mathcal{B})} = W_j(\mathcal{A},\mathcal{B})$.

  Clearly, $W_j(\mathcal{A},\mathcal{B}) \subseteq \widetilde{W_j(\mathcal{A},\mathcal{B})}$.

  Suppose $\widetilde{p} \in \widetilde{W_j(\mathcal{A},\mathcal{B})}$.

  Then, there exists $p \in W_j(\mathcal{A},\mathcal{B})$, such that $\widetilde{p} \subseteq p$.

  So the duplicator wins $G_j(\mathcal{A},\text{dom}(p),\mathcal{B},\text{ran}(p))$.

  A fortiori, the duplicator wins $G_j(\mathcal{A},\text{dom}(\widetilde{p}),\mathcal{B},\text{ran}(\widetilde{p}))$.

  This shows that $\widetilde{p} \in W_j(\mathcal{A},\mathcal{B})$.

Isomorphisms and Games

Theorem

For structures $\mathcal{A}$ and $\mathcal{B}$, $\overline{a} \in A^s$, $\overline{b} \in B^s$, and $m \geq 0$, the following are equivalent:

(i) The duplicator wins $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$;

(ii) $\overline{a} \mapsto \overline{b} \in W_m(\mathcal{A}, \mathcal{B})$ and $(W_j(\mathcal{A}, \mathcal{B}))_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$;

(iii) There is $(I_j)_{j \leq m}$, with $\overline{a} \mapsto \overline{b} \in I_m$, such that $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$;

(iv) $\mathcal{B} \vDash \varphi_{\overline{a}}^m[\overline{b}]$;

(v) $\overline{a}$ satisfies in $\mathcal{A}$ the same formulas of quantifier rank $\leq m$ as $\overline{b}$ in $\mathcal{B}$.

- By the definition of $W_m(\mathcal{A}, \mathcal{B})$ and a previous result, (i) implies (ii).

  Obviously, (ii) implies (iii).

  Therefore it suffices to show the implication (iii) $\Rightarrow$ (i), the remaining equivalences being clear from previous results.

Isomorphisms and Games ((iii) $\Rightarrow$ (i))

- Suppose that $(I_j)_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$ and $\overline{a} \mapsto \overline{b} \in I_m$.

  We describe a winning strategy in $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$ for the duplicator.

  In his i-th move he should choose the element e_i (or f_i, respectively), such that for

  $$p_i : \overline{a}e_1 \ldots e_i \mapsto \overline{b}f_1 \ldots f_i$$

  it is true that $p_i \subseteq q$, for some $q \in I_{m-i}$.

  This is always possible because of the back and forth properties of $(I_j)_{j\leq m}$.

  Looking at $i = m$ we see that the duplicator wins.

The Case $s = 0$

- For $s = 0$, in view of a previous proposition, the preceding theorem yields the following corollary.

Corollary

For structures $\mathcal{A}, \mathcal{B}$ and $m \geq 0$ the following are equivalent:

(i) The duplicator wins $G_m(\mathcal{A}, \mathcal{B})$;

(ii) $(W_j(\mathcal{A}, \mathcal{B}))_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$;

(iii) $\mathcal{A} \cong_m \mathcal{B}$;

(iv) $\mathcal{B} \vDash \varphi^m_{\mathcal{A}}$;

(v) $\mathcal{A} \equiv_m \mathcal{B}$.

- The equivalence of (iii) and (v) is known as **Fraïssé's Theorem**.

The Ehrenfeucht-Fraïssé Method

- Ehrenfeucht's Theorem and Fraïssé's Theorem are different formulations of the same fact.
- The preceding proof exhibits the close relationship between:
  - Sequences $(I_j)_{j \leq m}$;
  - Winning strategies for the duplicator in $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.
- Therefore, one often speaks of the **Ehrenfeucht-Fraïssé game** or the **Ehrenfeucht-Fraïssé method**.

Example

- Let τ be the empty vocabulary.

  Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures, i.e., nonempty sets.

  Suppose $\|A\| \geq m$ and $\|B\| \geq m$.

  Then $\mathcal{A} \cong_m \mathcal{B}$.

  In fact, $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$, with

  $$I_j := \{p \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) : \|\mathrm{dom}(p)\| \leq m - j\}.$$

- As a consequence the class $\mathrm{EVEN}[\tau]$ of finite τ-structures of even cardinality is not axiomatizable in first-order logic.

  In fact, for each $m > 0$, let $\mathcal{A}_m$ be a structure of cardinality m.

  Then, $\mathcal{A}_m \in \mathrm{EVEN}[\tau]$ iff $\mathcal{A}_{m+1} \notin \mathrm{EVEN}[\tau]$.

  But $\mathcal{A}_m \cong_m \mathcal{A}_{m+1}$.

  Now, we apply the Axiomatizability Theorem.

Example

- Let $\tau = \{<, \min, \max\}$ be a vocabulary for finite orderings.

  Let $m \geq 1$.

  Let $\mathcal{A}$ and $\mathcal{B}$ be finite orderings, with $\|A\| > 2^m$ and $\|B\| > 2^m$.

  We show, next, that $\mathcal{A} \cong_m \mathcal{B}$.

  I follows that the class of finite orderings of even cardinality is not axiomatizable in first-order logic.

  This statement remains true, if we consider orderings as $\{<, S, \min, \max\}$-structures.

Example (Cont'd)

- Given any ordering $\mathcal{C}$, we define its distance function d by

$$d(a,a') := \|\{b \in C : (a < b \le a') \text{ or } (a' < b \le a)\}\|.$$

For $j \ge 0$, we introduce the "truncated" j-distance function d_j on $C \times C$ by

$$d_j(a,a') := \begin{cases} d(a,a'), & \text{if } d(a,a') < 2^j, \\ \infty, & \text{otherwise.} \end{cases}$$

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are finite orderings with $\|A\|, \|B\| > 2^m$.

For $j \le m$, set

$$I_j := \{p \in \text{Part}(\mathcal{A},\mathcal{B}) : d_j(a,a') = d_j(p(a),p(a')) \text{ for } a,a' \in \text{dom}(p)\}.$$

Example (Cont'd)

Claim: $(I_j)_{j \le m} : \mathcal{A} \cong_m \mathcal{B}$.

By assumption on the cardinalities of $\mathcal{A}$ and $\mathcal{B}$, we have $\{(\min^A, \min^B), (\max^A, \max^B)\} \in I_j$, for every $j \le m$.

We prove the forth property of $(I_j)_{j \le m}$.

The back property is shown similarly.

Suppose $j < m$, $p \in I_{j+1}$, and $a \in A$. We distinguish two cases, depending on whether or not "there is an $a' \in \text{dom}(p)$, such that $d_j(a, a') < 2^j$".

- If the condition holds, there is exactly one $b \in B$ for which $p \cup \{(a, b)\}$ is a partial isomorphism preserving d_j-distances.
- Suppose the condition does not hold. Let $\text{dom}(p) = \{a_1, \ldots, a_r\}$ with $a_1 < \ldots < a_r$. We restrict ourselves to the case $a_i < a < a_{i+1}$, for some i. Then, $d_j(a_i, a) = \infty$ and $d_j(a, a_{i+1}) = \infty$. Hence, $d_{j+1}(a_i, a_{i+1}) = \infty$. Therefore, $d_{j+1}(p(a_i), p(a_{i+1})) = \infty$. Thus, there is a b, such that $p(a_i) < b < p(a_{i+1})$, $d_j(p(a_i), b) = \infty$, and $d_j(b, p(a_{i+1})) = \infty$. We can now verify that $q := p \cup \{(a, b)\}$ is a partial isomorphism in I_j.

Example

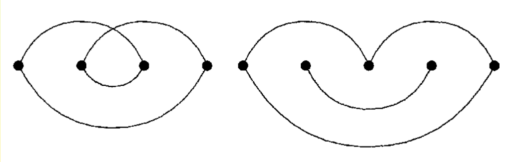
- Let $\tau = \{<, \min, \max\}$ be a vocabulary for finite orderings. Let $\sigma = \tau \cup \{E\}$, with a binary relation symbol E. For $n \geq 3$, let $\mathcal{A}_n$ be the ordered τ-structure with

$$A_n = \{0, \ldots, n\},$$

such that:
  - $\min^{\mathcal{A}_n} = 0$ and $\max^{\mathcal{A}_n} = n$;
  - $<^{\mathcal{A}_n}$ is the natural ordering on A_n;
  - E is interpreted as

$$E^{\mathcal{A}_n} = \{(i,j) : |i - j| = 2\} \cup \{(0, n), (n, 0), (1, n - 1), (n - 1, 1)\}.$$

$(A_n, E^{\mathcal{A}_n})$ is a graph that is connected iff n is odd.



Example (Cont'd)

- Let $m \geq 2$ and $\ell, k \geq 2^m$.

  Let I_j be the set of partial isomorphisms from $\mathcal{A}_\ell|_\tau$ to $\mathcal{A}_k|_\tau$ as in the preceding example.

  For $j \geq 2$ any $p \in I_j$ preserves E as well.

  I.e., $I_j \subseteq \text{Part}(\mathcal{A}_i, \mathcal{A}_k)$.

  Hence, $(I_{j+2})_{j \leq m-2} : \mathcal{A}_\ell \cong_{m-2} \mathcal{A}_k$.

  By the Axiomatizability Theorem, the class of finite connected ordered graphs is not first-order axiomatizable.

Example

- For $\ell \geq 1$, let $\mathcal{G}_\ell$ be the graph given by a cycle of length $\ell + 1$.

  To be precise, set:

$$\begin{array}{rcl} G_\ell & := & \{0, \ldots, \ell\}; \\ E^{G_\ell} & := & \{(i, i+1) : i < \ell\} \cup \{(i+1, i) : i < \ell\} \cup \{(0, \ell), (\ell, 0)\}. \end{array}$$

  Thus, for $\ell, k \in \mathbb{N}$, the disjoint union $\mathcal{G}_\ell \mathbin{\dot\cup} \mathcal{G}_k$ consists of:
  - A cycle of length $\ell + 1$;
  - A cycle of length $k + 1$.

  Note that for $\mathcal{A}_\ell$, as defined in the preceding example, we have:
  - $\mathcal{A}_\ell \backslash \{E\} \cong \mathcal{G}_\ell$ for ℓ odd;
  - $\mathcal{A}_\ell \backslash \{E\} \cong \mathcal{G}_{\frac{\ell}{2}-1} \mathbin{\dot\cup} \mathcal{G}_{\frac{\ell}{2}}$ for ℓ even.

Example (Cont'd)

Claim: If $\ell, k \geq 2^m$, then $\mathcal{G}_\ell \cong_m \mathcal{G}_k$ and $\mathcal{G}_\ell \cong_m \mathcal{G}_\ell \cup \mathcal{G}_\ell$.

For $j \in \mathbb{N}$, define the distance function d_j on a graph $\mathcal{G}$ by

$$d_j(a, a') := \begin{cases} d(a,b), & \text{if } d(a,b) < 2^{j+1} \\ \infty, & \text{otherwise} \end{cases},$$

where d denotes the distance function on $\mathcal{G}$ introduced previously.

We define I_j as the set of $p \in \text{Part}(\mathcal{G}_\ell, \mathcal{G}_\ell \cup \mathcal{G}_\ell)$, with:

- $\|\text{dom}(p)\| \leq m - j$;
- $d_j(a, b) = d_j(p(a), p(b))$, for $a, b \in \text{dom}(p)$.

We can verify that

$$(I_j)_{j \leq m} : \mathcal{G}_\ell \cong_m G_\ell \cup \mathcal{G}_\ell.$$

This proves that $\mathcal{G}_\ell$ and $\mathcal{G}_\ell \cup \mathcal{G}_\ell$ are m-isomorphic.

Example: Consequences (Cont'd)

Claim: The class CONN of connected finite graphs is not axiomatizable in first order logic.

By the Axiomatization Theorem, CONN is not axiomatizable, since for each m we have

$$\mathcal{G}_{2^m} \in \text{CONN}, \quad \mathcal{G}_{2^m} \mathbin{\dot\cup} \mathcal{G}_{2^m} \notin \text{CONN}, \quad \mathcal{G}_{2^m} \equiv_m \mathcal{G}_{2^m} \mathbin{\dot\cup} \mathcal{G}_{2^m}.$$

Claim: The global relation TC, the relation of transitive closure on the class GRAPH of finite graphs, is not first-order definable.

Suppose $\psi(x, y)$ is a first-order formula defining TC on GRAPH.

Then CONN would be the class of finite models of

$$\forall x \forall y(\neg x = y \rightarrow \psi(x, y))$$

together with the graph axioms.

Equivalence Invariance of Operations

Proposition

The product, the disjoint union, and the ordered sum preserve $\equiv_m$:

(a) If $\mathcal{A}_1 \equiv_m \mathcal{B}_1$ and $\mathcal{A}_2 \equiv_m \mathcal{B}_2$, then $\mathcal{A}_1 \times \mathcal{A}_2 \equiv_m \mathcal{B}_1 \times \mathcal{B}_2$;

(b) If $\mathcal{A}_1 \equiv_m \mathcal{B}_1$ and $\mathcal{A}_2 \equiv_m \mathcal{B}_2$, then $\mathcal{A}_1 \mathbin{\dot\cup} \mathcal{A}_2 \equiv_m \mathcal{B}_1 \mathbin{\dot\cup} \mathcal{B}_2$;

(c) If $\mathcal{A}_1 \equiv_m \mathcal{B}_1$ and $\mathcal{A}_2 \equiv_m \mathcal{B}_2$, then $\mathcal{A}_1 \triangleleft \mathcal{A}_2 \equiv_m \mathcal{B}_1 \triangleleft \mathcal{B}_2$.

- Suppose $\mathcal{A}_1 \equiv_m \mathcal{B}_1$ and $\mathcal{A}_2 \equiv \mathcal{B}_2$.

  By Ehrenfeucht's Theorem there are winning strategies S_1 and S_2 for the duplicator in $G_m(\mathcal{A}_1, \mathcal{B}_1)$ and $G_m(\mathcal{A}_2, \mathcal{B}_2)$, respectively.

  (a) The following gives a winning strategy for the duplicator in the game $G_m(\mathcal{A}_1 \times \mathcal{A}_2, \mathcal{B}_1 \times \mathcal{B}_2)$. We simultaneously play games in $G_m(\mathcal{A}_1, \mathcal{B}_1)$ and $G_m(\mathcal{A}_2, \mathcal{B}_2)$. Suppose that in his i-th move the spoiler chooses, say, $(a_1, a_2) \in A_1 \times A_2$. Let $b_1 \in B_1$ and $b_2 \in B_2$ be answers to a_1 and a_2 according to S_1 and S_2, respectively. Then the duplicator chooses (b_1, b_2).

Equivalence Invariance of Operations (Cont'd)

(b),(c) Let $* \in \{\uplus, \lhd\}$. The following represents a winning strategy for the duplicator in $G_m(\mathcal{A}_1 * \mathcal{A}_2, \mathcal{B}_1 * \mathcal{B}_2)$ (when describing it we use moves of plays in $G_m(\mathcal{A}_1, \mathcal{B}_1)$ and $G_m(\mathcal{A}_2, \mathcal{B}_2)$).

- Suppose that in his i-th move the spoiler selects, say, $a \in A_1 * A_2$.
- Then the duplicator responds by applying S_1 if $a \in A_1$, and S_2 if $a \in A_2$.

- The proof above yields more:

Corollary

(a) If $(\mathcal{A}_1, \overline{a}_1) \equiv_m (\mathcal{B}_1, \overline{b}_1)$ and $(\mathcal{A}_2, \overline{a}_2) \equiv_m (\mathcal{B}_2, \overline{b}_2)$, then $(\mathcal{A}_1 \uplus \mathcal{A}_2, \overline{a}_1, \overline{a}_2) \equiv_m (\mathcal{B}_1 \uplus \mathcal{B}_2, \overline{b}_1, \overline{b}_2)$.

(b) If $(\mathcal{A}_1, \overline{a}_1) \equiv_m (\mathcal{B}_1, \overline{b}_1)$ and $(\mathcal{A}_2, \overline{a}_2) \equiv_m (\mathcal{B}_2, \overline{b}_2)$, then $(\mathcal{A}_1 \lhd \mathcal{A}_2, \overline{a}_1, \overline{a}_2) \equiv_m (\mathcal{B}_1 \lhd \mathcal{B}_2, \overline{b}_1, \overline{b}_2)$.

Remark

- Let $\mathcal{Z} := (\mathbb{Z}, <)$ and $\mathcal{Q} := (\mathbb{Q}, <)$ be the integers and the rationals with their orderings. Let $\varphi := \exists x \exists y (x < y \wedge \forall z \neg (x < z \wedge z < y))$.

  We have $\mathcal{Z} \vDash \varphi$ and $\mathcal{Q} \nvDash \varphi$. Hence, $\mathcal{Z} \not\equiv_3 \mathcal{Q}$. Therefore, $\mathcal{Z} \not\cong_3 \mathcal{Q}$.

  The spoiler can "transform" this information into a winning strategy for the game $G_3(\mathcal{Z}, \mathcal{Q})$. This is given by the table on the right, where the selections of the spoiler are in red.

$\mathcal{Z}$	$\mathcal{Q}$
5	a
6	b
?	$\frac{a+b}{2}$

  Note that no third move of the duplicator will lead to a partial isomorphism (since for $a < b$ we have $a < \frac{a+b}{2} < b$ and there is no integer between 5 and 6). In this strategy of the spoiler:

  - His selections in $\mathcal{Z}$ correspond to the existential quantifiers in φ;
  - His selections in $\mathcal{Q}$ correspond to the universal quantifiers in φ.

  For this reason moves in $G_m(\mathcal{A}, \mathcal{B})$ in which the spoiler chooses an element of $\mathcal{A}$ (of $\mathcal{B}$) are sometimes called $\exists$-**moves** ($\forall$-**moves**).

On the Importance of the Isomorphism Types

- The formulas $\varphi^m_{\mathcal{A},\overline{a}}$, the m-isomorphism type of $\overline{a}$ in $\mathcal{A}$, will also play a crucial role in subsequent considerations.
- Their methodological importance stems from the following two facts:
  (1) They have a clear algebraic meaning.
  (2) Every first-order formula is equivalent to a disjunction of such formulas.
- Classical model theory has been characterized by the equation

$$\text{model theory} = \text{universal algebra} + \text{logic}.$$

- By (1) and (2) above, it is clear that the formulas $\varphi^j_{\mathcal{A},\overline{a}}$ provide a bridge between structures and first-order formulas, i.e., between the main notions from (universal) algebra and from (first-order) logic, respectively.
- Their value as a tool in model theory is therefore not surprising.

Alternative Way to View Isomorphism Types

- There is a more algebraic, sort of logic-free, way to define m-isomorphism types.
- For $\overline{a} = a_1 \dots a_s$ in $\mathcal{A}$, set:

$$\begin{array}{rcl} \mathrm{IT}^0(\mathcal{A}, \overline{a}) & := & \{\varphi : \mathcal{A} \vDash \varphi[\overline{a}], \varphi(v_1, \dots, v_s) \text{ atomic}\}; \\ \mathrm{IT}^{m+1}(\mathcal{A}, \overline{a}) & := & \{\mathrm{IT}^m(\mathcal{A}, \overline{a}a) : a \in A\}. \end{array}$$

- It can be verified that, for any $\mathcal{B}$ and $\overline{b} \in \mathcal{B}$,

$$\mathrm{IT}^m(\mathcal{A}, \overline{a}) = \mathrm{IT}^m(\mathcal{B}, \overline{b}) \quad \text{iff} \quad \varphi^m_{\mathcal{A}, \overline{a}} = \varphi^m_{\mathcal{B}, \overline{b}}.$$

Subsection 4

Hanf's Theorem

The Gaifman Graph of a Structure

- All vocabularies in this section will be relational unless stated otherwise.
- Let M be a nonempty subset of a structure $\mathcal{A}$.
  We denote by $\mathcal{M}$ the substructure of $\mathcal{A}$ with universe M.
- Given a structure $\mathcal{A}$, we define the binary relation E^A on A by

$$E^A := \{(a,b) : a \neq b \text{ and there are } R \text{ in } \tau \text{ and } \overline{c} \in A \text{ such that } R^A\overline{c} \text{ and } a \text{ and } b \text{ are components of the tuple } \overline{c}\}.$$

- The structure $\mathcal{G}(\mathcal{A}) := (A, E^A)$ is called the **Gaifman graph of** $\mathcal{A}$.
- Obviously, if $\mathcal{A}$ itself is a graph then $\mathcal{G}(\mathcal{A}) = \mathcal{A}$.

Balls and Ball Types

- For a in A and $r \in \mathbb{N}$, we denote by $S(r,a)$ (or $S^{\mathcal{A}}(r,a)$) the r-**ball** of a,
$$S(r,a) := \{b \in A : d(a,b) \le r\}.$$
- $\mathcal{S}(r,a)$ (or $\mathcal{S}^{\mathcal{A}}(r,a)$) stands for the substructure of $\mathcal{A}$ with universe $S(r,a)$.
- Note that for $b,c \in S(r,a)$, we have $d(b,c) \le 2r$.
- For $\overline{a} = a_1 \ldots a_s$, we set
$$S(r,\overline{a}) := S(r,a_1) \cup \cdots \cup S(r,a_s).$$
- We define the r-**ball type** of a point a in $\mathcal{A}$ to be the isomorphism type of $(\mathcal{S}(r,a),a)$.
- I.e., points a in $\mathcal{A}$ and b in $\mathcal{B}$ have the same r-ball type iff $(\mathcal{S}^{\mathcal{A}}(r,a),a) \cong (\mathcal{S}^{\mathcal{B}}(r,b),b)$.

Introducing Hanf's and Gaifman's Theorems

- We showed that certain graphs are m-isomorphic and hence, m-equivalent, using a sequence $(I_j)_{j \le m}$, where - in the terminology just introduced - for each $p \in I_j$ and $a \in \mathrm{dom}(p)$, there was an isomorphism of $\mathcal{S}(2^j - 1, a)$ onto $\mathcal{S}(2^j - 1, p(a))$ compatible with p.
- We use generalizations of this idea to show two further theorems on the expressive power of first-order logic.
  - The first one (Hanf's Theorem) is obtained by applying the idea just mentioned to graphs of structures.
  - In the second one (Gaifman's Theorem) the requirement of isomorphism of corresponding balls is weakened to ℓ-equivalence for a suitable ℓ.

Hanf's Theorem

Theorem (Hanf's Theorem)

Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures and let $m \in \mathbb{N}$. Suppose that for some $e \in \mathbb{N}$:

- The 3^m-balls in $\mathcal{A}$ and $\mathcal{B}$ have less than e elements;
- For each 3^m-ball type ι, one of the following holds:
  (i) $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements of 3^m-ball type ι;
  (ii) Both $\mathcal{A}$ and $\mathcal{B}$ have more than $m \cdot e$ elements of 3^m-ball type ι.

Then $\mathcal{A} \equiv_m \mathcal{B}$.

- For $n < \ell$ the ℓ-ball type of an element determines its n-ball type.
  So for $n < 3^m$ and every n-ball type ι, one of following holds:
  - $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements of n-ball type ι;
  - Both $\mathcal{A}$ and $\mathcal{B}$ have more than $m \cdot e$ elements of n-ball type ι.

  We show that $(I_j)_{j \le m} : \mathcal{A} \cong_m \mathcal{B}$, where I_j is the set

$$\{\overline{a} \mapsto \overline{b} \in \text{Part}(\mathcal{A},\mathcal{B}) : (\mathcal{S}(3^j,\overline{a}),\overline{a}) \cong (\mathcal{S}(3^j,\overline{b}),\overline{b}) \text{ and } \text{length}(\overline{a}) \le m-j\}.$$

Hanf's Theorem (Cont'd)

- For length$(\overline{a}) = 0$, we set $(\mathcal{S}(3^j, \overline{a}), \overline{a}) = \varnothing$ and agree that $\varnothing \cong \varnothing$.

  Therefore, we have $\varnothing \mapsto \varnothing \in I_m$.

  Concerning the back and forth properties it is enough, by symmetry, to prove the forth property. Thus, suppose that $0 \le j \le m$, $a \in A$ and $\overline{a} \mapsto \overline{b} \in I_{j+1}$, say, $\pi : (\mathcal{S}(3^{j+1}, \overline{a}), \overline{a}) \cong (\mathcal{S}(3^{j+1}, \overline{b}), \overline{b})$.

  Case 1: $a \in S(2 \cdot 3^{j+1}, \overline{a})$. Then $S(3^j, \overline{a}a) \subseteq S(3^{j+1}, \overline{a})$. Setting $b := \pi(a)$, we have $\pi : (\mathcal{S}(3^j, \overline{a}a), \overline{a}a) \cong (\mathcal{S}(3^j, \overline{b}b), \overline{b}b)$. Hence $\overline{a}a \mapsto \overline{b}b \in I_j$.

  Case 2: $a \notin S(2 \cdot 3^j, \overline{a})$ (and, hence, $S(3^j, a) \cap S(3^j, \overline{a}) = \varnothing$). Let ι be the 3^j-ball type of a. By hypothesis, $\mathcal{S}(2 \cdot 3^j, \overline{a})$ and $\mathcal{S}(2 \cdot 3^j, \overline{b})$ contain the same number of elements of 3^j-ball type ι. By our assumption on the cardinality of balls, this is $\le$ length$(\overline{a}) \cdot e \le m \cdot e$. Therefore, by (i) or (ii), there must be an element $b \notin S(2 \cdot 3^j, \overline{b})$, with 3^j-ball type ι. Choose $\pi' : (\mathcal{S}(3^j, a), a) = (\mathcal{S}(3^j, b), b)$. Then the corresponding restriction of $\pi \cup \pi'$ is an isomorphism of $(\mathcal{S}(3^j, \overline{a}a), \overline{a}a)$ onto $(\mathcal{S}(3^j, \overline{b}b), \overline{b}b)$.

An Application: Graph Connectedness

- Note that a graph $\mathcal{G}$ is connected if each nonempty subset of G closed under the graph relation E^G contains all elements of G.

  Equivalently, if $\mathcal{G}$ is a model of the "second-order sentence"

  $$\forall P((\exists x Px \wedge \forall x \forall y((Px \wedge Exy) \rightarrow Py)) \rightarrow \forall z Pz).$$

  Claim: The class of connected graphs is not axiomatizable by a second-order sentence of the form $\exists P_1 \cdots \exists P_r \psi$, where $P_1, \ldots, P_r$ are unary and ψ is first-order.

  For $\ell \geq 1$, let $\mathcal{D}_\ell = (D_\ell, E_\ell)$ be a digraph consisting of a cycle of length $\ell + 1$. E.g.,

  $$D_\ell := \{0, \ldots, \ell\}, \quad E_\ell := \{(i, i+1) : i < \ell\} \cup \{(\ell, 0)\}.$$

Lemma 1

Lemma 1

Suppose $\tau = \{E, P_1, \ldots, P_r\}$, where $P_1, \ldots, P_r$ are unary, and let $m \geq 0$. Then there is an $\ell_0 \geq 1$, such that for any $\ell \geq \ell_0$ and any τ-structure of the form $(\mathcal{D}_\ell, P_1, \ldots, P_r)$, there are $a, b \in D_\ell$ with disjoint and isomorphic 3^m-balls.

- For the structures under consideration any 3^m-ball contains exactly $2 \cdot 3^m + 1$ elements (note that $P_1, \ldots, P_r$ are unary and therefore do not influence the distances induced by the underlying digraphs).

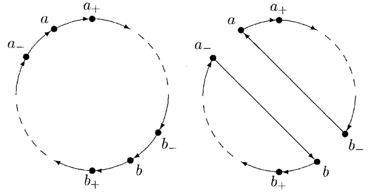
  Let i be the number of possible isomorphism types of 3^m-balls.

  Then in a structure of cardinality $\geq \ell_0 := (i+1)(2 \cdot 3^m + 1)$ there must be two points with disjoint 3^m-balls of the same isomorphism type.

Lemma 2

Lemma 2

Suppose $(\mathcal{D}_\ell, P_1, \ldots, P_r)$ is a τ-structure (τ as in Lemma 1) containing elements a and b with disjoint and isomorphic 3^m-balls. Denote by a_- and b_- the elements of D_ℓ with $E_\ell a_- a$ and $E_\ell b_- b$, respectively.



Let $(D_\ell, E'_\ell, P_1, \ldots, P_r)$ be the structure obtained by splitting the cycle $(\mathcal{D}_\ell, P_1, \ldots, P_r)$ into two cycles by removing the edges (a_-, a), (b_-, b) and adding edges (b_-, a), (a_-, b) instead; more formally:

$$E'_\ell := (E_\ell \backslash \{(a_-, a), (b_-, b)\}) \cup \{(b_-, a), (a_-, b)\}.$$

Then $(\mathcal{D}_\ell, P_1, \ldots, P_r) \cong_m (D_\ell, E'_\ell, P_1, \ldots, P_r)$.

- Immediate by Hanf's Theorem, since both structures have the same number of 3^m-balls of any given isomorphism type.

Lemma 3

Lemma 3

For $\tau = \{E, P_1, \ldots, P_r\}$ and $m \geq 0$, choose ℓ_0 according to Lemma 1. Let $\ell \geq \ell_0$ and $(\mathcal{G}_\ell, P_1, \ldots, P_r)$ be a τ-structure, where $\mathcal{G}_\ell$ is the Gaifman graph $\mathcal{G}(\mathcal{D}_\ell)$ of $\mathcal{D}_\ell$, that is, $\mathcal{D}_\ell$ is a cycle of length $\ell + 1$. Let $\mathcal{G}'_\ell$ be the Gaifman graph $\mathcal{G}((D_\ell, E'_\ell))$, where (D_ℓ, E'_ℓ) is defined as in Lemma 2. Then

$$(\mathcal{G}_\ell, P_1, \ldots, P_r) \equiv_m (\mathcal{G}'_\ell, P_1, \ldots, P_r).$$

- Note that a partial isomorphism between digraphs is a partial isomorphism of the associated graphs.

  So the result follows from the two preceding lemmas.

Finite Connected Graphs: A Negative Result

Proposition

The class of finite and connected graphs cannot be axiomatized by a formula of the form

$$\exists P_1 \cdots \exists P_r \psi,$$

where $P_1, \ldots, P_r$ are unary relation symbols and ψ is a first-order sentence over the vocabulary $\{E, P_1, \ldots, P_r\}$.

- Suppose for the sentence $\exists P_1 \cdots \exists P_r \psi$ and any finite graph $\mathcal{G}$,

  $$\mathcal{G} \text{ is connected iff for some } P_1, \ldots, P_r \subseteq G, (\mathcal{G}, P_1, \ldots, P_r) \vDash \psi.$$

  For $m := \mathrm{qr}(\psi)$, choose ℓ_0 as in Lemma 1.
  As $\mathcal{G}_{\ell_0}$ is connected, there are $P_1, \ldots, P_r$, with $(\mathcal{G}_{\ell_0}, P_1, \ldots, P_r) \vDash \psi$.
  Then, by Lemma 3, $(\mathcal{G}'_{\ell_0}, P_1, \ldots, P_r) \vDash \psi$.
  However, $\mathcal{G}'_{\ell_0}$ is not connected, a contradiction.

Finite Connected Graphs: A Positive Result

Proposition

The class of finite and connected graphs can he axiomatized by a formula of the form $\exists R\psi$, where R is binary and ψ is a first-order sentence over the vocabulary $\{E, R\}$.

- Let ψ be a sentence expressing that:
  - R is an irreflexive and transitive relation;
  - R has a minimal element;
  - Exy holds for any immediate R-successor y of x.

  I.e., ψ is the conjunction of

$$\forall x \neg Rxx \wedge \forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz);$$
$$\exists x \forall y(x = y \vee Rxy);$$
$$\forall x \forall y((Rxy \wedge \forall z \neg(Rxz \wedge Rzy)) \to Exy).$$

Finite Connected Graphs: A Positive Result (Cont'd)

- Let $\mathcal{G}$ be a graph. Suppose $\mathcal{G}$ is a model of $\exists R\psi$, say $(\mathcal{G}, R^A) \models \psi$. Then, for any element of $\mathcal{G}$, there is a path connecting it with the minimal element. Hence, $\mathcal{G}$ is connected.

  Conversely, suppose $\mathcal{G}$ is connected.

  Choose an arbitrary $a \in G$.

  For $n \in \mathbb{N}$, set

  $$L_n := \{b : d(a, b) = n\}.$$

  Take as R the transitive closure of

  $$\{(b, c) : E^G bc \text{ and, for some } n, b \in L_n \text{ and } c \in L_{n+1}\}.$$

  Then $(\mathcal{G}, R) \models \psi$.

Subsection 5

Gaifman's Theorem

Introducing Gaifman's Theorem

- Let τ be a relational signature.
- Let $\mathcal{A}$ be a τ-structure.
- A subset M of A is ℓ-**scattered**, if the distance (in the Gaifman graph $\mathcal{G}(\mathcal{A})$) between any two elements of M exceeds ℓ.
- Given $r, n \geq 1$ and a τ-formula $\psi(x)$, it is easy to write down a first-order sentence asserting that there is a $2r$-scattered subset M of cardinality at least n, such that $\mathcal{S}(r, a) \vDash \psi[a]$, for all $a \in M$.
- Note that due to $2r$-scatteredness, the balls $S(r, a)$, for $a \in M$ are pairwise disjoint.
- Gaifman's Theorem states that every first-order sentence is logically equivalent to a boolean combination of such sentences.
- This further formalizes the fact, already present in Hanf's Theorem, that first-order sentences only capture local properties of structures.

Distance Formulas

- Note that there is an τ-formula $\theta_n(x,y)$, such that for any τ-structure $\mathcal{A}$ and $a,b \in A$,

$$\mathcal{A} \vDash \theta_n(x,y)[a,b] \quad \text{iff} \quad d(a,b) \le n.$$

In fact, set $\theta_0(x,y) := x = y$.

Denoting by $\text{ar}(R)$ the arity of R, set

$$\begin{array}{rcl}\theta_{n+1}(x,y) & := & \theta_n(x,y) \vee \exists z(\theta_n(x,z) \wedge \bigvee_{R\in\tau} \exists u_1 \cdots \exists u_{\text{ar}(R)} \\ & & (Ru_1 \dots u_{\text{ar}(R)} \wedge \bigvee_{1\le i,j\le \text{ar}(R)}(u_i = z \wedge u_j = y))).\end{array}$$

- In formulas, we introduce the shorthand $d(x,y) \le n$ for $\theta_n(x,y)$.
- For $\overline{x} = x_1 \dots x_m$, let

$$d(\overline{x},y) \le n := (d(x_1,y) \le n \vee \cdots \vee d(x_m,y) \le n).$$

Relativization of Quantifiers

- Let $k \in \mathbb{N}$. With every τ-formula $\varphi = \varphi(\overline{x}, \overline{y})$, we associate a formula $\varphi^{S(k,\overline{x})}(\overline{x}, \overline{y})$, such that for any τ-structure $\mathcal{A}$, $\overline{a} \in A$, and $\overline{b} \in S(k, \overline{a})$,

$$\mathcal{A} \vDash \varphi^{S(k,\overline{x})}[\overline{a}, \overline{b}] \quad \text{iff} \quad S(k, \overline{a}) \vDash \varphi[\overline{a}, \overline{b}].$$

- To define $\varphi^{S(k,\overline{x})}(\overline{x}, \overline{y})$:
  - First, replace any bound occurrence in φ of a variable in $\overline{x}$ by a new variable;
  - Then, inductively relativize the quantifiers to $S(k, \overline{x})$.
    E.g.,

$$[\exists z \varphi]^{S(k,\overline{x})} := \exists z (d(\overline{x}, z) \leq k \wedge \varphi^{S(k,\overline{x})}).$$

Local Sentences

- Call a sentence **basic local** if it has the form

$$\exists x_1 \cdots \exists x_n \bigwedge_{1 \le i < j \le n} (d(x_i, x_j) > 2r \wedge \psi^{S(r,x_i)}(x_i)),$$

where $\psi = \psi(x)$ is a first-order formula.

- Note that for $\ell < k$,

$$\vDash \varphi^{S(\ell,\overline{x})} \leftrightarrow [\varphi^{S(k,\overline{x})}]^{S(\ell,\overline{x})} \quad \text{and} \quad \vDash \varphi^{S(\ell,\overline{x})} \leftrightarrow [\varphi^{S(\ell,\overline{x})}]^{S(k,\overline{x})}.$$

- In particular, any sentence of the form

$$\exists x_1 \cdots \exists x_n \bigwedge_{1 \le i < j \le n} (d(x_i, x_j) > 2r \wedge \psi^{S(\ell,x_i)}(x_i)),$$

with $\ell \le r$, is logically equivalent to a basic local sentence.

- A **local sentence** is a boolean combination of basic local sentences.

Gaifman's Theorem

Theorem (Gaifman's Theorem)

Every first-order sentence is logically equivalent to a local sentence.

- By a result we saw in the first set, it suffices to show

Lemma

Suppose $\mathcal{A}$ and $\mathcal{B}$ satisfy the same basic local sentences. Then $\mathcal{A} \equiv \mathcal{B}$.

- We show that $\mathcal{A} \cong_m \mathcal{B}$, for $m \in \mathbb{N}$.
  The argument parallels that for Hanf's Theorem.
  There, the sets I_j consisted of partial isomorphisms $\overline{a} \mapsto \overline{b}$, such that $\text{length}(\overline{a}) \leq m - j$ and $(\mathcal{S}(3^j, \overline{a}), \overline{a}) \cong (\mathcal{S}(3^j, \overline{b}), \overline{b})$.
  Here, we replace $\cong$ by $\equiv_{g(j)}$ and take balls of radius 7^j.
  The values $g(0), g(1), \ldots$ of g can be defined by induction.
  $g(j)$ only has to be greater than some values which one gets in the course of the proof.

Gaifman's Theorem (Cont'd)

- Let I_j comprise all the partial isomorphisms $\overline{a} \mapsto \overline{b}$ from $\mathcal{A}$ to $\mathcal{B}$, such that $\text{length}(\overline{a}) \leq m-j$ and $(\mathcal{S}(7^j,\overline{a}),\overline{a}) \equiv_{g(j)} (\mathcal{S}(7^j,\overline{b}),\overline{b})$.
  Again, in case $\text{length}(\overline{a}) = 0$, we set $(\mathcal{S}(7^j,\overline{a}),\overline{a}) = \varnothing$ and agree that $\varnothing \equiv_k \varnothing$, for all k. In particular, $\varnothing \mapsto \varnothing \in I_m$.
  We show $(I_j)_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$.
  By symmetry, we can restrict ourselves to the forth property.
  Suppose $0 \leq j < m$, $a \in A$, $\overline{a} \mapsto \overline{b} \in I_{j+1}$. Hence
  $$(\mathcal{S}(7^{j+1},\overline{a}),\overline{a}) \cong_{g(j+1)} (\mathcal{S}(7^{j+1},\overline{b}),\overline{b}).$$
  For $\overline{d}$ in a structure $\mathcal{D}$, let
  $$\psi^j_{\overline{d}}(\overline{x}) := \left[\varphi^{g(j)}_{\mathcal{S}(7^j,\overline{d}),\overline{d}}(\overline{x})\right]^{S(7^j,\overline{x})}.$$
  Recall that $\varphi^{\ell}_{\mathcal{D},\overline{d}}$ denotes the ℓ-isomorphism type of $\overline{d}$ in $\mathcal{D}$.
  So $\psi^j_{\overline{d}}(\overline{x})$ expresses that $(\mathcal{S}(7^j,\overline{d}),\overline{d}) \equiv_{g(j)} (\mathcal{S}(7^j,\overline{x}),\overline{x})$.

Gaifman's Theorem (Case 1)

- **Case 1**: Suppose $a \in \mathcal{S}(2 \cdot 7^j, \overline{a})$. Then

$$\mathcal{S}(7^{j+1}, \overline{a}) \vDash \exists z(d(\overline{a}, z) \le 2 \cdot 7^j \wedge \psi^j_{\overline{a}a}(\overline{a}z)).$$

We assume that the quantifier rank of this formula is $\le g(j+1)$.
(This gives us a first condition on the value of $g(j+1)$.)
Hence, by the hypothesis,

$$\mathcal{S}(7^{j+1}, \overline{b}) \vDash \exists z(d(\overline{b}, z) \le 2 \cdot 7^j \wedge \psi^j_{\overline{a}a}(\overline{b}z)).$$

So, for some b, we have

$$(\mathcal{S}(7^j, \overline{a}a), \overline{a}a) \equiv_{g(j)} (\mathcal{S}(7^j, \overline{b}b), \overline{b}b).$$

Therefore, $\overline{a}a \mapsto \overline{b}b \in I_j$.

Gaifman's Theorem (Case 2)

- **Case 2**: Suppose $a \notin S(2 \cdot 7^j, \overline{a})$, i.e., $S(7^j, \overline{a}) \cap S(7^j, a) = \varnothing$.

  For $s \geq 1$, the following formula $\delta_s(x_1, \ldots, x_s)$ expresses that $\{x_1, \ldots, x_s\}$ is a $4 \cdot 7^j$-scattered set of elements whose 7^j-ball has the same $g(j)$-isomorphism type as that of a:

  $$\delta_s := \bigwedge_{1 \leq \ell < k \leq s} d(x_\ell, x_k) > 4 \cdot 7^j \wedge \bigwedge_{1 \leq \ell \leq s} \psi_a^j(x_\ell).$$

  Compare the cardinalities (e and i below) of maximal $4 \cdot 7^j$-scattered sets in $\mathcal{S}(2 \cdot 7^j, \overline{a})$ and in $\mathcal{A}$, respectively, consisting of such elements. More precisely, let e and i be such that:

  1. $\mathcal{S}(7^{j+1}, \overline{a}) \vDash \exists x_1 \cdots \exists x_e (\bigwedge_{1 \leq k \leq e} d(\overline{a}, x_k) \leq 2 \cdot 7^j \wedge \delta_e)$;
  2. $\mathcal{S}(7^{j+1}, \overline{a}) \nvDash \exists x_1 \cdots \exists x_{e+1} (\bigwedge_{1 \leq k \leq e+1} d(\overline{a}, x_k) \leq 2 \cdot 7^j \wedge \delta_{e+1})$;
  3. $\mathcal{A} \vDash \exists x_1 \cdots \exists x_i \ \delta_i$;
  4. $\mathcal{A} \nvDash \exists x_1 \cdots \exists x_{i+1} \ \delta_{i+1}$.

  If no such i exists, set $i = \infty$.

Gaifman's Theorem (Case 2 Cont'd)

- Note that e is bounded by the length of $\overline{a}$ (and hence by m), since any two elements of the same ball of radius $2 \cdot 7^j$ have a distance at most $4 \cdot 7^j$.

  Clearly, $e \leq i$.

  Claim: The corresponding numbers e and i determined in $\mathcal{S}(7^{j+1}, \overline{b})$ and $\mathcal{B}$, respectively, are the same.

  Concerning $\mathcal{B}$, this holds since the sentences in 3 and 4 are basic local up to logical equivalence.

  Concerning $\mathcal{S}(7^{j+1}, \overline{b})$, note that:
  - $(\mathcal{S}(7^{j+1}, \overline{a}), \overline{a}) \equiv_{g(j+1)} (\mathcal{S}(7^{j+1}, \overline{b}), \overline{b})$;
  - $g(j+1)$ is greater than the quantifier rank of the sentences in 1 and 2. (This gives a second condition on the value of $g(j+1)$; recall that e is bounded by m.)

Gaifman's Theorem (Case 2.1)

- **Case 2.1**: $e = i$.

  Then all elements satisfying $\psi^j_a(x)$ have distance from a

  $$\leq 4 \cdot 7^j + 2 \cdot 7^j = 6 \cdot 7^j < 7^{j+1}.$$

  Suppose to the contrary that one such a' satisfies $d(\overline{a}, a') > 6 \cdot 7^j$.
  Then a' together with e many witnesses for 1 show that $i \geq e + 1$.
  In particular, this holds for a.
  Since $a \notin S(2 \cdot 7^{j+1}, \overline{a})$,

  $$\mathcal{S}(t^{j+1}, \overline{a}) \vDash \exists z(2 \cdot 7^j < d(\overline{a}, z) \leq 6 \cdot 7^j \wedge \psi^j_a(z) \wedge \psi^j_{\overline{a}}(\overline{a})).$$

  Then, by the hypothesis,

  $$\mathcal{S}(7^{j+1}, \overline{b}) \vDash \exists z(2 \cdot 7^j < d(\overline{b}, z) < 6 \cdot 7^j \wedge \psi^j_a(z) \wedge \psi^j_{\overline{a}}(\overline{b})).$$

  (This gives us a third condition on the value of $g(j+1)$.)

Gaifman's Theorem (Case 2.1 Cont'd)

- Thus, there is b, with $2 \cdot 7^j < d(\overline{b}, b) < 6 \cdot 7^j$, such that

$$(\mathcal{S}(7^j, a), a) \equiv_{g(j)} (\mathcal{S}(7^j, b), b).$$

Moreover,

$$(\mathcal{S}(7^j, \overline{a}), \overline{a}) \equiv_{g(j)} (\mathcal{S}(7^j, \overline{b}), \overline{b}).$$

Note that the universes of the structures on the left sides in the displayed equivalences are disjoint, and the same applies to the right sides.

So, by the fact that disjoint unions preserve $\equiv_m$, we obtain

$$(\mathcal{S}(7^j, \overline{a}a), \overline{a}a) \equiv_{g(j)} (\mathcal{S}(7^j, \overline{b}b), \overline{b}b).$$

Thus, $\overline{a}a \mapsto \overline{b}b \in I_j$.

Gaifman's Theorem (Case 2.2)

- **Case 2.2**: $e < i$.

  Then
  $$\mathcal{B} \vDash \exists x_1 \cdots \exists x_{e+1}\ \delta_{e+1}.$$
  Hence there must be an element b in B, such that
  $$S(7^j, \overline{b}) \cap S(7^j, b) = \varnothing \quad \text{and} \quad \mathcal{B} \vDash \psi_a^j(x)[b].$$
  In particular,
  $$(\mathcal{S}(7^j, a), a) \equiv_{g(j)} (\mathcal{S}(7^j, b), b).$$
  Now one can argue as at the end of the preceding case.

Homomorphisms

- In the rest of this subsection all structures are assumed to be finite.
- Recall that τ is a relational vocabulary.
- Given τ-structures $\mathcal{A}$ and $\mathcal{B}$, a mapping $h : A \to B$ is a **homomorphism** if, for all $R \in \tau$ and $\overline{a} \in A$,

$$R^{\mathcal{A}}\overline{a} \quad \text{implies} \quad R^{\mathcal{B}}h(\overline{a}).$$

- The homomorphism is said to be **strict** if, in addition, for all $R \in \tau$ and $\overline{a} \in A$, with $R^{\mathcal{B}}h(\overline{a})$, there is $\overline{e} \in A$, such that $R^{\mathcal{A}}\overline{e}$ and $h(\overline{e}) = h(\overline{a})$.
- A sentence φ is **preserved under (strict) homomorphisms** if for all $\mathcal{A}, \mathcal{B}$ and any (strict) homomorphism $h : A \to B$,

$$\mathcal{A} \vDash \varphi \quad \text{implies} \quad \mathcal{B} \vDash \varphi.$$

Preservation of Existential Positive Sentences

Proposition

Every existential positive sentence, that is, every sentence built up from atomic formulas with the connectives $\wedge$ and $\vee$, and the quantifier $\exists$, is preserved under homomorphisms.

- We show by structural induction on a positive formula $\varphi(\overline{x})$ that

$$\mathcal{A} \vDash \varphi(\overline{x})[\overline{a}] \quad \text{implies} \quad \mathcal{B} \vDash \varphi(\overline{x})[h(\overline{a})].$$

  - $\mathcal{A} \vDash x_i = x_j[\overline{a}]$ iff $a_i = a_j$ implies $h(a_i) = h(a_j)$ iff $\mathcal{B} \vDash x_i = x_j[h(\overline{a})]$;
  - $\mathcal{A} \vDash R\overline{x}[\overline{a}]$ iff $R^{\mathcal{A}}\overline{a}$ implies $R^{\mathcal{B}}[h(\overline{a})]$ iff $\mathcal{B} \vDash R\overline{x}[h(\overline{a})]$;
  - $\mathcal{A} \vDash (\phi \wedge \psi)(\overline{x})[\overline{a}]$ iff $\mathcal{A} \vDash \phi(\overline{x})[\overline{a}]$ and $\mathcal{A} \vDash \psi(\overline{x})[\overline{a}]$ imply $\mathcal{B} \vDash \phi(\overline{x})[h(\overline{a})]$ and $\mathcal{B} \vDash \psi(\overline{x})[h(\overline{a})]$ iff $\mathcal{B} \vDash (\phi \wedge \psi)(\overline{x})[h(\overline{a})]$;
  - Similarly for $\vee$;
  - $\mathcal{A} \vDash \exists x \varphi(x)[\overline{a}]$ iff $\mathcal{A} \vDash \varphi(x)\left[\overline{a}\frac{a}{x}\right]$, for some $a \in A$, implies $\mathcal{B} \vDash \varphi(x)\left[h(\overline{a})\frac{h(a)}{x}\right]$, for some $a \in A$, implies $\mathcal{B} \vDash \exists x \varphi(x)[h(\overline{a})]$.

Minimal Models

- A model $\mathcal{A}$ of a sentence φ is said to be **minimal** if no proper substructure is a model of φ, i.e., if $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B} \vDash \varphi$ imply $\mathcal{B} = \mathcal{A}$.

Proposition

For any φ, every model of φ contains a minimal model, that is, if $\mathcal{B} \vDash \varphi$, then there is $\mathcal{A} \subseteq \mathcal{B}$, such that $\mathcal{A}$ is a minimal model of φ (recall that we restrict ourselves to finite structures).

- Let $\mathcal{B}$ be a model of φ.

  If it is minimal, we are done.

  Otherwise, there exists $\mathcal{B}_1 \subset \mathcal{B}$, such that $\mathcal{B}_1 \vDash \varphi$.

  We continue in the same way.

  Since $\mathcal{B}$ is finite, the process must stop with a substructure $\mathcal{A}$ of $\mathcal{B}$, which is a minimal model of φ.

Application of Gaifman's Theorem

Theorem

If φ is preserved under strict homomorphisms, there are ℓ and m, such that no minimal model of φ contains an ℓ-scattered subset of cardinality m.

- By Gaifman's Theorem, φ is logically equivalent to a boolean combination of basic local sentences $\varphi_1, \ldots, \varphi_k$. Suppose

$$\varphi_i = \exists x_1 \cdots \exists x_{n_i} \bigwedge_{1 \leq s < t \leq n_i} (d(x_s, x_t) > 2r_i \wedge \psi_i^{S(r_i, x_s)}(x_s)).$$

Set $r := \max\{r_i : 1 \leq i \leq k\}$, $\ell := 2r$ and $m := 2^k + 1$.

Let $\mathcal{A}$ be a minimal model of φ.

Application of Gaifman's Theorem (Cont'd)

Claim: $\mathcal{A}$ contains no ℓ-scattered subset of cardinality m.

By contradiction, suppose that M is ℓ-scattered and $\|M\| \geq m$.

For $i = 1, \ldots, k$, let $\rho_i(u)$ express

$$\text{“there is } v \text{ such that } d(u,v) \leq r_i \text{ and } \psi_i^{S(r_i,v)}(v)\text{”}.$$

By choice of m, there are $a, a' \in M$, such that $a \neq a'$ and for $i = 1, \ldots, k$,

$$\mathcal{A} \vDash \rho_i[a] \quad \text{iff} \quad \mathcal{A} \vDash \rho_i[a'].$$

Let $\mathcal{B}$ be the substructure of $\mathcal{A}$ with universe $A \backslash \{a\}$.

Then $\mathcal{B} \nvDash \varphi$, since $\mathcal{A}$ is a minimal model of φ.

Application of Gaifman's Theorem (Cont'd)

- Set $n := \max\{n_i : i = 1, \ldots, k\}$, $\mathcal{B}_n := \dot\bigcup_{j=1}^{n} \mathcal{B}$ (the disjoint union of n copies of $\mathcal{B}$), and $\mathcal{A}_n := \mathcal{A} \mathbin{\dot\cup} \mathcal{B}_n$.
  The projection of $\mathcal{B}_n$ to $\mathcal{B}$ is a strict homomorphism.
  Therefore, since $\mathcal{B} \nvDash \varphi$, $\mathcal{B}_n \nvDash \varphi$ either.
  The inclusion map of $\mathcal{A}$ to $\mathcal{A}_n$ is also a strict homomorphism.
  Thus, since $\mathcal{A} \vDash \varphi$, we have $\mathcal{A}_n \vDash \varphi$.
  We obtain the desired contradiction, if we show that for $i = 1, \ldots, k$,

$$\mathcal{A}_n \vDash \varphi_i \quad \text{iff} \quad \mathcal{B}_n \vDash \varphi_i.$$

($\Leftarrow$) Fix i. Suppose first that $\mathcal{B}_n \vDash \varphi_i$.
Then, for some $b \in B_n$, $\mathcal{S}^{B_n}(r_i, b) \vDash \psi_i[b]$.
View b as an element of $\mathcal{B}$.
Then the B_n-part of $\mathcal{A}_n$ contains n_i - even n - copies of the element b, which are pairwise at infinite distance.
Since $\mathcal{S}^{B_n}(r_i, b) \cong \mathcal{S}^{\mathcal{A}_n}(r_i, b)$, we obtain $\mathcal{A}_n \vDash \varphi_i$.

Application of Gaifman's Theorem (Conclusion)

(⇒) Assume now that $\mathcal{A}_n \vDash \varphi_i$.

Choose $e \in A_n$, such that $\mathcal{S}^{A_n}(r_i, e) \vDash \psi_i[e]$.

- If $a \notin S^A(r_i, e)$, then $\mathcal{S}^{A_n}(r_i, e) \cong \mathcal{S}^B(r_i, e)$.
  Now we argue as above.
- Assume that $a \in S^A(r_i, e)$.
  Then, $\mathcal{A} \vDash \rho_i[a]$. Hence, $\mathcal{A} \vDash \rho_i[a']$.
  Thus, there is $e' \in A$, such that $d(e', a') \le r$ and $\mathcal{S}^A(r_i, e') \vDash \psi_i[e']$.
  Now
  $$d(e', a) \ge d(a', a) - d(a', e') > \ell - r = 2r - r \ge r_i.$$
  Therefore, $a \notin \mathcal{S}^A(r_i, e')$.
  Hence, $\mathcal{S}^A(r_i, e') \cong \mathcal{S}^B(r_i, e')$.
  So $\mathcal{B}_n$ contains n_i copies of e'.
  Thus, $\mathcal{B}_n \vDash \varphi_i$.