Finite Model Theory

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1. More on Games
   - Second-Order Logic
   - Infinitary Logics $L_{\infty\omega}$ and $L_{\omega_1\omega}$
   - The Logics FO^s and $L^s_{\infty\omega}$
   - Logics with Counting Quantifiers
   - Failure of Classical Theorems in the Finite

Subsection 1

Second-Order Logic

Second Order Logic

- **Second order logic**, SO, is an extension of first-order logic which allows to quantify over relations.
- In addition to the symbols of first-order logic, its alphabet contains, for each $n \geq 1$, countably many n-ary **relation** (or **predicate**) **variables** $V_1^n, V_2^n, \ldots$.
- To denote relation variables we use letters $X, Y, \ldots$.
- We define the set of second-order formulas of vocabulary τ to be the set generated by the rules for first-order formulas extended by:
  - If X is n-ary and $t_1, \ldots, t_n$ are terms, then $Xt_1 \ldots t_n$ is a formula.
  - If φ is a formula and X is a relation variable, then $\exists X \varphi$ is a formula.

Free Variables and Satisfaction

- The **free occurrence** of a variable or of a relation variable in a second order formula is defined in the obvious way.
- The notion of **satisfaction** is extended canonically.
- Then, given $\varphi = \varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_k)$ with free (individual and relation) variables among $x_1, \ldots, x_n$, $Y_1, \ldots, Y_k$, a τ-structure $\mathcal{A}$, elements $a_1, \ldots, a_n \in A$, and relations $R_1, \ldots, R_k$ over A of arities corresponding to $Y_1, \ldots, Y_k$, respectively,

$$\mathcal{A} \vDash \varphi[a_1, \ldots, a_n, R_1, \ldots, R_k]$$

means that $a_1, \ldots, a_n$ together with $R_1, \ldots, R_k$ satisfy φ in $\mathcal{A}$.

Example

- For any τ the class EVEN$[\tau]$ of finite τ-structures of even cardinality is axiomatizable in second-order logic (but not in first-order logic, as we already saw).

  In fact, EVEN$[\tau] = \text{Mod}(\varphi)$, where φ is a sentence expressing

  > "there is a binary relation which is an equivalence relation having only equivalence classes with exactly two elements".

  E.g.,

$$\begin{aligned}\exists X(&\forall x Xxx \wedge \forall x \forall y(Xxy \to Xyx)\\ &\wedge \forall x \forall y \forall z((Xxy \wedge Xyz) \to Xxz)\\ &\wedge \forall x \exists^{=1} y(Xxy \wedge y \neq x)).\end{aligned}$$

Monadic Second Order Logic

- We are mainly interested in the fragment MSO of second order logic known as **monadic second order logic**.
- In formulas of MSO only unary relation variables ("set variables") are allowed.
- We write

$$\mathcal{A} \equiv_m^{\mathrm{MSO}} \mathcal{B}$$

  if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same monadic second order sentences of quantifier rank $\leq m$ (the **quantifier rank** is the maximal number of nested first-order and second-order quantifiers).

MSO Ehrenfeucht-Fraïssé Games

- As in first-order logic, $\equiv_m^{\text{MSO}}$ can be characterized by an Ehrenfeucht-Fraïssé game, MSO$-G_m(\mathcal{A},\mathcal{B})$.
- The rules are the same as in the first-order Ehrenfeucht-Fraïssé game, but now in every move the spoiler can decide whether to make a point move or a set move.
  - The point moves are as the moves in the first-order case.
  - In a set move:
    - The spoiler chooses a subset $P \subseteq A$ or $Q \subseteq B$;
    - The duplicator answers by a subset $Q \subseteq B$ or $P \subseteq A$, respectively.
- After m moves, elements $a_1,\ldots,a_r$ and subsets $P_1,\ldots,P_s$ in A, and corresponding elements $b_1,\ldots,b_r$ and subsets $Q_1,\ldots,Q_s$ in B (with $m = r+s$) have been chosen.
- The duplicator wins if $\overline{a} \mapsto \overline{b} \in \text{Part}((\mathcal{A},P_1,\ldots,P_s),(\mathcal{B},Q_1,\ldots,Q_s))$.

Ehrenfeucht-Fraïssé Theorem

Theorem

$\mathcal{A} \equiv_m^{\mathrm{MSO}} \mathcal{B}$ iff the duplicator wins MSO$-G_m(\mathcal{A},\mathcal{B})$.

- The following is an outline of the proof of the theorem.

  Given $\mathcal{A}$, $\overline{a}$ ($= a_1 \ldots a_r$) in A, and $\overline{P}$ ($= P_1 \ldots P_s$) a sequence of subsets of A, define the formulas $\psi^j_{\overline{a},\overline{P}}$, similar to the j-isomorphism type $\varphi^j_{\overline{a}}$, but now taking into account also the second-order set quantifiers.

$$\begin{aligned}
\psi^0_{\overline{a},\overline{P}} &:= \bigwedge\{\varphi(v_1,\ldots,v_r,V_1,\ldots,V_s) : \varphi \text{ atomic or negated atomic}, \\
&\qquad\qquad \mathcal{A} \vDash \varphi[\overline{a},\overline{P}]\}; \\
\psi^{j+1}_{\overline{a},\overline{P}} &:= \bigwedge_{a\in A} \exists v_{r+1} \psi^j_{\overline{a}a,\overline{P}} \wedge \forall v_{r+1} \bigvee \psi^j_{\overline{a}a,\overline{P}} \\
&\qquad \wedge \bigwedge_{P\subseteq A} \exists V_{s+1} \psi^j_{\overline{a},\overline{P}P} \wedge \forall V_{s+1} \bigvee_{P\subseteq A} \psi^j_{\overline{a},\overline{P}P}.
\end{aligned}$$

Ehrenfeucht-Fraïssé Theorem (Cont'd)

- One can show the equivalence of:
  (i) The duplicator wins MSO$-G_m((\mathcal{A},\overline{P},\overline{a}),(\mathcal{B},\overline{Q},\overline{b}))$;
  (ii) $\mathcal{B} \vDash \psi^m_{\overline{a},\overline{P}}[\overline{b},\overline{Q}]$;
  (iii) $\overline{a},\overline{P}$ satisfies in $\mathcal{A}$ the same formulas of MSO of quantifier rank $\leq m$ as $\overline{b},\overline{Q}$ in $\mathcal{B}$.

m-Equivalence is an Equivalence Relation

Proposition

For a fixed vocabulary and $m \in \mathbb{N}$, the relation $\equiv_m^{\mathrm{MSO}}$ is an equivalence relation with finitely many equivalence classes.

- We may show, by induction of j, that, for varying $\mathcal{A}$, $\overline{a}$ and $\overline{P}$, there are only finitely many different $\psi^j_{\overline{a},\overline{P}}$.

  Since the number of those determine the number of equivalence classes of $\equiv_m^{\mathrm{MSO}}$, the relation $\equiv_m^{\mathrm{MSO}}$ has finitely many equivalence classes.

Equivalence and Operations

Proposition

The disjoint union and the ordered sum preserve the relation $\equiv_m^{\mathrm{MSO}}$, i.e., for relational τ we have:

(a) If $\mathcal{A}_1 \equiv_m^{\mathrm{MSO}} \mathcal{B}_1$ and $\mathcal{A}_2 \equiv_m^{\mathrm{MSO}} \mathcal{B}_2$, then $\mathcal{A}_1 \mathbin{\dot\cup} \mathcal{A}_2 \equiv_m^{\mathrm{MSO}} \mathcal{B}_1 \mathbin{\dot\cup} \mathcal{B}_2$.

(b) If $\mathcal{A}_1 \equiv_m^{\mathrm{MSO}} \mathcal{B}_1$ and $\mathcal{A}_2 \equiv_m^{\mathrm{MSO}} \mathcal{B}_2$, then $\mathcal{A}_1 \triangleleft \mathcal{A}_2 \equiv_m^{\mathrm{MSO}} \mathcal{B}_1 \triangleleft \mathcal{B}_2$.

- Let $* \in \{\dot\cup, \triangleleft\}$. Assume $\mathcal{A}_1 \equiv_m^{\mathrm{MSO}} \mathcal{B}_1$, $\mathcal{A}_2 \equiv_m^{\mathrm{MSO}} \mathcal{B}_2$.

  By hypothesis and the last theorem there are winning strategies S_1 and S_2 for the duplicator in the games MSO$-G_m(\mathcal{A}_1, \mathcal{B}_1)$ and MSO$-G_m(\mathcal{A}_2, \mathcal{B}_2)$, respectively.

  Then the following represents a winning strategy for the duplicator in MSO$-G_m(\mathcal{A}_1 * \mathcal{A}_2, \mathcal{B}_1 * \mathcal{B}_2)$ (when describing it we use moves of plays in MSO$-G_m(\mathcal{A}_1, \mathcal{B}_1)$ and MSO$-G_m(\mathcal{A}_2, \mathcal{B}_2)$).

Equivalence and Operations (Cont'd)

- Suppose, first, that the i-th move of the spoiler is a point move where he selects, say, $a \in A_1 * A_2$.

  Then the duplicator gets his answer by applying:
  - S_1, if $a \in \mathcal{A}_1$;
  - S_2, if $a \in A_2$.
- Now assume that the spoiler selects, say, $P \subseteq A_1 \cup A_2$.

  Set $P_1 := P \cap A_1$ and $P_2 := P \cap A_2$.

  Let Q_1 and Q_2 be the selections of the duplicator according to S_1 and S_2, respectively.

  Then, in the game MSO$-G_m(\mathcal{A}_1 * \mathcal{A}_2, \mathcal{B}_1 * \mathcal{B}_2)$, the duplicator chooses $Q_1 \cup Q_2$.

Prenex Normal Form

- An (M)SO-formula is in **prenex normal form** if it written as

$$Q_1\alpha_1 \cdots Q_s\alpha_s\psi,$$

  where:
  - $Q_1, \ldots, Q_s \in \{\forall, \exists\}$;
  - $\alpha_1, \ldots, \alpha_s$ are first order or second order variables;
  - ψ is quantifier free.
- Equivalences that govern (M)SO-formulas include:
  - $\vDash \neg\exists X\varphi \leftrightarrow \forall X\neg\varphi$;
  - $\vDash (\varphi \vee \forall Y\psi) \leftrightarrow \forall Y(\varphi \vee \psi)$, if Y is not free in φ.
- An induction using such equivalences shows that each (M)SO-formula is logically equivalent to an (M)SO-formula in prenex normal form.

Prenex Normal Form (Cont'd)

- In addition, the following logical equivalences hold:

$$\models \exists x Q_1\alpha_1 \cdots Q_s\alpha_s\psi \leftrightarrow \exists X Q_1\alpha_1 \cdots Q_s\alpha_s(\exists^{=1} xXx \wedge \forall x(Xx \rightarrow \psi))$$

and

$$\models \forall x Q_1\alpha_1 \cdots Q_s\alpha_s\psi \leftrightarrow \forall X Q_1\alpha_1 \cdots Q_s\alpha_s(\exists^{=1} xXx \rightarrow \forall x(Xx \rightarrow \psi)).$$

- So every (M)SO-formula is logically equivalent to one in prenex normal form in which each second order quantifier precedes all first order quantifiers.

(M)Σ_n^1 and (M)Π_n^1 Formulas

- A formula in prenex normal form is called a (M)Σ_n^1 **formula**, if the string of second-order quantifiers consists of n consecutive blocks, where:
  - In each block all quantifiers are of the same type i.e., all universal or all existential;
  - Adjacent blocks contain quantifiers of different type;
  - The first block is existential.
- A formula in prenex normal form is called a (M)Π_n^1 **formula**, if the string of second-order quantifiers consists of n consecutive blocks, where:
  - In each block all quantifiers are of the same type;
  - Adjacent blocks contain quantifiers of different type;
  - The first block is universal.

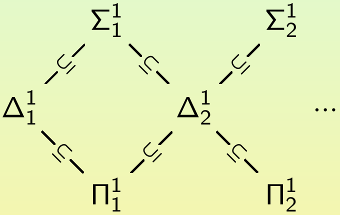
  Example: Consider the following formula, with quantifier-free φ,

$$\exists X \exists Y \forall Z \forall x \exists y \varphi.$$

It is a Σ_2^1-formula.

Relations Between Classes of Formulas

- The negation of a Σ^1_n-formula is logically equivalent to a Π^1_n-formula.
- The negation of a Π^1_n-formula is equivalent to a Σ^1_n-formula.
- Denoting by Δ^1_n the set of formulas that are logically equivalent to both a Σ^1_n-formula and a Π^1_n-formula, we have up to logical equivalence



- This can easily be verified by adding dummy variables.
- The same inclusions hold for the monadic classes.

Comments on the Inclusions Between Classes of Formulas

- It can be shown that for arbitrary models all the inclusions above are proper (this also holds for MSO).
- The question to what extent the hierarchies are proper in the finite is related to important questions of complexity theory (to come later).

  Example: We have seen that the class of finite, connected graphs is $\mathrm{M}\Pi^1_1$-axiomatizable but not $\mathrm{M}\Sigma^1_1$-axiomatizable.
- It follows that, in the finite, $\mathrm{M}\Sigma^1_1 \neq \mathrm{M}\Pi^1_1$.

Subsection 2

Infinitary Logics $L_{\infty\omega}$ and $L_{\omega_1\omega}$

Infinitary Logics $L_{\infty\omega}$ and $L_{\omega_1\omega}$

- The **infinitary logics** $L_{\infty\omega}$ and $L_{\omega_1\omega}$ allow arbitrary and countable disjunctions (and hence conjunctions), respectively.
- More formally, let τ be a vocabulary.
- The class of $L_{\infty\omega}$-**formulas** over τ is given by the following clauses:
  - It contains all atomic first order formulas over τ;
  - If φ is a formula, then so is $\neg\varphi$;
  - If φ is a formula and x a variable, then $\exists x\varphi$ is a formula;
  - If Ψ is a set of formulas, then $\bigvee\Psi$ is a formula.
- For $L_{\omega_1\omega}$ we replace the last clause by:
  - If Ψ is a *countable set* of formulas then $\bigvee\Psi$ is a formula.

Semantics of Infinitary Logics

- The semantics is a direct extension of the semantics of first order logic with $\bigvee\Psi$ being interpreted as the disjunction over all formulas in Ψ.
- Neglecting the interpretation of the free variables,

$$\mathcal{A} \models \bigvee \Psi \quad \text{iff} \quad \text{for some } \psi \in \Psi,\ \mathcal{A} \models \psi.$$

- We set

$$\bigwedge \Psi := \neg \bigvee \{\neg\psi : \psi \in \Psi\}.$$

- Then $\bigwedge\Psi$ is interpreted as the conjunction over all formulas in Ψ.
- By identifying $(\varphi \vee \psi)$ with $\bigvee\{\varphi, \psi\}$, we see that $L_{\infty\omega}$ and $L_{\omega_1\omega}$ are extensions of first order logic.

Examples

(a) For any τ, the models of the $L_{\omega_1\omega}$-sentence

$$\bigvee\{\varphi_{=n} : n \geq 1\},$$

where $\varphi_{=n}$ is a first order sentence expressing that the universe has cardinality n, are the finite τ-structures.

- The $L_{\omega_1\omega}$-sentence $\bigvee\{\varphi_{=2n} : n \geq 1\}$ axiomatizes the class $\text{EVEN}[\tau]$.
- If M is any nonempty set of positive natural numbers, then the class of models of the $L_{\omega_1\omega}$-sentence $\bigvee\{\varphi_{=k} : k \in M\}$ corresponds to the query "$\|A\| \in M$?".
- In particular, we see that nonrecursive queries are $L_{\omega_1\omega}$-definable.

Examples (Cont'd)

(b) Any class of finite structures is axiomatizable in $L_{\omega_1\omega}$.

In fact, let K be a class of finite structures.

Choose a set Φ of first-order sentences such that $K = \text{Mod}(\Phi)$.

Then $K = \text{Mod}(\varphi)$, for the $L_{\omega_1\omega}$-sentence $\varphi := \bigwedge \Phi$.

Examples (Cont'd)

(c) "Connectivity" is a property of graphs expressible in $L_{\omega_1\omega}$.

In fact, let $\varphi_n(x,y)$ be a first order formula saying that there is a path from x to y of length n,

$$\varphi_n(x,y) := \exists z_0 \cdots \exists z_n (z_0 = x \wedge z_n = y \wedge E z_0 z_1 \wedge \cdots \wedge E z_{n-1} z_n).$$

Then "connectivity" is expressed in $L_{\omega_1\omega}$ by

$$\forall x \forall y (\neg x = y \to \bigvee \{\varphi_n(x,y) : n \geq 1\}).$$

Free Variables and Sentences

- $L_{\infty\omega}$-**sentences** are $L_{\infty\omega}$-formulas without free variables.
- Note that $L_{\infty\omega}$-formulas may have infinitely many free variables.

  Example:

  $$\bigvee\{\neg v_i = v_j : 1 \le i < j\}.$$

- On the other hand, subformulas of $L_{\infty\omega}$-*sentences* only have finitely many free variables.
- In the following we restrict ourselves to $L_{\infty\omega}$-formulas with only finitely many free variables.

Equivalence of $L_{\infty\omega}$ and $L_{\omega_1\omega}$ in the Finite

Proposition

(a) In the finite, every $L_{\infty\omega}$-formula $\varphi(\overline{x})$ is equivalent to an $L_{\omega_1\omega}$-formula $\psi(\overline{x})$.

(b) Assume $\mathcal{A}$ and $\mathcal{B}$ are finite. For every $L_{\infty\omega}$-formula $\varphi(\overline{x})$, there is an FO-formula $\psi(\overline{x})$, such that

$$\mathcal{A} \vDash \forall\overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})) \text{ and } \quad \mathcal{B} \vDash \forall\overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$$

In both cases, (a) and (b), the formula ψ can be chosen such that $\text{free}(\psi) \subseteq \text{free}(\varphi)$ and every variable occurring in ψ (free or bound) occurs in φ.

- The proofs are by induction over the rules for $L_{\infty\omega}$-formulas.

  The translation procedure preserves the "structure" of formulas and only replaces infinitary disjunctions by countable ones in Part (a) and by finite ones in Part (b).

Equivalence of $L_{\infty\omega}$ and $L_{\omega_1\omega}$ in the Finite (Cont'd)

- In the main step, suppose that

$$\varphi(\overline{x}) = \bigvee\{\varphi_i(\overline{x}) : i \in I\}$$

is an $L_{\infty\omega}$-formula.

In Part (a), consider all finite $\mathcal{C}$, with universe $\{1, 2, \ldots, \|C\|\}$.

In Part (b) suppose $\mathcal{C} \in \{\mathcal{A}, \mathcal{B}\}$ has universe $\{1, 2, \ldots, \|C\|\}$.

For each $\overline{c} \in C$, if there exists an $i \in I$, such that $\mathcal{C} \vDash \varphi_i[\overline{c}]$, choose such an i.

Let I_0 be the set of i's chosen in this way.

Then I_0 is countable in Part (a) and finite in Part (b).

Moreover, in Part (a), $\bigvee\{\varphi_i(\overline{x}) : i \in I\}$ and $\bigvee\{\varphi_i(\overline{x}) : i \in I_0\}$ are equivalent in the finite.

And, similarly, in Part (b), $\bigvee\{\varphi_i(\overline{x}) : i \in I\}$ and $\bigvee\{\varphi_i(\overline{x}) : i \in I_0\}$ are equivalent in $\mathcal{A}$ and $\mathcal{B}$.

An Improvement

- Since every finite structure can be characterized in first order logic, we obtain the following improvement of Part (a).

Proposition

In the finite, every $L_{\infty\omega}$-formula $\varphi(\overline{x})$ is equivalent to a countable disjunction - and hence to a countable conjunction - of first order formulas. In fact, in the finite, $\varphi(\overline{x})$ is equivalent to

$$\bigvee\left\{\varphi_{\mathcal{A},\overline{a}}^{\|A\|+1}(\overline{x}) : \mathcal{A} \text{ finite, } \overline{a} \in A, \mathcal{A} \vDash \varphi[\overline{a}]\right\}.$$

- For simplicity we restrict ourselves to sentences. Let $\mathcal{B}$ be a finite structure. If $\mathcal{B} \vDash \varphi$, then $\varphi_{\mathcal{B}}^{\|B\|+1}$ is a member of the disjunction. So the disjunction is satisfied by $\mathcal{B}$.
  Conversely, suppose $\mathcal{B}$ satisfies the disjunction. Then, for some finite $\mathcal{A}$, with $\mathcal{A} \vDash \varphi$, we have $\mathcal{B} \vDash \varphi_{\mathcal{A}}^{\|A\|+1}$. Thus, by a previous theorem, $\mathcal{A} \cong \mathcal{B}$. Therefore, $\mathcal{B} \vDash \varphi$.

Equivalence and Infinitary Games

- We say that $\mathcal{A}$ and $\mathcal{B}$ are $L_{\infty\omega}$-**equivalent**, written

$$\mathcal{A} \equiv^{L_{\infty\omega}} \mathcal{B},$$

if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same $L_{\infty\omega}$-sentences.

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be structures, $\overline{a} \in A^s$, and $\overline{b} \in B^s$. The game $G_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$ is the same as the game $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$ up to the fact that now each player has to make infinitely many moves. Thus, in the course of a play of $G_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$, elements $e_1, e_2, \ldots \in A$ and $f_1, f_2, \ldots \in B$ are chosen.
The duplicator **wins the play** if $\overline{a}e_1 \ldots e_i \mapsto \overline{b}f_1 \ldots f_i \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$, for all i.
The **spoiler wins** if $\overline{a}e_1 \ldots e_i \mapsto \overline{b}f_1 \ldots f_i \notin \mathrm{Part}(\mathcal{A}, \mathcal{B})$, for some i.
The **duplicator wins** $G_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$ if he has a winning strategy.

Characterization of Equivalence via Games

Lemma

Suppose that the duplicator wins $G_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$. Then:

(a) $\overline{a} \mapsto \overline{b} \in \text{Part}(\mathcal{A}, \mathcal{B})$;

(b) For $a \in A$, there is $b \in B$, such that the duplicator wins $G_\infty(\mathcal{A}, \overline{a}a, \mathcal{B}, \overline{b}b)$;

(c) For $b \in B$, there is $a \in A$, such that the duplicator wins $G_\infty(\mathcal{A}, \overline{a}a, \mathcal{B}, \overline{b}b)$.

- Immediate from the definition.

Partial Isomorphisms

Definition

(a) $\mathcal{A}$ and $\mathcal{B}$ are said to be **partially isomorphic**, written

$$\mathcal{A} \cong_{\text{part}} \mathcal{B},$$

if there is a nonempty set I of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$ with the back and forth properties:

- For every $p \in I$ and every $a \in A$ there is $q \in I$ with $q \supseteq p$ and $a \in \text{dom}(q)$;
- For every $p \in I$ and every $b \in B$ there is $q \in I$ with $q \supseteq p$ and $b \in \text{ran}(q)$.

We then write $I : \mathcal{A} \cong_{\text{part}} \mathcal{B}$.

(b) The set of winning positions for the duplicator is

$$\begin{array}{rcl} W_\infty(\mathcal{A},\mathcal{B}) & = & \{\overline{a} \mapsto \overline{b} : s \in \mathbb{N}, \overline{a} \in A^s, \overline{b} \in B^s, \\ & & \text{the duplicator wins } G_\infty(\mathcal{A},\overline{a},\mathcal{B},\overline{b})\}. \end{array}$$

Equivalence, Games and Partial Isomorphisms

Theorem

For structures $\mathcal{A}$ and $\mathcal{B}$, $\overline{a} \in A^s$ and $\overline{b} \in B^s$ the following are equivalent:

(i) The duplicator wins $G_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$;

(ii) $\overline{a} \mapsto \overline{b} \in W_\infty(\mathcal{A}, \mathcal{B})$ and $W_\infty(\mathcal{A}, \mathcal{B}) : \mathcal{A} \cong_{\text{part}} \mathcal{B}$;

(iii) There is a set I with $\overline{a} \mapsto \overline{b} \in I$, such that $I : \mathcal{A} \cong_{\text{part}} \mathcal{B}$;

(iv) $\overline{a}$ and $\overline{b}$ satisfy the same formulas of $L_{\infty\omega}$ in $\mathcal{A}$ and $\mathcal{B}$, respectively, i.e., if $\varphi(x_1, \ldots, x_s)$ is a formula of $L_{\infty\omega}$, then $\mathcal{A} \vDash \varphi[\overline{a}]$ iff $\mathcal{B} \vDash \varphi[\overline{b}]$.

(i)⇒(ii) is covered by the preceding lemma.

(ii)⇒(iii) Obvious.

(iii)⇒(i) A set I with $\overline{a} \mapsto \overline{b} \in I$ and $I : \mathcal{A} \cong_{\text{part}} \mathcal{B}$ can be viewed as a winning strategy for the duplicator for the game $G_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.

Equivalence, Games and Partial Isomorphisms (Cont'd)

(iii)⇒(iv) Let I be as in (iii). By (transfinite) induction on the quantifier rank of the $L_{\infty\omega}$-formula $\varphi(x_1,\ldots,x_s)$ we prove that, for any $e_1\ldots e_s \mapsto f_1\ldots f_s \in I$,

$$\mathcal{A} \models \varphi[\overline{e}] \quad \text{iff} \quad \mathcal{B} \models \varphi[\overline{f}].$$

The case of quantifier rank 0 has been already covered.

For any quantifier rank, note that the class of formulas satisfying the equivalence contains the atomic formulas and is closed under $\neg$ and $\vee$.

Suppose that $\varphi(x_1,\ldots,x_s) = \exists y\psi(x_1,\ldots,x_s,y)$.

Assume, for example, that $\mathcal{A} \models \varphi[e_1,\ldots,e_s]$.

Then, there exists $a \in A$, such that $\mathcal{A} \models \psi[e_1,\ldots,e_s,a]$.

The forth property of I yields $b \in B$, with $e_1\ldots e_s a \mapsto f_1\ldots f_s b \in I$.

Since $\text{qr}(\psi) < \text{qr}(\varphi)$, by the induction hypothesis, $\mathcal{B} \models \psi[f_1,\ldots,f_s,b]$.

Hence, $\mathcal{B} \models \varphi[f_1,\ldots,f_s]$.

Equivalence, Games and Partial Isomorphisms (Cont'd)

(iv)⇒(iii) Suppose that (iv) holds. Let I be the set of all partial isomorphisms $e_1 \ldots e_r \mapsto f_1 \ldots f_r$ (with $r \geq 0$) such that, for all $L_{\infty\omega}$-formulas $\varphi(x_1, \ldots, x_r)$,

$$\mathcal{A} \vDash \varphi[\overline{e}] \quad \text{iff} \quad \mathcal{B} \vDash \varphi[\overline{f}].$$

By (iv), $\overline{a} \mapsto \overline{b} \in I$.

We show that I has the back and forth properties.

Let $e_1 \ldots e_r \mapsto f_1 \ldots f_r \in I$ and $a \in I$.

For each $b \in B$, if there is a formula $\varphi(x_1, \ldots, x_r, x)$ of $L_{\infty\omega}$ such that

$$\mathcal{A} \vDash \varphi(\overline{x}, x)[\overline{e}a] \quad \text{and} \quad \mathcal{B} \vDash \neg\varphi(\overline{x}, x)[\overline{f}b],$$

let $\varphi_b(\overline{x}, x)$ be such a formula.

Otherwise, set $\varphi_b(\overline{x}, x) := x = x$.

Equivalence, Games and Partial Isomorphisms (Cont'd)

- Since $\mathcal{A} \vDash \exists x \bigwedge\{\varphi_b : b \in B\}[\overline{e}]$, we have

$$\mathcal{B} \vDash \exists x \bigwedge\{\varphi_b : b \in B\}[\overline{f}].$$

Hence, there is $b' \in B$, such that

$$\mathcal{B} \vDash \bigwedge\{\varphi_b : b \in B\}[\overline{f}b'].$$

Using the definition of $\varphi_{b'}$, one easily sees that $\overline{e}a$ and $\overline{f}b'$ satisfy the same formulas of $L_{\infty\omega}$ in $\mathcal{A}$ and $\mathcal{B}$, respectively.

Hence, $\overline{e}a \mapsto \overline{f}b' \in I$.

The back property is proven similarly.

The Case Without Parameters

- Note that $W_\infty(\mathcal{A},\mathcal{B}) \neq \varnothing$ iff $\varnothing \mapsto \varnothing \in W_\infty(\mathcal{A},\mathcal{B})$.

  Therefore, by the preceding theorem, we get

Corollary

For $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

(i) The duplicator wins $G_\infty(\mathcal{A},\mathcal{B})$;

(ii) $W_\infty(\mathcal{A},\mathcal{B}) : A \cong_{\text{part}} \mathcal{B}$;

(iii) $\mathcal{A} \cong_{\text{part}} \mathcal{B}$;

(iv) $\mathcal{A} \equiv^{L_{\infty\omega}} \mathcal{B}$.

Countable Structures

Lemma

Let $\mathcal{A}$ and $\mathcal{B}$ be countable.

(a) If $\mathcal{A} \cong_{\text{part}} \mathcal{B}$ then $\mathcal{A} \cong \mathcal{B}$.

(b) If $I : \mathcal{A} \cong_{\text{part}} \mathcal{B}$ and $p_0 \in I$, then p_0 can be extended to an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

- Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$. It suffices to show (b). Suppose $I : \mathcal{A} \cong_{\text{part}} \mathcal{B}$ and $p_0 \in I$. By repeated application of the back and forth properties, we get $p_1, p_2, \ldots$ in I, such that $p_0 \subseteq p_1 \subseteq \cdots$ and such that $a_1 \in \text{dom}(p_1)$, $b_1 \in \text{ran}(p_2)$, $a_2 \in \text{dom}(p_3)$, ... Then $\bigcup_{n \geq 0} p_n$ is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Corollary

If $\mathcal{A}$ and $\mathcal{B}$ are countable and $L_{\infty\omega}$-equivalent then they are isomorphic.

- The claim follows from the preceding corollary and lemma.

Example

- Let τ be relational.

  For $r \geq 0$, let Δ_{r+1} be the set

  $$\Delta_{r+1} := \{\varphi(v_1, \dots, v_r, v_{r+1}) : \varphi \text{ has the form } R\overline{x}, \text{ where } R \in \tau \text{ and where } v_{r+1} \text{ occurs in } \overline{x}\}.$$

  For a subset Φ of Δ_{r+1}, with $\Phi^c := \Delta_{r+1} \backslash \Phi$, the sentence

  $$\chi_\Phi := \forall v_1 \cdots \forall v_r (\bigwedge_{1 \leq i < j \leq r} v_i \neq v_j \to \exists v_{r+1} (\bigwedge_{1 \leq i \leq r} v_i \neq v_{r+1} \wedge \bigwedge_{\varphi \in \Phi} \varphi \wedge \bigwedge_{\varphi \in \Phi^c} \neg\varphi)),$$

  is called an **extension axiom**.

  More precisely, it is called an $(r+1)$-**extension axiom**.

  The set T_{rand} of all extension axioms is called the **random structure theory**.

  Clearly, every model of T_{rand} is infinite.

Example (Cont'd)

Claim: Any two models of T_{rand} are $L_{\infty\omega}$-equivalent.

Thus, for each $L_{\infty\omega}$-sentence φ, $T_{\text{rand}} \vDash \varphi$ or $T_{\text{rand}} \vDash \neg\phi$.

We prove that the extension axioms in T_{rand} guarantee, for any models $\mathcal{A}$ and $\mathcal{B}$ of T_{rand}, that the set

$$I := \{\overline{a} \mapsto \overline{b} : \overline{a} \in A, \overline{b} \in B \text{ and } \varphi^0_{\mathcal{A},\overline{a}} = \varphi^0_{\mathcal{B},\overline{b}}\}$$

has the back and forth properties.

Let $\overline{a} \mapsto \overline{b} \in I$, where $\overline{a} = a_1 \dots a_r$ and $a_1, \dots, a_r$ can be assumed to be distinct. Also let, say, a_{r+1} be in $A \backslash \{a_1, \dots, a_r\}$.

Set $\Phi := \{\varphi(v_1, \dots, v_{r+1}) : \varphi \in \Delta_{r+1}, \mathcal{A} \vDash \varphi[\overline{a}a_{r+1}]\}$.

Now $\mathcal{B} \vDash \chi_\Phi$. So there exists $b_{r+1} \in B$, such that $\varphi^0_{\mathcal{B},\overline{b}b_{r+1}} = \varphi^0_{\mathcal{A},\overline{a}a_{r+1}}$.

This show that $\overline{a}a_{r+1} \mapsto \overline{b}b_{r+1} \in I$.

Moreover, since τ is relational, the empty partial isomorphism is in I.

Hence, $I : \mathcal{A} \cong_{\text{part}} \mathcal{B}$.

Example (Cont'd)

Claim: T_{rand} has a countable model and, hence, by the preceding corollary, an (up to isomorphism) unique countable model $\mathcal{R}$, the so-called **infinite random structure**.

Let $(\alpha_n)_{n\geq 0}$ be an enumeration of all pairs $(\overline{m}, \chi)$, where:

- $\overline{m}$ is a tuple of distinct natural numbers;
- χ is an $(r+1)$-extension axiom, where $r := \text{length}(m)$.

Suppose that for $\alpha_n = (\overline{m}, \chi)$ all entries of $\overline{m}$ are not greater than n.

By induction on n we define structures $\mathcal{A}_n$, with $A_n = \{0, \ldots, n\}$, and $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$, such that $\mathcal{A} := \bigcup_{n\geq 0} \mathcal{A}_n$ is a model of T_{rand}.

- Let $\mathcal{A}_0 = (A_0, (\varnothing)_{R\in\tau})$ (each relation symbol is interpreted as $\varnothing$);
- Suppose $\mathcal{A}_n$ has been defined and $\alpha_n = (m_1, \ldots, m_r, \chi)$ with $\chi = \chi_\Phi$. Define $\mathcal{A}_{n+1}$, with universe A_{n+1}, such that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ and such that, for $\varphi \in \Delta_{r+1}$, $\mathcal{A}_{n+1} \vDash \varphi[m_1, \ldots, m_r, n+1]$ iff $\varphi \in \Phi$ (note that v_{r+1} occurs in every formula of Δ_{r+1}).

This ensures that $\mathcal{A} := \bigcup_{n\geq 0} \mathcal{A}_n$ is a model of χ.

Spectra of Sentences

- For a sentence φ, let the **spectrum** Spec(φ) of φ be the set

$$\text{Spec}(\varphi) := \{m \geq 1 : \text{there is } \mathcal{A} \vDash \varphi \text{ with } \|A\| = m\}.$$

Proposition

For any first-order sentence φ, at least one of Spec(φ) or Spec($\neg\varphi$) is cofinite, i.e., there exists n_0, such that

$$\{n : n \geq n_0\} \subseteq \text{Spec}(\varphi) \quad \text{or} \quad \{n : n \geq n_0\} \subseteq \text{Spec}(\neg\varphi).$$

- Let Φ be the set consisting of the following sentences:
  (i) $\varphi_{\geq m}$, for all $m \geq 1$;
  (ii) $\forall \overline{x} R \overline{x}$, for all $R \in \tau$;
  (iii) $c = d$, for all $c, d \in \tau$.

  Clearly, Φ is satisfiable.

Spectra of Sentences (Cont'd)

- Any two models $\mathcal{A}$ and $\mathcal{B}$ of Φ are partially isomorphic via

$$I := \{p \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) : \mathrm{dom}(p) \text{ finite}\}.$$

Hence, by a previous corollary, they are elementarily equivalent.

Therefore, given a first-order sentence φ, we have

$$\Phi \vDash \varphi \quad \text{or} \quad \Phi \vDash \neg\varphi.$$

Say $\Phi \vDash \varphi$.

By Compactness, there is a finite $\Phi_0 \subseteq \Phi$, such that $\Phi_0 \vDash \varphi$.

Let n_0 be larger than any m such that $\varphi_{\geq m}$ is in Φ_0.

Then Φ_0 and, hence, φ, has a model of cardinality n for each $n \geq n_0$.

Subsection 3

The Logics FO^s and $L^s_{\infty\omega}$

Introduction

- In first-order logic FO, every finite structure $\mathcal{A}$ can be characterized up to isomorphism by a first-order sentence $\varphi_{\mathcal{A}}$ which, in general, needs $\|A\|+1$ variables.
- Hence, an arbitrary class K of finite structures can be axiomatized in $L_{\infty\omega}$ by the sentence
$$\bigvee\{\varphi_{\mathcal{A}} : \mathcal{A} \in K\}$$
which, in general, contains infinitely many variables.
- Since every class of finite structures is axiomatizable in it, $L_{\infty\omega}$ is too powerful in the finite to yield new general principles.
- This motivates the restriction to formulas of $L_{\infty\omega}$ containing only finitely many variables.

$L^s_{\infty\omega}$ and FO^s

- Fix $s \geq 1$.
- $L^s_{\infty\omega}$ and FO^s denote by the fragments of $L_{\infty\omega}$ and FO, respectively, containing only formulas, whose free and bound variables are among $v_1, \ldots, v_s$.
- Moreover, we set
$$L^\omega_{\infty\omega} := \bigcup_{s\geq 1} L^s_{\infty\omega}.$$
- We have $\mathrm{FO} = \bigcup_{s\geq 1} \mathrm{FO}^s$.
- On the other hand, $L^\omega_{\infty\omega} \neq L_{\infty\omega}$.
- E.g., the formula
$$\bigvee\{\varphi_{=n} : n \geq 1\}$$
belongs to $L_{\infty\omega}$ but not to $L^\omega_{\infty\omega}$.
- In examples, we write $x = v_1$, $y = v_2$, $z = v_3$, etc.

Example

- Let $\tau = \{<\}$. There are FO^2-formulas $\psi_n(x)$ and χ_n such that for orderings $\mathcal{A}$, $a \in A$:
  - For $n \geq 0$,
  $$\mathcal{A} \vDash \psi_n[a] \quad \text{iff} \quad a \text{ is the } n\text{-th element of } <^A;$$
  - For $n \geq 1$,
  $$\mathcal{A} \vDash \chi_n \quad \text{iff} \quad \|A\| = n.$$

  In fact, define inductively:
  $$\begin{array}{rcl}\psi_0(x) & := & \forall y \neg y < x;\\ \psi_{n+1}(x) & := & \forall y(y < x \leftrightarrow \bigvee_{i \leq n} \exists x(x = y \wedge \psi_i(x))).\end{array}$$

  Moreover, set
  $$\chi_n := \exists x \psi_{n-1}(x) \wedge \neg \exists x \psi_n(x).$$

Example

- For each $n \geq 1$, there is an FO^3-formula $\varphi_n(x,y)$ that in digraphs expresses that there is a path of length at most n from x to y.

  Define inductively:

$$\begin{array}{rcl} \varphi_1(x,y) & := & Exy; \\ \varphi_{n+1}(x,y) & := & \varphi_n(x,y) \vee \exists z(Ezy \wedge \exists y(y = z \wedge \varphi_n(x,y))). \end{array}$$

Example: Improving the Quantifier Rank (Cont'd)

- Concerning the quantifier rank we can do better than in ψ_n and φ_n.

  Let φ be an $L^s_{\infty\omega}$-formula and π a permutation of $1, \ldots, s$.

  By simultaneously replacing both the free and the bound occurrences of $v_1, \ldots, v_s$ by $v_{\pi(1)}, \ldots, v_{\pi(s)}$ one obtains a formula

$$\varphi\begin{pmatrix} v_{\pi(1)} & \cdots & v_{\pi(s)} \\ v_1 & \cdots & v_s \end{pmatrix}.$$

  Clearly,

$$\mathcal{A} \models \varphi\begin{pmatrix} v_{\pi(1)} & \cdots & v_{\pi(s)} \\ v_1 & \cdots & v_s \end{pmatrix}[\overline{a}] \quad \text{iff} \quad \mathcal{A} \models \varphi[a_{\pi(1)}, \ldots, a_{\pi(s)}].$$

Example: Improving the Quantifier Rank (Cont'd)

- Now in the preceding examples we can replace ψ_n by the formula ψ'_n of quantifier rank $\leq n+1$, where:

$$\begin{array}{rcl}\psi'_0 & := & \psi_0;\\ \psi'_{n+1} & := & \forall y \left(y < x \leftrightarrow \bigvee_{i \leq n} \psi'_i \left(\begin{array}{c} yx \\ xy \end{array} \right) \right).\end{array}$$

Moreover, we can replace φ_n by the formula φ'_n of quantifier rank $\leq n+1$, where:

$$\begin{array}{rcl}\varphi'_1 & := & \varphi_1;\\ \varphi'_{n+1} & := & \varphi'_n \vee \exists z \left(Ezy \wedge \varphi'_n \left(\begin{array}{c} xzy \\ xyz \end{array} \right) \right).\end{array}$$

.

Passing from $L^s_{\infty\omega}$ to FO^s

- Using previous results, we get

Proposition

Assume $\mathcal{A}$ and $\mathcal{B}$ are finite. For every $L^s_{\infty\omega}$-formula φ, there is an FO^s-formula ψ with $\mathrm{free}(\psi) \subseteq \mathrm{free}(\varphi)$ such that

$$\mathcal{A} \vDash \forall x_1 \cdots \forall x_s(\varphi \leftrightarrow \psi) \quad \text{iff} \quad \mathcal{B} \vDash \forall x_1 \cdots \forall x_s(\varphi \leftrightarrow \psi).$$

Corollary

If $\mathcal{A}$ and $\mathcal{B}$ are finite then $\mathcal{A} \equiv^s \mathcal{B}$ implies $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$.

- Given an $L^s_{\infty\omega}$-sentence φ, choose ψ according to the preceding proposition. Then

$$\mathcal{A} \vDash \varphi \text{ iff } \mathcal{A} \vDash \psi \text{ iff } \mathcal{B} \vDash \psi \text{ iff } \mathcal{B} \vDash \varphi.$$

Characterization of Formulas in FO^s

- If $\varphi \in \mathrm{FO}^s$ (or, $\varphi \in L^s_{\infty\omega}$), then every subformula of φ contains at most s free variables (namely, at most $v_1, \ldots, v_s$).
- This property characterizes the formulas of FO^s up to logical equivalence.

Proposition

Assume $s \geq 1$. If every subformula of $\varphi(v_1, \ldots, v_s) \in \mathrm{FO}$ has at most s free variables, then φ is logically equivalent to a formula of FO^s. The statement remains true, if we replace FO and FO^s by $L_{\infty\omega}$ and $L^s_{\infty\omega}$, respectively.

- By induction (on the quantifier rank) we associate with every formula $\varphi(v_1, \ldots, v_s)$ all of whose subformulas have at most s free variables, a formula $\varphi^* \in \mathrm{FO}^s$, with $\mathrm{free}(\varphi) = \mathrm{free}(\varphi^*)$, such that

$$\vDash \varphi \leftrightarrow \varphi^*.$$

Characterization of Formulas in FO^s (Cont'd)

- For atomic φ, set $\varphi^* := \varphi$.

  For $\varphi = \neg\chi$, set $\varphi^* = \neg\chi^*$.

  For $\varphi = (\chi_1 \vee \chi_2)$, set $\varphi^* = (\chi_1^* \vee \chi_2^*)$.

  Let $\varphi = \exists y\chi$. Then, $\text{free}(\chi) \subseteq \{v_1, \ldots, v_s, y\}$ and $\|\text{free}(\chi)\| \leq s$.
  - If $y \notin \text{free}(\chi)$, then $\chi = \chi(v_1, \ldots, v_s)$. Then, χ^* is defined by induction hypothesis. We set $\varphi^* := \chi^*$.
  - If $y \in \text{free}(\chi)$ and $y \in \{v_1, \ldots, v_s\}$ then, again, $\chi = \chi(v_1, \ldots, v_s)$. So χ^* is again defined. We set $\varphi^* := \exists y\chi^*$.
  - If $y \in \text{free}(x)$ and $y \notin \{v_1, \ldots, v_s\}$, then, there is an i such $v_i \notin \text{free}(\chi)$. Set $\chi_0 := \chi\begin{pmatrix} v_i & y \\ y & v_i \end{pmatrix}$ (as before, $\chi_0 := \chi\begin{pmatrix} v_i & y \\ y & v_i \end{pmatrix}$ is obtained from x by simultaneously replacing all occurrences of y and v_i by v_i and y, respectively). Then, $\vDash \varphi \leftrightarrow \exists v_i\chi_0$, $\text{free}(\chi_0) \subseteq \{v_1, \ldots, v_s\}$, and every subformula of χ_0 has at most s free variables. Thus, χ_0^* is defined. We set $\varphi^* := \exists v_i\chi_0^*$.

Pebble Games: Example

- Consider the formula

$$\varphi = \exists x \exists y (x < y \wedge \exists x y < x).$$

- Let $\mathcal{A} := (\{a, b\}, <)$ and $\mathcal{B} := (\{c, d, e\}, <)$ be the orderings with $a < b$ and $c < d < e$.
- Since $\mathcal{A} \vDash \neg\varphi$ and $\mathcal{B} \vDash \varphi$, the spoiler has a winning strategy in $G_3(\mathcal{A}, \mathcal{B})$.
- How is the fact that φ only contains two variables reflected in the course of a play?
- A play won by the spoiler is given in the table, where his selections are in red. There is no third move of the duplicator leading to a partial isomorphism.

	$\mathcal{A}$	$\mathcal{B}$
first move	a	c
second move	b	d
third move	?	e

Pebble Games: Example (Cont'd)

- The strategy of the spoiler consists in choosing, for the first two quantifiers $\exists x \exists y$, the elements c for x and d for y in $\mathcal{B}$ in order to have $\mathcal{B} \vDash (x < y \wedge \exists x y < x)[c, d]$.
- The only selections for the duplicator leading to a partial isomorphism are a for x and b fox y.
- For the second quantifier $\exists x$, the spoiler selects in $\mathcal{B}$ the element e, thereby getting a witness for $\mathcal{B} \vDash \exists x y < x[d]$.
- Obviously the old value c for x is no longer relevant.

Pebble Games: Example (Cont'd)

- Therefore, the play above may be represented more informatively by

	first move		second move		third move	
	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{B}$
x-box	a	c	a	c	?	e
y-box	$*$	$*$	b	d	b	d

- In the table, the x-boxes and the y-boxes always contain the actual value for x and y, respectively, and $*$ stands for an empty box.

Partial Isomorphisms

- Fix a vocabulary τ.
- By convention, $*$ will not belong to the universe of any structure.
- For $\overline{a} \in (A \cup \{*\})^s$, $\overline{a} = a_1 \dots a_s$, define the **support** $\mathrm{supp}(\overline{a})$ of $\overline{a}$ by

$$\mathrm{supp}(\overline{a}) := \{i : a_i \in A\}.$$

- If $a \in A$, let $\overline{a}\frac{a}{i}$ denote $a_1 \dots a_{i-1} a a_{i+1} \dots a_s$.
- For $a \in (A \cup \{*\})^s$ and $\overline{b} \in (B \cup \{*\})^s$, we say that $\overline{a} \mapsto \overline{b}$ is an s-**partial isomorphism** from $\mathcal{A}$ to $\mathcal{B}$, if:
  - $\mathrm{supp}(\overline{a}) = \mathrm{supp}(\overline{b})$;
  - $\overline{a}' \mapsto \overline{b}'$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$, where $\overline{a}'$ and $\overline{b}'$ are the subsequences of $\overline{a}$ and $\overline{b}$ with indices in the support.

Pebble Games

- Let $\mathcal{A}$ and $\mathcal{B}$ be structures.
- Let $\overline{a} \in (A \cup \{*\})^s$ and $\overline{b} \in (B \cup \{*\})^s$, with $\text{supp}(\overline{a}) = \text{supp}(\overline{b})$.
- In the **pebble game** $G^s_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$ we have:
  - s pebbles $\alpha_1, \ldots, \alpha_s$ for $\mathcal{A}$;
  - s pebbles $\beta_1, \ldots, \beta_s$ for $\mathcal{B}$.
- Initially, α_i is placed on a_i if $a_i \in A$, and off the board if $a_i = *$.

  Similarly, β_i is placed on $b_i \in B$ or off the board if $b_i = *$.
- Each play consists of m moves.
  - In his j-th move, the spoiler selects a structure, $\mathcal{A}$ or $\mathcal{B}$, and a pebble for this structure (being off the board or already placed on an element).
  - If he selects $\mathcal{A}$ and α_i, he places α_i on some element of $\mathcal{A}$. Then the duplicator places β_i on some element of $\mathcal{B}$.
  - If the spoiler selects $\mathcal{B}$ and β_i, he places β_i on an element of $\mathcal{B}$. The duplicator places α_i on some element of $\mathcal{A}$.
- Note that there may be several pebbles on the same element.

Winning and Losing a Pebble Game

- The **duplicator wins the pebble game** $G^s_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$ if, for each $j \leq m$, we have that $\overline{e} \mapsto \overline{f}$ is an s-partial isomorphism, where:
  - $\overline{e} = e_1 \ldots e_s$ are the elements marked by $\alpha_1, \ldots, \alpha_s$ after the j-th move ($e_i = *$ in case α_i is off the board);
  - $\overline{f} = f_1 \ldots f_s$ are the corresponding values given by $\beta_1, \ldots, \beta_s$.
- For $j = 0$ this means that $\overline{a} \mapsto \overline{b}$ is an s-partial isomorphism.
- The pebble game $G^s_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$, with infinitely many moves, is defined similarly.
- $G^s_m(\mathcal{A}, \mathcal{B})$ abbreviates $G^s_m(\mathcal{A}, * \ldots *, \mathcal{B}, * \ldots *)$.
- $G^s_\infty(\mathcal{A}, \mathcal{B})$ abbreviates $G^s_\infty(\mathcal{A}, * \ldots *, \mathcal{B}, * \ldots *)$.

Logics and Pebble Games

- We show that logics and the games fit together.
- When writing $\mathcal{A} \vDash \varphi[\overline{a}]$ for $\overline{a} \in (A \cup \{*\})^s$ we assume that the free variables of φ have indices in $\mathrm{supp}(\overline{a})$, i.e., $z \in \mathrm{supp}(\overline{a})$ if $v_i \in \mathrm{free}(\varphi)$.

Theorem

For structures $\mathcal{A}$ and $\mathcal{B}$, and for $\overline{a} \in (A \cup \{*\})^s$ and $\overline{b} \in (B \cup \{*\})^s$, with $\mathrm{supp}(\overline{a}) = \mathrm{supp}(\overline{b})$, the following hold:

(a) $\overline{a}$ satisfies in $\mathcal{A}$ the same FO^s-formulas of quantifier rank $\leq m$ as $\overline{b}$ in $\mathcal{B}$ iff the duplicator wins $G^s_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.

(b) $\overline{a}$ satisfies in $\mathcal{A}$ the same $L^s_{\infty\omega}$-formulas as $\overline{b}$ in $\mathcal{B}$ iff the duplicator wins $G^s_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.

In particular:

(c) $\mathcal{A} \equiv^s_m \mathcal{B}$ iff the duplicator wins $G^s_m(\mathcal{A}, \mathcal{B})$.

(d) $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$ iff the duplicator wins $G^s_\infty(\mathcal{A}, \mathcal{B})$.

Examples

(a) Let $\tau = \emptyset$.

Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures (i.e., sets).

If $\|A\|, \|B\| \geq s$, then the duplicator wins $G^s_\infty(\mathcal{A}, \mathcal{B})$.

Equivalently, $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$.

For arbitrary $\mathcal{A}$ and $\mathcal{B}$, the duplicator wins $G^s_\infty(\mathcal{A}, \mathcal{B})$ iff he wins $G^s_s(\mathcal{A}, \mathcal{B})$.

Examples (Cont'd)

(b) Suppose $\ell \geq 3$.

Let $\mathcal{G}_\ell$ be the graph consisting of one cycle of length $\ell + 1$.

Let $\mathcal{G}_\ell \uplus \mathcal{G}_\ell$ be the graph consisting of two cycles of length $\ell + 1$.

Then the duplicator wins $G^2_\infty(\mathcal{G}_\ell, \mathcal{G}_\ell \uplus \mathcal{G}_\ell)$.

Hence, by the theorem, $\mathcal{G}_\ell \equiv^{L^2_{\infty\omega}} \mathcal{G}_\ell \uplus \mathcal{G}_\ell$.

On the other hand, $\mathcal{G}_\ell \not\equiv^{L^3_{\infty\omega}} \mathcal{G}_\ell \uplus \mathcal{G}_\ell$.

Consider the $L^3_{\infty\omega}$-sentence

$$\forall x \forall y (x = y \vee \bigvee_{n>0} \varphi_n(x, y)),$$

where $\varphi_n(x, y)$ was introduced in a previous example.

It expresses connectivity.

Moreover, the spoiler wins $G^3_\infty(\mathcal{G}_\ell, \mathcal{G}_\ell \uplus \mathcal{G}_\ell)$.

s-m-Isomorphisms

Definition

Structures $\mathcal{A}$ and $\mathcal{B}$ are s-m-**isomorphic**, written $\mathcal{A} \cong^s_m \mathcal{B}$, iff there is a sequence $\{I_j\}_{j \leq m}$ of nonempty sets of s-partial isomorphisms with the following properties:

s-**forth property**: For $j < m$, $\overline{a} \mapsto \overline{b} \in I_{j+1}$, $1 \leq i \leq s$, and $a \in A$, there is $b \in B$, such that $\overline{a}\frac{a}{i} \mapsto \overline{b}\frac{b}{i} \in I_j$.

s-**back property**: For $j < m$, $\overline{a} \mapsto \overline{b} \in I_{j+1}$, $1 \leq i \leq s$, and $b \in B$, there is $a \in A$, such that $\overline{a}\frac{a}{i} \mapsto \overline{b}\frac{b}{i} \in I_j$.

We then write $(I_j)_{j \leq m} : \mathcal{A} \cong^s_m \mathcal{B}$.

- The notions s-**partially isomorphic**, $\mathcal{A} \cong^s_{\text{part}} \mathcal{B}$, and $I : \mathcal{A} \cong^s_{\text{part}} \mathcal{B}$ are defined similarly.

s-m-Isomorphism Types

- For $m \in \mathbb{N}$, any structure $\mathcal{A}$, and $\overline{a} \in (A \cup \{*\})^s$, the s-m-**isomorphism type** $\psi^m_{\overline{a}} (= {}^s\psi^m_{\mathcal{A},\overline{a}})$ **of $\overline{a}$ in $\mathcal{A}$** is given by:

$$\psi^0_{\overline{a}}(\overline{v}) := \bigwedge\{\psi : \psi \text{ atomic or negated atomic, and } \mathcal{A} \vDash \psi[\overline{a}]\}$$

(recall that when writing $\mathcal{A} \vDash \psi[\overline{a}]$ we assume that the free variables of ψ have indices in $\text{supp}(\overline{a})$);

$$\psi^{m+1}_{\overline{a}} := \psi^0_{\overline{a}} \wedge \bigwedge_{1 \le i \le s} \left(\bigwedge_{a \in A} \exists v_i \psi^m_{\overline{a}\frac{a}{i}} \wedge \forall v_i \bigvee_{a \in A} \psi^m_{\overline{a}\frac{a}{i}}\right).$$

- In particular, $\psi^m_{\mathcal{A}} := \psi^m_{*\ldots*}$ is an FO^s-sentence of quantifier rank m.

s-Partial Isomorphisms for Winning Positions

- The set $W^s_m(\mathcal{A},\mathcal{B})$ of s-partial isomorphisms corresponding to winning positions in the game $G^s_m(\mathcal{A},\overline{a},\mathcal{B},\overline{b})$ is given by

$$W^s_m(\mathcal{A},\mathcal{B}) := \{\overline{a} \mapsto \overline{b} : \text{the duplicator wins } G^s_m(\mathcal{A},\overline{a},\mathcal{B},\overline{b})\}.$$

- The set $W^s_\infty(\mathcal{A},\mathcal{B})$ of s-partial isomorphisms corresponding to winning positions in the game $G^s_\infty(\mathcal{A},\overline{a},\mathcal{B},\overline{b})$ is given by

$$W^s_\infty(\mathcal{A},\mathcal{B}) := \{\overline{a} \mapsto \overline{b} : \text{the duplicator wins } G^s_\infty(\mathcal{A},\overline{a},\mathcal{B},\overline{b})\}.$$

Formulas of $\mathcal{FO}^s$, Games, Isomorphisms and Types

Theorem

Let $\overline{a} \in (A \cup \{*\})^s$ and $\overline{b} \in (B \cup \{*\})^s$, with $\text{supp}(\overline{a}) = \text{supp}(\overline{b})$.

(a) The following are equivalent:
   (i) The duplicator wins $G^s_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$;
   (ii) $\overline{a} \mapsto \overline{b} \in W^s_m(\mathcal{A}, \mathcal{B})$ and $(W^s_j(\mathcal{A}, \mathcal{B}))_{j \leq m} : \mathcal{A} \cong^s_m \mathcal{B}$;
   (iii) There is $(I_j)_{j \leq m}$ with $\overline{a} \mapsto \overline{b} \in I_m$, such that $(I_j)_{j \leq m} : \mathcal{A} \cong^s_m \mathcal{B}$;
   (iv) $\mathcal{B} \models \psi^m_{\overline{a}}[\overline{b}]$;
   (v) $\overline{a}$ satisfies in $\mathcal{A}$ the same FO^s-formulas of quantifier rank $\leq m$ as $\overline{b}$ in $\mathcal{B}$.

(b) The following are equivalent:
   (i) The duplicator wins $G^s_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$;
   (ii) $\overline{a} \mapsto \overline{b} \in W^s_\infty(\mathcal{A}, \mathcal{B})$ and $W^s_\infty(\mathcal{A}, \mathcal{B}) : \mathcal{A} \cong^s_{\text{part}} \mathcal{B}$;
   (iii) There is I, with $\overline{a} \mapsto \overline{b} \in I$, such that $I : \mathcal{A} \cong^s_{\text{part}} \mathcal{B}$;
   (iv) $\overline{a}$ satisfies in $\mathcal{A}$ the same $L^s_{\infty\omega}$-formulas as $\overline{b}$ in $\mathcal{B}$.

The Special Case of Empty Sequences

Corollary

(a) The following are equivalent:

  (i) The duplicator wins $G^s_m(\mathcal{A},\mathcal{B})$;
  (ii) $(W^s_j(\mathcal{A},\mathcal{B}))_{j\leq m}:\mathcal{A}\cong^s_m\mathcal{B}$;
  (iii) $\mathcal{A}\cong^s_m\mathcal{B}$;
  (iv) $\mathcal{B}\vDash\psi^m_{\mathcal{A}}$;
  (v) $\mathcal{A}\equiv^s_m\mathcal{B}$.

(b) The following are equivalent:

  (i) The duplicator wins $G^s_\infty(\mathcal{A},\mathcal{B})$;
  (ii) $W^s_\infty(\mathcal{A},\mathcal{B}):\mathcal{A}\cong^s_{\text{part}}\mathcal{B}$;
  (iii) $\mathcal{A}\cong^s_{\text{part}}\mathcal{B}$;
  (iv) $\mathcal{A}\equiv^{L^s_{\infty\omega}}\mathcal{B}$.

Example

- For $r \geq 0$, let Δ_{r+1} be the set

$$\Delta_{r+1} := \{\varphi(v_1, \ldots, v_r, v_{r+1}) : \varphi \text{ has the form } R\overline{x}, \text{ where } R \in \tau \text{ and where } v_{r+1} \text{ occurs in } \overline{x}\}.$$

For a subset Φ of Δ_{r+1}, with $\Phi^c := \Delta_{r+1} \backslash \Phi$, recall that the sentence

$$\chi_\Phi := \forall v_1 \cdots \forall v_r (\bigwedge_{1 \leq i < j \leq r} v_i \neq v_j \rightarrow \exists v_{r+1} (\bigwedge_{1 \leq i \leq r} v_i \neq v_{r+1} \wedge \bigwedge_{\varphi \in \Phi} \varphi \wedge \bigwedge_{\varphi \in \Phi^c} \neg\varphi)),$$

is called an $(r+1)$-extension axiom.

For $s \geq 1$, let ϵ_s be the conjunction of the finitely many r-extension axioms with $r \leq s$. Clearly, $\epsilon_s \in FO^s$. We have:

- Every model of ϵ_s has at least s elements.
- Every two models $\mathcal{A}$ and $\mathcal{B}$ of ϵ_s are s-partially isomorphic.

Therefore, by the corollary, any two models of ϵ_s are $L^s_{\infty\omega}$-equivalent.

Consequently, or every $L^s_{\infty\omega}$-sentence φ, either $\epsilon_s \vDash \varphi$ or $\epsilon_s \vDash \neg\varphi$.

Axiomatizability in $L^s_{\infty\omega}$ and $L^\omega_{\infty\omega}$

Theorem

Let K be a class of finite structures.

(a) For $s \geq 1$ the following are equivalent:
 (i) K is not axiomatizable in $L^s_{\infty\omega}$;
 (ii) There are finite $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \in K$, $\mathcal{B} \notin K$, and $\mathcal{A} \cong^s_{\text{part}} \mathcal{B}$.

(b) The following are equivalent:
 (i) K is not axiomatizable in $L^\omega_{\infty\omega}$;
 (ii) For every $s \geq 1$, there are finite $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \in K$, $\mathcal{B} \notin K$ and $\mathcal{A} \cong^s_{\text{part}} \mathcal{B}$.

- Clearly, (b) follows from (a). To show (ii)⇒(i) in (a), suppose by contradiction, that $K = \text{Mod}(\varphi)$, for some $\varphi \in L^s_{\infty\omega}$. Choose $\mathcal{A}$ and $\mathcal{B}$ as given by (ii). Since $\mathcal{A} \in K$, $\mathcal{A} \vDash \varphi$. Since $\mathcal{B} \notin K$, $\mathcal{B} \nvDash \varphi$. Since $\mathcal{A} \cong^s_{\text{part}} \mathcal{B}$, $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$. This gives a contradiction.

Axiomatizability in $L^s_{\infty\omega}$ and $L^\omega_{\infty\omega}$ (Cont'd)

- Conversely, suppose that the condition in (ii) is not satisfied. Then for all finite $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{A} \in K \text{ and } \mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B} \text{ imply } \mathcal{B} \in K.$$

  Claim: $K = \mathrm{Mod}(\varphi)$ for the $L^s_{\infty\omega}$-sentence $\varphi := \bigvee_{\mathcal{A}\in K} \bigwedge_{m\geq 0} \psi^m_{\mathcal{A}}$.

  Clearly, $K \subseteq \mathrm{Mod}(\varphi)$, since $\mathcal{B} \vDash \bigwedge_{m\geq 0} \psi^m_{\mathcal{B}}$ holds for any $\mathcal{B}$.

  To obtain $\mathrm{Mod}(\varphi) \subseteq K$, assume that $\mathcal{B}$ is a finite model of φ.

  Then, for some $\mathcal{A} \in K$ and all $m \geq 0$, we have $\mathcal{B} \vDash \psi^m_{\mathcal{A}}$.

  This gives $\mathcal{A} \equiv^s_m \mathcal{B}$. Thus, $A \equiv^s \mathcal{B}$.

  By a previous theorem, we get $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$.

  By hypothesis, we get $\mathcal{B} \in K$.

Example

- Let τ be the empty vocabulary.

  For $s \geq 1$, consider structures $\mathcal{A}$ and $\mathcal{B}$ with

  $$\|A\| = s \quad \text{and} \quad \|B\| = s + 1.$$

  We have:
  - $\mathcal{A} \in \text{EVEN}[\tau]$ iff $\mathcal{B} \notin \text{EVEN}[\tau]$;
  - $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$, by a previous example.

  We conclude that $\text{EVEN}[\tau]$ is not $L^{\omega}_{\infty\omega}$-axiomatizable.

The Equivalence Relation $\sim$

- For simplicity, let τ be relational.
- Let $\mathcal{A}$ be a τ-structure.
- The binary relation $\sim$ defined on A^s by

$$\overline{a} \sim \overline{b} \quad \text{iff} \quad \overline{a} \text{ and } \overline{b} \text{ satisfy the same } L^s_{\infty\omega}\text{-formulas in } \mathcal{A}$$

  is an equivalence relation on A^s.
- By a previous theorem, $\overline{a} \sim \overline{b}$ iff the duplicator wins $G^s_\infty(\mathcal{A}, \overline{a}, \mathcal{A}, \overline{b})$.
- Let $[\overline{a}]$ denote the equivalence class of $\overline{a}$.
- Let

$$A/s := \{[\overline{a}] : \overline{a} \in A^s\}$$

  be the set of equivalence classes.

The s-Invariant $\mathcal{A}/s$ of $\mathcal{A}$

- We endow A/s with a τ/s-structure $\mathcal{A}/s$.
- For every $[\overline{a}] \in A/s$, the relations on A/s capture the properties of $\overline{a}$ in any game $G^s_\infty(\mathcal{A}, \overline{a}, \ldots)$.
- The relation symbols in τ/s (and their meaning in $\mathcal{A}/s$) are:
  - For every k-ary $R \in \tau \cup \{=\}$ and any $i_1, \ldots, i_k$ with $1 \le i_1, \ldots, i_k \le s$, a unary relation symbol $R_{i_1 \ldots i_k}$;

    $$R^{A/s}_{i_1 \ldots i_k} := \{[\overline{a}] : \overline{a} \in A^s, R^{\mathcal{A}} a_{i_1} \ldots a_{i_k}\}$$

    (the $R^{A/s}_{i_1 \ldots i_k}$ capture the isomorphism type of $\overline{a}$);
  - For $i = 1, \ldots, s$ a binary relation symbol S_i;

    $$S^{A/s}_i := \{([\overline{a}], [\overline{a}']) : \overline{a}, \overline{a}' \in A^s, \text{ there is } a \in A \text{ such that } [\overline{a}'] = [\overline{a}\tfrac{a}{i}]\}$$

    ($S^{A/s}_i$ encodes the possible moves of the i-th pebble).
- $\mathcal{A}/s$ is called the s-**invariant** of $\mathcal{A}$.

Equivalence and s-Invariant Structures

Theorem

For structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$ iff $\mathcal{A}/s \cong \mathcal{B}/s$.

- Suppose, first, that $\pi : \mathcal{A}/s \cong \mathcal{B}/s$. Set

$$I := \{\overline{a} \mapsto \overline{b} : \overline{a} \in A^s, \overline{b} \in B^s, \pi([\overline{a}]) = [\overline{b}]\}.$$

We show that $I : \mathcal{A} \cong^s_{\text{part}} \mathcal{B}$. This guarantees $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$.

I is a nonempty set of s-partial isomorphisms (use the $R_{i_1 \dots i_k}$'s).

For the s-forth property, assume that $\overline{a} \mapsto \overline{b} \in I$, $1 \leq i \leq s$, and $a \in A$.

Then $S_i^{A/s}[\overline{a}][\overline{a}\frac{a}{i}]$. Hence $S_i^{B/s}[\overline{b}]\pi([\overline{a}\frac{a}{i}])$.

By the definition of $S_i^{B/s}$, there is $b \in B$, such that $[\overline{b}\frac{b}{i}] = \pi([\overline{a}\frac{a}{i}])$.

Hence $\overline{a}\frac{a}{i} \mapsto \overline{b}\frac{b}{i} \in I$.

Equivalence and s-Invariant Structures (Converse)

- Conversely suppose that $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$. Then $W^s_\infty(\mathcal{A},\mathcal{B}) : \mathcal{A} \cong^s_{\text{part}} \mathcal{B}$.

  For $\overline{a} \in A^s$ and $\overline{b} \in B^s$ set

  $$\pi([\overline{a}]) := [\overline{b}] \quad \text{iff} \quad \overline{a} \mapsto \overline{b} \in W^s_\infty(\mathcal{A},\mathcal{B}).$$

  Hence, by a previous theorem, $\pi([\overline{a}]) = [\overline{b}]$ iff $\overline{a}$ in $\mathcal{A}$ satisfies the same $L^s_{\infty\omega}$-formulas as $\overline{b}$ in $\mathcal{B}$.

  - By this equivalence and by the definition of $\sim$, π is well-defined and injective.
  - $\text{dom}(\pi) = A/s$, by the s-forth property of $W^s_\infty(\mathcal{A},\mathcal{B})$.
  - $\text{ran}(\pi) = B/s$, by the s-back property of $W^s_\infty(\mathcal{A},\mathcal{B})$.

  Obviously, π is compatible with the interpretations of the $R_{i_1 \ldots i_k}$.

  It is also compatible with the interpretations of the S_i (use once more the s-back and s-forth properties of $W^s_\infty(\mathcal{A},\mathcal{B})$).

  Therefore, $\pi : \mathcal{A}/s \cong \mathcal{B}/s$.

Results Involving FO^s

- For finite structures we can replace $L^s_{\infty\omega}$ by FO^s.
- Recall that, for finite $\mathcal{A}$, $\mathcal{B}$, for every $L^s_{\infty\omega}$-formula φ, there is an FO^s-formula ψ, with $\mathrm{free}(\psi) \subseteq \mathrm{free}(\varphi)$, such that

$$\mathcal{A} \vDash \forall x_1 \cdots \forall x_s (\varphi \leftrightarrow \psi) \quad \text{and} \quad \mathcal{A} \vDash \forall x_1 \cdots \forall x_s (\varphi \leftrightarrow \psi).$$

- So, by the theorem, we get

Proposition

Let $\mathcal{A}$ and $\mathcal{B}$ be finite structures, $\overline{a} \in (A \cup \{*\})^s$ and $\overline{b} \in (B \cup \{*\})^s$, with $\mathrm{supp}(\overline{a}) = \mathrm{supp}(\overline{b})$. If for all $\varphi \in \mathrm{FO}^s$,

$$\mathcal{A} \vDash \varphi[\overline{a}] \quad \text{iff} \quad \mathcal{B} \vDash \varphi[\overline{b}],$$

then for all $\varphi \in L^s_{\infty\omega}$,

$$\mathcal{A} \vDash \varphi[\overline{a}] \quad \text{iff} \quad \mathcal{B} \vDash \varphi[\overline{b}].$$

Results Involving FO^s

Corollary

(a) Let $\mathcal{A}$ be a finite structure and $\sim$ be defined as before. Then, for $\overline{a}, \overline{b} \in A^s$,

$$\overline{a} \sim \overline{b} \quad \text{iff} \quad \overline{a} \text{ and } \overline{b} \text{ satisfy the same } \mathrm{FO}^s\text{-sentences.}$$

(b) For finite structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \equiv^s \mathcal{B}$ iff $\mathcal{A}/s \cong \mathcal{B}/s$.

(a) This is immediate from the preceding proposition.

(b) This follows from the preceding theorem and the fact, shown previously, that, for finite $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{A} \equiv^s \mathcal{B} \quad \text{implies} \quad \mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}.$$

Hierarchy of Winning Sets

Proposition

Let $\mathcal{A}$ and $\mathcal{B}$ be structures. Then:

(a) $W^s_0(\mathcal{A},\mathcal{B}) \supseteq W^s_1(\mathcal{A},\mathcal{B}) \supseteq \cdots$.

(b) If $\mathcal{A}$ and $\mathcal{B}$ are finite, then there is an $m \leq (\|A\|+1)^s \cdot (\|B\|+1)^s$, such that $W^s_m(\mathcal{A},\mathcal{B}) = W^s_{m+1}(\mathcal{A},\mathcal{B})$.

(c) For $m \geq 0$, if $W^s_m(\mathcal{A},\mathcal{B}) = W^s_{m+1}(\mathcal{A},\mathcal{B})$ and $W^s_m(\mathcal{A},\mathcal{B})$ is nonempty, then $W^s_m(\mathcal{A},\mathcal{B}) : \mathcal{A} \cong_{\text{part}} \mathcal{B}$.

(a) Follows immediately from the definition of the $W^s_j(\mathcal{A},\mathcal{B})$.

(b) Follows from (a), since there are at most $(\|A\|+1)^s \cdot (\|B\|+1)^s$ s-partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

Hierarchy of Winning Sets (Cont'd)

(c) Suppose that $W^s_m(\mathcal{A},\mathcal{B}) = W^s_{m+1}(\mathcal{A},\mathcal{B})$.

We claim that $W^s_m(\mathcal{A},\mathcal{B})$ has the s-back and the s-forth property.

We show, say, the s-forth property.

Let $\overline{a} \mapsto \overline{b} \in W^s_m(\mathcal{A},\mathcal{B})$, $1 \leq i \leq s$, and $a \in A$.

By assumption, $\overline{a} \mapsto \overline{b} \in W^s_{m+1}(\mathcal{A},\mathcal{B})$.

Thus, there is $b \in B$, such that $\overline{a}\frac{a}{i} \mapsto \overline{b}\frac{b}{i} \in W^s_m(\mathcal{A},\mathcal{B})$.

By hypothesis, $W^s_m(\mathcal{A},\mathcal{B}) \neq \varnothing$.

So we have $W^s_m(\mathcal{A},\mathcal{B}) : \mathcal{A} \cong_{\text{part}} \mathcal{B}$.

The s-Rank of a Finite Structure

- Fix a finite structure $\mathcal{A}$.
- Let $\overline{a}$, $\overline{b}$ range over $(A \cup \{*\})^s$.
- By the proposition we know that:
  - $W^s_0(\mathcal{A},\mathcal{A}) \supseteq W^s_1(\mathcal{A},\mathcal{A}) \supseteq \cdots \supseteq W^s_m(\mathcal{A},\mathcal{A}) \supseteq \cdots$
  - There exists j, such that $W^s_j(\mathcal{A},\mathcal{A}) = W^s_{j+1}(\mathcal{A},\mathcal{A})$.
- The minimal such j is called the s-**rank** $r(\mathcal{A})$ of $\mathcal{A}$, $r(\mathcal{A}) = r(s,\mathcal{A})$.

The s-Scott Formula of a Tuple in a Finite Structure

- Fix a finite structure $\mathcal{A}$.
- Let $\overline{a} \in (A \cup \{*\})^s$.
- Consider the formula

$$\sigma_{\overline{a}} := \psi_{\overline{a}}^{r(\mathcal{A})} \wedge \bigwedge_{\overline{b} \in (\mathcal{A} \cup \{*\})^s} \forall v_1 \cdots \forall v_s (\psi_{\overline{b}}^{r(\mathcal{A})} \to \psi_{\overline{b}}^{r(\mathcal{A})+1})$$

(more exactly, $\sigma_{\overline{a}} = {}^s\sigma_{\mathcal{A},\overline{a}}$).

- $\sigma_{\overline{a}}$ is called the s-**Scott formula of** $\overline{a}$ **in** $\mathcal{A}$.
- It is an FO^s-formula of quantifier rank $r(\mathcal{A}) + 1 + s$.
- In particular, $\sigma_{\mathcal{A}} := \sigma_{*\ldots*}$ is an FO^s-sentence.
- We show that it captures the whole $L^s_{\infty\omega}$-theory of $\mathcal{A}$.

Property of the s-Scott Formula

Theorem

Let $\mathcal{A}$ be finite.

(a) For any structure $\mathcal{B}$, $\mathcal{B} \vDash \sigma_{\mathcal{A}}$ iff $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$.

(b) For $\overline{a} \in (\mathcal{A} \cup \{*\})^s$, any structure $\mathcal{B}$ and $\overline{b} \in (\mathcal{B} \cup \{*\})^s$ with $\mathrm{supp}(\overline{a}) = \mathrm{supp}(\overline{b})$, $\mathcal{B} \vDash \sigma_{\overline{a}}[\overline{b}]$ iff $\overline{a}$ satisfies in $\mathcal{A}$ the same $L^s_{\infty\omega}$-formulas as $\overline{b}$ in $\mathcal{B}$.

- We only prove Part (a).
  Since $\mathcal{A} \vDash \sigma_{\mathcal{A}}$, we have that $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$ implies $\mathcal{B} \vDash \sigma_{\mathcal{A}}$.
  Next, supose $\mathcal{B} \vDash \psi^{r(\mathcal{A})}_{\mathcal{A}} \wedge \bigwedge_{\overline{b} \in (\mathcal{A} \cup \{*\})^s} \forall v_1 \cdots \forall v_s (\psi^{r(\mathcal{A})}_{\overline{b}} \to \psi^{r(\mathcal{A})+1}_{\overline{b}})$.
  Since $\mathcal{B} \vDash \psi^{r(\mathcal{A})}_{\mathcal{A}}$, we get $* \ldots * \mapsto * \ldots * \in W^s_{r(\mathcal{A})}(\mathcal{A}, \mathcal{B})$.
  Since the second conjunct holds in $\mathcal{B}$, $W^s_{r(\mathcal{A})}(\mathcal{A}, \mathcal{B}) \subseteq W^s_{r(\mathcal{A})+1}(\mathcal{A}, \mathcal{B})$.
  Hence, $W^s_{r(\mathcal{A})}(\mathcal{A}, \mathcal{B}) = W^s_{r(\mathcal{A})+1}(\mathcal{A}, \mathcal{B})$. Therefore, by the preceding proposition, $W^s_{r(\mathcal{A})}(\mathcal{A}, \mathcal{B}) : \mathcal{A} \cong^s_{\mathrm{part}} \mathcal{B}$. Thus, $\mathcal{A} \equiv^{L^s_{\infty\omega}} \mathcal{B}$.

A Consequence

Corollary

In the finite, each $L^s_{\infty\omega}$-formula φ is equivalent to a countable disjunction of FO^s-formulas. In fact, φ is equivalent to the $L^s_{\infty\omega}$-formula

$$\bigvee\{\sigma_{\overline{a}} : \mathcal{A} \text{ finite}, \overline{a} \in A, \mathcal{A} \vDash \varphi[\overline{a}]\}.$$

Moreover, if K is any class of finite structures, then φ and

$$\bigvee\{\sigma_{\overline{a}} : \mathcal{A} \in K, \overline{a} \in A, \mathcal{A} \vDash \varphi[\overline{a}]\}$$

are equivalent in all structures of K.

Boundedness

- Let K be a class of finite structures.
- We say that K is s-**bounded** if the set $\{r(\mathcal{A}) : \mathcal{A} \in K\}$ of s-ranks of structures in K is bounded.
- The class K is **bounded** if it is s-bounded for every $s \geq 1$.

Theorem

Let K he a class of finite structures.

(a) For $s \geq 1$ the following are equivalent:

  (i) K is s-bounded.
  (ii) On K, every $L^s_{\infty\omega}$-formula is equivalent to an FO^s-formula.
  (iii) On K, every $L^s_{\infty\omega}$-formula is equivalent to an FO-formula.

(b) K is hounded iff FO and $L^\omega_{\infty\omega}$ have the same expressive power on K.

- Part (b) is a consequence of (a). So it suffices to prove (a).
  Assume K is s-bounded. Set $m := \sup\{r(\mathcal{A}) : \mathcal{A} \in K\} < \infty$.
  Thus, for $\mathcal{A} \in K$ and $\overline{a}$ in $\mathcal{A}$ the quantifier rank of $\sigma_{\overline{a}}$ is $\leq m + s + 1$.

Boundedness (Cont'd)

- Let φ be any $L^s_{\infty\omega}$-formula.

  Then the disjunction in the preceding corollary is a disjunction of formulas of quantifier rank $\leq m+s+1$. Hence, it is a finite one.

  This shows that (i) implies (ii).

  The implication from (ii) to (iii) is trivial.

  Finally, we show that (iii) implies (i)

  Assume, towards a contradiction, that K is not s-bounded.

  Let $\mathcal{A}_0, \mathcal{A}_1, \ldots$ be structures in K of pairwise distinct s-rank.

  For $M \subseteq \mathbb{N}$, let $\varphi_M := \bigvee\{\sigma_{\mathcal{A}_i} : i \in M\}$.

  By the preceding proposition, if $L, M \subseteq \mathbb{N}$ and $L \neq M$, then it is not the case that

  $$K \vDash \varphi_L \leftrightarrow \varphi_M.$$

  Hence on K, $L^s_{\infty\omega}$ contains uncountably many pairwise nonequivalent sentences. So on K, $L^s_{\infty\omega}$ is more expressive than FO.

Example

- Suppose $\tau = \varnothing$.

  Let $\mathcal{A}$ be a τ-structure.

  Then $W_0^s(\mathcal{A}, \mathcal{A}) = W_1^s(\mathcal{A}, \mathcal{A})$.

  Hence, $r(\mathcal{A}) = 0$.

  So the class K of finite τ-structures is bounded.

  Therefore, FO and $L^\omega_{\infty\omega}$ have the same expressive power on K.

Subsection 4

Logics with Counting Quantifiers

Example

- To express in first-order logic that there are, say, seven elements with the property $\varphi(x)$ we need, in general, at least seven quantifiers:

$$\exists x_1 \cdots \exists x_7 (\varphi(x_1) \wedge \cdots \wedge \varphi(x_7) \wedge \bigwedge_{1 \leq i < j \leq 7} \neg x_i = x_j).$$

- By a previous example, we see that in case $\varphi(x) := x = x$ we really need seven quantifiers.

The Logics FO(C) and $L_{\infty\omega}(C)$

- Let FO(C), **first-order logic with counting quantifiers**, be the logic obtained from FO by adding, for every $\ell \geq 1$, a new quantifier $\exists^{\geq \ell}$ with the intended interpretation "there exist at least ℓ".
- Let $L_{\infty\omega}(C)$ (for short, $C_{\infty\omega}$), $L_{\infty\omega}$ **with counting quantifiers**, be the logic obtained from $L_{\infty\omega}$ by adding, for every $\ell \geq 1$, a new quantifier $\exists^{\geq \ell}$ with the intended interpretation "there exist at least ℓ".
- More precisely, extend the calculus of formulas for first-order or infinitary logic by the following rule:
  - If φ is a formula and $\ell \geq 1$, then $\exists^{\geq \ell} x \varphi$ is a formula.
- $\exists^{\geq \ell} x$ is considered as a new quantifier and not as an abbreviation.

The Logics FO(C) and $L_{\infty\omega}(C)$ (Cont'd)

- For the interpretation of these quantifiers we add, for $\varphi = \varphi(\overline{x}, x)$ and $\overline{a} \in A$, the clause

$$\mathcal{A} \models \exists^{\geq \ell} x \varphi[\overline{a}] \quad \text{iff} \quad \|\{b \in A : \mathcal{A} \models \varphi[\overline{a}, b]\}\| \geq \ell.$$

- Since the quantifiers $\exists^{\geq \ell}$ are first-order definable, the languages FO(C) and $C_{\infty\omega}$ have the same expressive power as FO and $L_{\infty\omega}$, respectively.

The Logics $\mathrm{FO}(C)^s$, $C^s_{\infty\omega}$ and $C^\omega_{\infty\omega}$

- The situation concerning expressive power changes if we restrict to $\mathrm{FO}(C)^s$ and $C^s_{\infty\omega}$, the fragments consisting of the formulas with variables among $v_1, \ldots, v_s$.

  Example: $\exists^{\geq \ell} x x = x$ is a sentence in $\mathrm{FO}(C)^1$ not equivalent to any sentence in FO^1.

  The sentence

  $$\bigvee_{\ell \geq 1} (\exists^{\geq 2\ell} x x = x \wedge \neg \exists^{\geq 2\ell+1} x x = x)$$

  is a $C^1_{\infty\omega}$-sentence axiomatizing the class $\mathrm{EVEN}[\tau]$ of structures of even cardinality that is not equivalent to any sentence of $L^\omega_{\infty\omega}$.
- Define

  $$C^\omega_{\infty\omega} := \bigcup_{s \geq 1} C^s_{\infty\omega}.$$

The Quantifiers $\exists^{\geq \ell}$ versus $\exists^{=\ell}$

- For $\ell \geq 1$, set
$$\exists^{=\ell} x\varphi := \exists^{\geq \ell} x\varphi \wedge \neg \exists^{\geq \ell+1} x\varphi.$$
- Let
$$\exists^{=0} x\varphi := \forall x \neg \varphi.$$
- Then $\exists^{\geq \ell} x\varphi$ is equivalent to $\neg \bigvee_{j<\ell} \exists^{=j} x\varphi$.
- Hence, we would obtain logics of the same expressive power when adding the quantifiers $\exists^{=\ell}$ instead of $\exists^{\geq \ell}$.

Examples

(a) Suppose $\mathcal{A}$ and $\mathcal{B}$ are finite structures, such that $\mathcal{A} \equiv^{\mathrm{FO}(C)^1} \mathcal{B}$.

That is, $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of $\mathrm{FO}(C)^1$.

Then $\|A\| = \|B\|$.

It suffices to observe that $\exists^{=\|A\|} x x = x$ is a sentence in $\mathrm{FO}(C)^1$.

(b) Let $\tau = \{<\}$.

Consider the sentence of $\mathrm{FO}[\tau]$

$$\forall x \neg x < x \wedge \forall x \forall y \forall z ((x < y \wedge y < z) \to x < z) \wedge$$
$$\forall x \forall y \forall z ((y < x \wedge z < x) \to (y < z \vee y = z \vee z < y)).$$

It asserts that $<$ is irreflexive and transitive, and the predecessors of any element are linearly ordered.

Its finite models are called **finite $<$-forests**.

Examples (Cont'd)

- For a <-forest $\mathcal{A}$ and $a \in \mathcal{A}$ the **height** $h_{\mathcal{A}}(a)$ is defined by

$$h_{\mathcal{A}}(a) := \|\{b \in A : b < a\}\|.$$

The **height** $h(\mathcal{A})$ is defined by

$$h(\mathcal{A}) := \max\{h_{\mathcal{A}}(a) : a \in A\}.$$

The element a is a **root** if $h_{\mathcal{A}}(a) = 0$.

Every finite <-forest can be characterized, up to isomorphism, in $\text{FO}(C)^2$.

Claim: For every finite <-forest $\mathcal{A}$, there is a sentence φ in $\text{FO}(C)^2$, such that for all finite <-forests $\mathcal{B}$, $\mathcal{B} \vDash \varphi$ iff $\mathcal{B} \cong \mathcal{A}$.

Examples (Cont'd)

- To prove this one shows by induction on the height, that for <-forests $\mathcal{A}$, with exactly one root, there is a formula $\psi_{\mathcal{A}}(x)$ in $\mathrm{FO}(C)^2$, such that, for any <-forest $\mathcal{B}$ and $b \in B$,

$$B \vDash \psi_{\mathcal{A}}[b] \quad \text{iff} \quad \mathcal{B}_b \cong \mathcal{A},$$

  where $\mathcal{B}_b$ is the substructure of $\mathcal{B}$ with universe $\{b' \in B : b = b' \vee b < b'\}$.

  In the induction step, for $\mathcal{A}$ with root a, $\psi_{\mathcal{A}}(x)$ gives:
  - The number of elements of A;
  - For any isomorphism type of some $\mathcal{A}_b$ with $b \in A \backslash \{a\}$, the number of trees $\mathcal{A}_c$, with $c \in A \backslash \{a\}$, that are of this type.

(c) For $s \geq 1$ there are <-forests $\mathcal{A}$ and $\mathcal{B}$ that satisfy the same sentences in FO^s but are not isomorphic. E.g., <-forests consisting only of roots, the first one having s roots, the second one $s + 1$ roots.

Ehrenfeucht-Fraïssë Games for Counting Quantifers

- We consider the pebble games $\text{C-G}_m^s(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$, with m moves, and $\text{C-G}_\infty^s(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$, with infinitely many moves.
- In each of these games, each move consists of two steps:
  1. The spoiler chooses:
     - One of the two structures, say $\mathcal{A}$;
     - A corresponding pebble, say α_i;
     - A subset X of A.

     The duplicator must answer with a subset Y of B, with $\|Y\| = \|X\|$.
  2. The spoiler places β_i on some element $b \in Y$.
     The duplicator answers by placing α_i on some $a \in X$.
     (X and Y can now be forgotten.)
- The definition for winning is given as in the previous pebble games.

  It only takes into consideration the chosen elements, not the subsets.

Explanation

- We explain the significance of the two steps of a move in C-$G^s_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$, and C-$G^s_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.
- Suppose that the spoiler attempts to show that

$$\mathcal{A} \vDash \exists^{\geq \ell} x \varphi(x), \text{ but not } \mathcal{B} \vDash \exists^{\geq \ell} x \varphi(x).$$

- He chooses a subset X consisting of ℓ elements witnessing that $\mathcal{A} \vDash \exists^{\geq \ell} x \varphi(x)$.
- The duplicator claims that the elements of the subset Y witness that $\mathcal{B} \vDash \exists^{\geq \ell} x \varphi(x)$.
- According to the spoilers conviction, there is a $b \in Y$ with not $\mathcal{B} \vDash \varphi[b]$.
- The duplicator means that some element a in X behaves as b.

Equivalence and Games

- In a way that parallels previous results, one can show the following

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be structures, $\overline{a} \in (A \cup \{*\})^s$ and $\overline{b} \in (B \cup \{*\})^s$, with $\text{supp}(\overline{a}) = \text{supp}(\overline{b})$.

(a) The following are equivalent:

  (i) For all $\varphi(\overline{x}) \in \text{FO}(C)^s_m$, $\mathcal{A} \vDash \varphi[\overline{a}]$ iff $\mathcal{B} \vDash \varphi[\overline{b}]$.
  (ii) The duplicator wins $\text{C-G}^s_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.

(b) $\mathcal{A} \equiv^{\text{FO}(C)^s_m} \mathcal{B}$ iff the duplicator wins $\text{C-G}^s_m(\mathcal{A}, \mathcal{B})$.

(c) The following are equivalent:

  (i) For all $\varphi(\overline{x}) \in C^s_{\infty\omega}$, $\mathcal{A} \vDash \varphi[\overline{a}]$ iff $\mathcal{B} \vDash \varphi[\overline{b}]$.
  (ii) The duplicator wins $\text{C-G}^s_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$.

(d) $\mathcal{A} \equiv^{C^s_{\infty\omega}} \mathcal{B}$ iff the duplicator wins $\text{C-G}^s_\infty(\mathcal{A}, \mathcal{B})$.

Colored Graphs and Color Types

- Let $C_1, C_2, \dots$ be unary relation symbols, the "color relations".
- A **colored graph** is, for some r, an $\{E, C_1, \dots, C_r\}$-structure $\mathcal{G}$, where, for $\mathcal{G} = (G, E^G, C_1^G, \dots, C_r^G)$, the following holds:
  - (G, E^G) is a graph;
  - $C_1^G \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} C_r^G = G$, i.e., each vertex satisfies exactly one color relation.
- For $a \in G$, the **color type** $\mathrm{ct}(a)$ is defined as

$$\mathrm{ct}(a) := (i, n_1, \dots, n_r),$$

  where:
  - $a \in C_i^G$;
  - $n_j := \|\{b \in C_j^G : E^G ab\}\|$.
- $\mathcal{G}$ is **stable** if for $a, b \in G$, $\mathrm{ct}(a) = \mathrm{ct}(b)$ iff $a, b \in C_i$, for some i.

Property of Stable Colored Graphs

Proposition

Let $\mathcal{G} = (G, E^G, C_1^G, \dots, C_r^G)$ be a stable colored graph and $a, b \in G$. Then the following are equivalent:

(i) For $j = 1, \dots, r$, $a \in C_j$ iff $b \in C_j$.

(ii) The duplicator has a winning strategy in the game $\text{C-G}_\infty^2(\mathcal{G}, a*, \mathcal{G}, b*)$.

- Suppose, first, that the duplicator wins $\text{C-G}_\infty^2(\mathcal{G}, a*, \mathcal{G}, b*)$.

  Then, by the preceding theorem, for all $\varphi(x)$ in $C_{\infty\omega}^2$, $\mathcal{G} \vDash \varphi[a]$ iff $\mathcal{G} \vDash \varphi[b]$.

  Taking $\varphi(x)$ to be $C_j(x)$, we obtain $\mathcal{G} \vDash C_j[a]$ iff $\mathcal{G} \vDash C_j[b]$.

  Equivalently, $a \in C_j^G$ iff $b \in C_j^G$.

Property of Stable Colored Graphs (Cont'd)

- Suppose, conversely, that, for all j, $a \in C_j^G$ iff $b \in C_j^G$.

  It suffices to show that a and b satisfy the same $C_{\infty\omega}^2$ formulas in $\mathcal{G}$.

  We do this by induction on the structure of a $C_{\infty\omega}^2$ formula.

  The hypothesis ensures that $\mathcal{G} \vDash C_j[a]$ iff $\mathcal{G} \vDash C_j[b]$.

  By the definition of E and stability, $(a,a),(b,b) \notin E^G$, $(a,b) \in E^G$ iff $(b,a) \in E^G$ and $\mathcal{G} \vDash E[a,c]$ iff $\mathcal{G} \vDash E[b,c]$, for all $c \neq a,b$.
  - The cases of $\neg$ and $\bigvee$ are easy.
  - Consider now the formula $\exists y \varphi(x,y)$. Assume $\mathcal{G} \vDash \exists y \varphi(x,y)[a]$. Then there exists c, such that $\mathcal{G} \vDash \varphi[a,c]$.
    - If $c = a$, then, by the induction hypothesis, $\mathcal{G} \vDash \varphi[b,b]$. So $\mathcal{G} \vDash \exists y \varphi(x,y)[b]$.
    - If $c = b$, then, by the induction hypothesis, $\mathcal{G} \vDash \varphi[b,a]$. So $\mathcal{G} \vDash \exists y \varphi(x,y)[b]$.
    - If $c \neq a,b$, then, by the induction hypothesis, $\mathcal{G} \vDash \varphi[b,c]$. So $\mathcal{G} \vDash \exists y \varphi(x,y)[b]$.
  - $\exists^{\geq \ell} y \varphi(x,y)$ may be handled similarly.

From a Colored Graph to a Stable Colored Graph

- We introduce a process of **color refinement** leading from a colored graph $\mathcal{G} = (G, E^G, C_1^G, \ldots, C_r^G)$ to a stable colored graph.
- Let $m := \|\{\mathrm{ct}(a) : a \in G\}\|$.
- Order the set $\{\mathrm{ct}(a) : a \in G\}$ lexicographically.
- Set $\mathcal{G}' := (G, E^G, C_1', \ldots, C_m')$, where C_k' is the set of elements $a \in G$, such that $\mathrm{ct}(a)$ is the k-th element in this ordering.
- Clearly, each C_i^G is the union of some C_k'.
- Let C_k' be the color class of elements of color type $(i, n_1, \ldots, n_r)$.
- C_k' is definable in $\mathcal{G}$ by a formula of $C^2_{\infty\omega}$ of quantifier rank ≤ 1

$$C_k' = \left\{ a \in G : \mathcal{G} \vDash \left(C_i x \wedge \bigwedge_{j=1,\ldots,r} \exists^{=n_j} y (Exy \wedge C_j y) \right) [a] \right\}.$$

- Note that we have extended the definition of **quantifier rank** for first-order logic by the clause $\mathrm{qr}(\exists^{\geq \ell} x \varphi) := 1 + \mathrm{qr}(\varphi)$.

The Stable Colored Refinement of a Colored Graph

- Obviously, $\mathcal{G}$ is stable if $m = r$, i.e., if there is no proper color refinement.
- If $\mathcal{G}'$ is not stable, we define $\mathcal{G}^{(2)} := (\mathcal{G}')'$.
- We continue by defining

$$\mathcal{G}^{(i+1)} := (\mathcal{G}^{(i)})', \quad i = 2, 3, \ldots.$$

- Since each C_i^G is the union of some C_k', after finitely many, say n, steps, we must reach a stable colored graph $\mathcal{G}^{(n)}$.
- $\mathcal{G}^{(n)}$ is called the **stable colored refinement** of $\mathcal{G}$.

Colored Graph and Colored Refinement

Theorem

For elements a and b of a colored graph $\mathcal{G}$ the following are equivalent:

(i) a, b are in the same color class of the stable colored refinement of $\mathcal{G}$.

(ii) For all $\varphi(x) \in C^2_{\infty\omega}$, $\mathcal{G} \vDash \varphi[a]$ iff $\mathcal{G} \vDash \varphi[b]$.

- An induction, using the formula defining C'_k shows that each color class of $\mathcal{G}^{(n)}$ is definable by a $C^2_{\infty\omega}$-formula of quantifier rank $\leq n$.

  This fact yields (ii)⇒(i) of the theorem.

  Conversely, note that a winning strategy for the colored refinement of $\mathcal{G}$ is a winning strategy for $\mathcal{G}$.

  Consequently, (i)⇒(ii) follows from the preceding proposition.

A Consequence

- For a graph $\mathcal{G} = (G, E^G)$, let the stable colored refinement be that of the colored graph (G, E^G, G).

Corollary

For elements a and b of a graph $\mathcal{G}$ the following are equivalent:

(i) a, b are in the same color class of the stable colored refinement of $\mathcal{G}$.

(ii) For all $\varphi(x) \in C^2_{\infty\omega}$, $\mathcal{G} \vDash \varphi[a]$ iff $\mathcal{G} \vDash \varphi[b]$.

Subsection 5

Failure of Classical Theorems in the Finite

Introduction

- Many known first-order logic results and techniques fail in the finite:
  - The Compactness Theorem fails (as we have seen);
  - There is no sound and complete proof calculus;
  - Ultraproducts and saturated structures become useless;
  - Beth's Definability Theorem and Craig's Interpolation Theorem fail when restricted to finite structures.
- Nevertheless, new methods and results intrinsic to the finite compensate for this loss.
  - Combinatorics has a strong impact, in particular, in connection with probabilities;
  - The restriction to the finite motivates the use of other languages, for example languages that are able to grasp notions of recursion or induction, building a bridge to computational aspects.

Implicit and Explicit Definability

- Let $\mathcal{L}$ be any logic considered so far, e.g., FO, $L_{\infty\omega}$, ...
- Let R be an n-ary relation symbol not contained in the vocabulary τ.
- An $\mathcal{L}[\tau \cup \{R\}]$-sentence φ **defines** R **implicitly (in the finite)** if every (finite) τ-structure $\mathcal{A}$ has at most one expansion $(\mathcal{A}, R^A)$ to a $\tau \cup \{R\}$-structure satisfying φ.
- We say that R is **explicitly definable (in the finite) relative to** φ, if there is an $\mathcal{L}[\tau]$-formula $\psi(\overline{x})$ such that

$$\varphi \vDash_{(\text{fin})} \forall \overline{x}(R\overline{x} \leftrightarrow \psi(\overline{x})).$$

The Beth Property

- Obviously, if R is explicitly definable relative to φ then φ defines R implicitly.
- We say that $\mathcal{L}$ has the **Beth property** (**in the finite**) if the converse holds.
- That is, $\mathcal{L}$ has the Beth property (in the finite) if whenever an $\mathcal{L}$-sentence φ defines a relation symbol implicitly (in the finite), then there is an explicit definition of it (in the finite) relative to φ.

Failure of the Beth Property in the Finite

Proposition

First-order logic does not have the Beth property in the finite.

- We consider orderings in the vocabulary $\tau := \{<, S, \min, \max\}$.

  Let R be a unary relation symbol.

  Let φ be the conjunction of:
  - The ordering axioms;
  - The following sentence fixing R as the set of even points,

  $$\neg R \min \wedge \forall x \forall y (Sxy \to (Rx \leftrightarrow \neg Ry)).$$

  Clearly, φ defines R implicitly in the finite.

  Suppose, for some FO$[\tau]$-formula $\psi(x)$, $\varphi \vDash_{\text{fin}} \forall x (Rx \leftrightarrow \psi(x))$.

  Then $\psi(\max)$ together with the ordering axioms would define the class of finite orderings of even cardinality.

  This contradicts non-axiomatizability of finite orderings of even cardinality in first-order logic.

The Craig Interpolation Property

- The Beth property is a consequence of the interpolation property (or, Craig property).
- The logic $\mathcal{L}$ has the **interpolation property (in the finite)** iff for all vocabularies σ and τ and any $\mathcal{L}$-sentences φ and ψ in the vocabularies σ and τ, respectively, such that $\varphi \models_{(\mathsf{fin})} \psi$, there is an **interpolant**, that is, an $\mathcal{L}[\sigma \cap \tau]$-sentence χ, such that

$$\varphi \models_{(\mathsf{fin})} \chi \quad \text{and} \quad \chi \models_{(\mathsf{fin})} \psi.$$

- Craig's Theorem states that first-order logic has the interpolation property.

Closure Under Order-Invariant Sentences in the Finite

- Let $\mathcal{L}$ be a logic.
- Let K be a class of finite τ-structures.
- It may happen that K is axiomatizable in $\mathcal{L}$, if we equip the structures in K with an arbitrary ordering.
- Consider the vocabulary $\tau \cup \{<\}$.
- Define
$$K_< := \{(A, <) : \mathcal{A} \in K, < \text{ an ordering on } \mathcal{A}\}.$$
- Then a sentence φ of $\mathcal{L}[\tau \cup \{<\}]$ may exists, such that
$$K_< = \text{Mod}(\varphi).$$
- The logic $\mathcal{L}$ is said to be **closed under order-invariant sentences in the finite**, whenever, in this situation, there is an $\mathcal{L}[\tau]$-sentence ψ such that $\text{Mod}(\psi) = K$.

Interpolation and Closure Under Order-Invariant Sentences

- Claim: A logic with the interpolation property is closed under order-invariant sentences in the finite.

  Suppose $\varphi = \varphi(<)$ axiomatizes $K_<$.

  Let $<'$ is a new binary relation symbol.

  Then

  $$\varphi(<) \vDash_{\text{fin}} (\text{“}<' \text{ is an ordering”} \rightarrow \varphi(<')).$$

  By hypothesis, there is an interpolant ψ.

  That is, there exists ψ in $\mathcal{L}$, such that

  $$\varphi(<) \vDash_{\text{fin}} \psi \quad \text{and} \quad \psi \vDash_{\text{fin}} (\text{“}<' \text{ is an ordering”} \rightarrow \varphi(<')).$$

  Clearly, $\text{Mod}(\psi) = K$.

The Interpolation Property in the Finite

Proposition

(a) First-order logic is not closed under order-invariant sentences in the finite.

(b) First-order logic does not have the interpolation property in the finite.

- Part (b) follows from Part (a) by the claim. We sketch a proof of (a). Let K be the class of finite Boolean algebras with an even number of atoms.
  - Using the Ehrenfeucht-Fraïssé method, one can show that K is not axiomatizable in first-order logic.
  - However, $K_<$ is axiomatizable in first-order logic.
    In fact, let φ be the conjunction of:
    - The axioms for Boolean algebras;
    - The axioms for orderings;
    - A sentence expressing that there is an element containing exactly the atoms at an even position (in the ordering induced on the atoms) and containing the last atom.

Universal and Existential Formulas

- Call a first-order formula **universal** (**existential**) if it is built up from atomic and negated atomic formulas using only the connectives $\wedge, \vee$ and the universal (existential) quantifier.
- If φ is a universal sentence, a simple inductive proof shows that φ is **preserved under substructures**, i.e.,

$$\mathcal{B} \subseteq \mathcal{A} \text{ and } \mathcal{A} \vDash \varphi \text{ imply } \mathcal{B} \vDash \varphi.$$

- If φ is existential then it is **preserved under extensions**, i.e.,

$$\mathcal{B} \subseteq \mathcal{A} \text{ and } \mathcal{B} \vDash \varphi \text{ imply } \mathcal{A} \vDash \varphi.$$

- In classical model theory one proves that every FO-sentence preserved under substructures is logically equivalent to a universal FO-sentence.

Universal Formulas and Substructures in the Finite

- We create an FO-sentence that, in the finite, is preserved under substructures but is not equivalent to a universal first-order sentence.
- Let the universal sentence φ_0 be the conjunction of:
  - The ordering axioms in $\{<, \min, \max\}$;
  - The following sentence expressing that R is a "partial successor relation",

$$\forall x \forall y (Rxy \to x < y) \land \forall x \forall y \forall z ((Rxy \land x < z) \to (y = z \lor y < z)).$$

- Let φ_1 be the sentence

$$\forall x (\neg x = \max \to \exists y Rxy)$$

expressing that R is the "total" successor relation.

Universal Formulas and Substructures (Cont'd)

- For finite structures $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{A} \vDash \varphi_0,\ \mathcal{B} \vDash (\varphi_0 \wedge \varphi_1) \text{ and } \mathcal{B} \subseteq \mathcal{A} \text{ imply } \mathcal{A} = \mathcal{B}.$$

- Using a new unary relation symbol Q, we set

$$\varphi := \varphi_0 \wedge (\varphi_1 \to \exists y Q y).$$

Claim: In finite models, φ is preserved under substructures.

Suppose $(\mathcal{A}, Q^A) \vDash \varphi$ and $(\mathcal{B}, Q^B) \subseteq (\mathcal{A}, Q^B)$.

Since φ_0 is universal, $\mathcal{B} \vDash \varphi_0$.

If $\mathcal{B} \nvDash \varphi_1$, then $(\mathcal{B}, Q^B) \vDash \varphi$.

If $\mathcal{B} \vDash \varphi_1$, then $\mathcal{B} = \mathcal{A}$. Therefore, $(\mathcal{B}, Q^B) = (\mathcal{A}, Q^A)$.

Hence, $(\mathcal{B}, Q^B) \vDash \varphi$.

Universal Formulas and Substructures (Cont'd)

- Claim: φ is not equivalent to a universal first-order sentence.

  Assume, to the contrary, that $\psi = \forall x_1 \cdots \forall x_n \chi$, with quantifier-free χ, is a universal first-order sentence with

  $$\vDash_{\text{fin}} \varphi \leftrightarrow \psi.$$

  Consider a $\{<, \min, \max, R\}$-structure $\mathcal{A}$ with $n+3$ elements.
  - $(\mathcal{A}, <^A, \min^A, \max^A)$ is an ordering;
  - R^A is the successor relation.

  Set $Q^A = \varnothing$. Then $(\mathcal{A}, Q^A) \nvDash \varphi$.

  Hence, $(\mathcal{A}, Q^A) \vDash \exists x_1 \cdots \exists x_n \neg \chi$, say, $(\mathcal{A}, Q^A) \vDash \neg \chi[a_1, \ldots, a_n]$.

  Choose $a \in A \backslash \{a_1, \ldots, a_n, \min^A, \max^A\}$ and set $Q' = \{a\}$.

  Since χ is quantifier-free, $(\mathcal{A}, Q') \vDash \neg \chi[a_1, \ldots, a_n]$.

  Therefore, $(\mathcal{A}, Q') \nvDash \forall x_1 \cdots \forall x_n \chi$.

  On the the other hand, $(\mathcal{A}, Q') \vDash \varphi$.

  Hence φ and ψ are not equivalent in the finite.

Universal and Existential Formulas in the Finite

Proposition

(a) There is a first-order sentence which, in the finite, is preserved under substructures but not equivalent to a universal first-order sentence.

(b) There is a first-order sentence which, in the finite, is preserved under extensions but not equivalent to an existential first-order sentence.

(a) By the preceding example.

(b) Let φ be according to (a). Then $\neg\varphi$ is preserved under extensions and not equivalent to an existential sentence.

Monotone Formulas

- Fix a relation symbol R of τ of arity r.
- A sentence φ is **monotone in** R (**in the finite**) if

$$(\mathcal{A}, R_1) \vDash \varphi \ (A \text{ finite}) \text{ and } R_1 \subseteq R_2 \subseteq A^r \text{ imply } (\mathcal{A}, R_2) \vDash \varphi.$$

- A first-order formula φ is **positive** in R if φ is built up from atomic formulas using $\neg, \wedge, \vee, \forall, \exists$ and any occurrence of the relation symbol R in φ is within the scope of an even number of negation symbols.
- An inductive argument shows that a sentence positive in R is monotone.
- While any first-order sentence monotone in R is logically equivalent to a formula positive in R, this is no longer true in the finite.

Proposition

There is a first-order sentence which, in the finite, is monotone in R, but not equivalent to a first-order sentence positive in R.