Finite Model Theory

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Subsection 1

0-1 Laws for FO and $L^{\omega}_{\infty\omega}$

Labeled Structures

- Let τ be a fixed vocabulary.
- Let K be a class of τ-structures.
- Let $L_n(K)$ be the number of structures in K with universe $\{1, 2, \ldots, n\}$,

$$L_n(K) := \|\{\mathcal{A} \in K : A = \{1, \ldots, n\}\}\|.$$

- Sometimes, structures $\mathcal{A}$ with $A = \{1, \ldots, n\}$ are called **labeled structures**, since every element in such a structure is labeled with a natural number.
- $L_n(K)$ is the number of labeled structures in K of cardinality n.
- If K is the class of models of a sentence φ, we write $L_n(\varphi) := L_n(K)$.
- If K is the class of all τ-structures, we write $L_n(\tau) := L_n(K)$.

Labeled Asymptotic Probabilities and 0-1 Laws

- Let $\ell_n(K)$ be the fraction of structures with universe $\{1,\ldots,n\}$ which are in K,
$$\ell_n(K) := \frac{L_n(K)}{L_n(\tau)}.$$
- In case it exists, $\ell(K) := \lim_{n\to\infty}\ell_n(K)$ is called the **labeled asymptotic probability** of K.
- $\ell_n(\varphi)$ stands for $\ell_n(\mathrm{Mod}(\varphi))$.
- $\ell(\varphi)$ stands for $\ell(\mathrm{Mod}(\varphi))$.
- If $\ell(\varphi) = 1$ we say that φ holds in **almost all** finite structures or that φ **almost surely** holds.
- A class Ψ of sentences of a logic is said to satisfy the **labeled 0-1 law** if $\ell(\varphi) = 1$ or $\ell(\varphi) = 0$ holds for every $\varphi \in \Psi$ or, equivalently, for $\varphi \in \Psi$ either φ or $\neg\varphi$ holds in almost all finite structures.

Example

- Suppose $\tau = \{P, c\}$, where P is unary.

  For any τ-structure (A, P^A, c^A),

  $$(A, P^A, c^A) \vDash Pc \quad \text{iff} \quad (A, A \backslash P^A, c^A) \nvDash Pc.$$

  Thus, we see that $\ell_n(Pc) = \frac{1}{2}$.

  It follows that $\ell(Pc) = \frac{1}{2}$.

Example

- Let $\tau := \{f\}$ be the "vocabulary" with a unary function symbol f.

  Consider the first-order sentence

  $$\forall x f(x) \neq x,$$

  expressing that f has no fixed-point.

  Note that, on the universe $\{1, \ldots, n\}$:

  - One can assign the values of f independently;
  - For each argument i, the $n-1$ possible values $\neq i$ do not lead to a fixed-point.

  It follows that

  $$\ell_n(\forall x f(x) \neq x) = \left(\frac{n-1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n.$$

  Hence, $\ell(\forall x f(x) \neq x) = e^{-1}$.

Example

- If K is the class EVEN$[\tau]$ of all τ-structures of even finite cardinality, then
$$\ell_n(K) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$
Hence, $\ell(K)$ does not exist.
Therefore, $\ell(\varphi)$ does not exist for the following sentences:
  - The second order sentence φ expressing

    "there is a binary relation which is an equivalence relation having only equivalence classes with exactly two elements";
  - The $L_{\omega_1\omega}$-sentence
$$\bigvee_{k\geq 1} \varphi_{=2k}.$$

Example

- Let τ be a relational vocabulary.

  Let K be a class of τ-structures.

  Construct a "random" structure of vocabulary τ on $\{1, \ldots, n\}$ by the following experiment:

  > For every m-ary relation symbol R in τ and for every $i_1, \ldots, i_m \in \{1, \ldots, n\}$, toss a fair coin to decide whether $Ri_1 \ldots i_m$ is true.

  Then $\ell_n(K)$ is the probability for the outcome $\mathcal{A}$ of the experiment to belong to K.

Extension Axioms Revisited

- In the following suppose that τ is relational.
- Recall that an $(r+1)$-extension axiom is a sentence

$$\chi_{\Phi} = \forall v_1 \cdots \forall v_r \left(\bigwedge_{1 \le i < j \le r} v_i \neq v_j \to \exists v_{r+1} \left(\bigwedge_{1 \le i \le r} v_i \neq v_{r+1} \wedge \bigwedge_{\varphi \in \Phi} \varphi \wedge \bigwedge_{\varphi \in \Phi^c} \neg\varphi \right)\right),$$

where Φ is a subset of

$$\begin{aligned} \Delta_{r+1} \quad = \quad & \{\varphi(v_1, \ldots, v_r, v_{r+1}) : \varphi \text{ has the form } R\overline{x}, \text{ where } R \in \tau \\ & \text{and where } v_{r+1} \text{ occurs in } \overline{x}\}. \end{aligned}$$

Asymptotic Behavior of Extension Axioms

Lemma

Any extension axiom holds in almost all finite structures.

- Given Φ, we show that the asymptotic labeled probability $\ell(\chi_\Phi)$ equals 1.

  Let $a_1, \ldots, a_r$ be distinct elements in a structure $\mathcal{A}$.

  Imagine the following experiment:
  - Add a further object a to $\mathcal{A}$ as a new element;
  - Randomly fix the truth values of $R\overline{b}$ for any R in τ and any sequence $\overline{b}$ in $A \cup \{a\}$ containing a.

  Let δ be the probability that $a_1, \ldots, a_r, a$ satisfies

$$\Phi \cup \{\neg\varphi : \varphi \in \Phi^c\}.$$

Asymptotic Behavior of Extension Axioms (Cont'd)

- Clearly, if c is the number of subsets of Δ_{r+1}, then $\delta = \frac{1}{c}$.

  In particular, $\delta > 0$.

  Thus,

$$\begin{array}{rcl}
\ell_n(\neg\chi_\Phi) & = & \ell_n(\exists v_1 \cdots \exists v_r (\bigwedge_{1 \le i < j \le r} v_i \neq v_j \wedge \\
 & & \forall v_{r+1} (\bigvee_{1 \le i \le r} v_i = v_{r+1} \vee \bigvee_{\varphi \in \Phi} \neg\varphi \vee \bigvee_{\varphi \in \Phi^c} \varphi))) \\
 & \le & n^r \left(\frac{c-1}{c}\right)^{n-r} \\
 & = & n^r (1-\delta)^{n-r}.
\end{array}$$

  Therefore,

$$\ell(\neg\chi_\Phi) = \lim_{n\to\infty} \ell_n(\neg\chi_\Phi) = 0.$$

Random Theories and 0-1 Laws

- Recall that $T_{\text{rand}}(= T_{\text{rand}}(\tau))$ denotes the set of extension axioms.

Corollary

Let φ be a first-order sentence.

(a) If $T_{\text{rand}} \models \varphi$, then $\ell(\varphi) = 1$.

(b) If $T_{\text{rand}} \models \neg\varphi$, then $\ell(\varphi) = 0$.

(a) Suppose that $T_{\text{rand}} \models \varphi$.

By compactness, $T_0 \models \varphi$, for some finite subset T_0 of T_{rand}.

Now T_0 is a set of extension axioms.

Thus, $\ell(\bigwedge T_0) = 1$, by the preceding lemma.

Hence, $\ell(\varphi) = 1$.

(b) Suppose, next, $T_{\text{rand}} \models \neg\varphi$. Then, by Part (a), $\ell(\neg\varphi) = 1$.

Therefore, $\ell(\varphi) = 0$.

Finite Sets of Extension Axioms

- For $s \geq 1$, let ϵ_s be the conjunction of the finitely many r-extension axioms with $r \leq s$.

Corollary

Let φ be an $L^{\omega}_{\infty\omega}$-sentence.

(a) If $\epsilon_s \vDash \varphi$, then $\ell(\varphi) = 1$.

(b) If $\epsilon_s \vDash \neg\varphi$, then $\ell(\varphi) = 0$.

- This follows from $\ell(\epsilon_s) = 1$.

0-1 Law for FO and $L^{\omega}_{\infty\omega}$

Theorem

Let τ be relational. Then both $\text{FO}[\tau]$ and $L^{\omega}_{\infty\omega}[\tau]$ satisfy the labeled 0-1 law.

- From previous work on infinitary games and on pebble games, respectively, we know that:
  - For φ in $\text{FO}[\tau]$,
    $$T_{\text{rand}} \vDash \varphi \quad \text{or} \quad T_{\text{rand}} \vDash \neg\varphi;$$
  - For $\varphi \in L^{s}_{\infty\omega}[\tau]$,
    $$\epsilon_s \vDash \varphi \quad \text{or} \quad \epsilon_s \vDash \neg\varphi.$$

So the assertions follow from the preceding corollaries.

Subsection 2

Parametric Classes

Labeled Asymptotic Conditional Probabilities

- Suppose that K and H are classes of τ-structures.
- Define the **labeled probability** $\ell_n(K|H)$ by

$$\ell_n(K|H) := \frac{L_n(K \cap H)}{L_n(H)}.$$

- If it exists, $\ell(K|H) := \lim_{n\to\infty} \ell_n(K|H)$ is called the **labeled asymptotic probability of K with respect to H**.
- Notations such as $\ell_n(\varphi|H)$ or $\ell_n(K|\tau)$ should be self-explaining.

  Example:
  (a) We obviously have $\ell_n(K|\tau) = \ell_n(K)$.
  (b) $\ell_n(\mathrm{CONN}|\mathrm{GRAPH})$ is the number of connected graphs on $\{1,\ldots,n\}$ divided by the total number of graphs on $\{1,\ldots,n\}$.

Parametric Sentences and Parametric Classes

- Let τ be relational.
- A first-order sentence φ is called **parametric** if it is a conjunction of sentences
$$\forall \text{ distinct } x_1 \dots x_s \psi,$$
where $s \geq 1$ and ψ is a boolean combination of formulas of the form $Ry_1 \dots y_n$ with $R \in \tau$ and $\{y_1, \dots, y_n\} = \{x_1, \dots, x_s\}$.
- Note that s cannot exceed the maximum of the arities of relation symbols in τ.
- A class K of structures is said to be **parametric**, if $K = \mathrm{Mod}(\varphi)$, where φ is a parametric sentence.

Examples

(a) The classes of graphs and digraphs are parametric classes.

They are axiomatized, respectively, by the parametric sentences

$$\forall x \neg Rxx \wedge \forall \text{ distinct } xy(Rxy \to Ryx) \quad \text{and} \quad \forall x \neg Rxx.$$

(b) The class of **tournaments** is axiomatized by the parametric sentence

$$\forall x \neg Rxx \wedge \forall \text{ distinct } xy(Rxy \leftrightarrow \neg Ryx).$$

(c) For relational τ the class K of all τ-structures is parametric.

K is the class of models of the empty conjunction.

Examples (Cont'd)

(d) Note that

$$\forall \text{ distinct } xyz((Rxy \wedge Ryz) \to Rxz)$$

is not a parametric sentence (since, e.g., $\{x, y\} \neq \{x, y, z\}$).

From our analysis of parametric classes it will become clear that, for example, the classes of transitive relations, equivalence relations, partial orderings, and orderings are not parametric, transitivity being the only obstacle.

(e) For R k-ary, consider the parametric sentence

$$\forall \text{ distinct } x_1 \dots x_k (Rx_1 \dots x_k \wedge \neg Rx_1 \dots x_k).$$

It is true in all structures of cardinality $< k$, but has no model of cardinality $\geq k$.

Nontrivial Parametric Sentences

- Let k be the maximum of the arities of the relation symbols in τ.
- A parametric sentence (and its model class) is called **nontrivial**, if it has a model of cardinality $\geq k$.

  Claim: Nontrivial parametric sentences have arbitrarily large models.

  Suppose that φ_0 is a nontrivial parametric sentence.

  Suppose B is any nonempty set.

  We present a procedure that stepwise fixes the relations on B.

  We show that it leads to a model of φ_0 with universe B.

Nontrivial Parametric Sentences (Cont'd)

- For any $s \leq k$ and any distinct $b_1, \ldots, b_s \in B$:
  - Choose an arbitrary model $\mathcal{A}$ of φ_0 of cardinality $\geq s$;
  - Choose distinct $a_1, \ldots, a_s \in A$.

  Define, for all R in τ, the "$\{b_1, \ldots, b_s\}$-part of R^B" as a copy of the "$\{a_1, \ldots, a_s\}$-part of R^A".

  More precisely, we define this part of R^B, such that for $\varphi(v_1, \ldots, v_s) = Ry_1 \ldots y_n$, with $\{y_1, \ldots, y_n\} = \{v_1, \ldots, v_s\}$,

  $$\mathcal{B} \vDash \varphi[b_1, \ldots, b_s] \quad \text{iff} \quad \mathcal{A} \vDash \varphi[a_1, \ldots, a_s].$$

  In $\mathcal{B}$, every s-tuple of distinct elements behaves as some s-tuple in some model of φ_0.

  As φ_0 is parametric, $\mathcal{B}$ itself is a model of φ_0.

Extension Axioms Compatible with φ_0

- Consider an $(r+1)$-extension axiom

$$\forall \text{ distinct } v_1 \ldots v_r \exists v_{r+1} \left(\bigwedge_{1 \le i \le r} \neg v_i = v_{r+1} \wedge \bigwedge_{\varphi \in \Phi} \varphi \wedge \bigwedge_{\varphi \in \Phi^c} \neg \varphi \right).$$

- It is called **compatible with** a sentence φ_0 if

$$\{\varphi_0\} \cup \left\{ \exists v_1 \cdots \exists v_r \exists v_{r+1} \left(\bigwedge_{1 \le i < j \le r+1} \neg v_i = v_j \wedge \bigwedge_{\varphi \in \Phi} \varphi \wedge \bigwedge_{\varphi \in \Phi^c} \neg \varphi \right) \right\}$$

is satisfiable.

- Let $T_{\text{rand}}(\varphi_0)$ be the set of sentences consisting of φ_0 and of all extension axioms compatible with φ_0.

Property of Extension Axioms Compatible with φ_0

Claim: Any two models of $T_{\text{rand}}(\varphi_0)$ are partially isomorphic and hence, $L_{\infty\omega}$-equivalent. Therefore,

$$T_{\text{rand}}(\varphi_0) \vDash \psi \quad \text{or} \quad T_{\text{rand}}(\varphi_0) \vDash \neg\psi$$

holds for any $L_{\infty\omega}$-sentence ψ.

We proved in a previous example that any two models of T_{rand} are partially isomorphic.

That proof also works for $T_{\text{rand}}(\varphi_0)$.

Countable Model of T_{rand}

Claim: $T_{\text{rand}}(\varphi_0)$ has an (up to isomorphism) unique countable model $\mathcal{R}(\varphi_0)$.

We gave a proof leading to a countable model of T_{rand}.

The proof can be transferred to $T_{\text{rand}}(\varphi_0)$, since by the construction process described above the corresponding $\mathcal{A}_i$'s can be chosen as models of φ_0.

Models of φ_0^s

- For $s \geq 1$, we denote by φ_0^s the conjunction of φ_0 with the finitely many r-extension axioms with $r \leq s$ that are compatible with φ_0.

  Claim: Any two models of φ_0^s are s-partially isomorphic and hence, $L_{\infty\omega}^s$-equivalent. Therefore, for any $L_{\infty\omega}^s$-sentence ψ,

  $$\varphi_0^s \vDash \psi \quad \text{or} \quad \varphi_0^s \vDash \neg\psi.$$

  Similar argument as above.

Asymptotic Probability of Compatible Extension Axioms

Claim: If ψ is an extension axiom compatible with φ_0, then

$$\ell(\psi|\varphi_0) = 1.$$

We showed that any extension axiom holds in almost all finite structures.

We argue similarly, but restrict to models of φ_0 and take as c the number of subsets Φ of Δ_{r+1} that correspond to $(r+1)$-extension axioms compatible with φ_0.

Given Φ, distinct $a_1, \ldots, a_r$, and a new a, we can satisfy Φ by $a_1 \ldots a_r a$, applying the construction procedure described above to any $a_{i_1} \ldots a_{i_m}$, with $1 \leq i_1 < i_2 < \cdots < i_m \leq r$.

0-1 Law for Nontrivial Parametric Classes

- Let H be a class of structures and Ψ a class of sentences.
- We say that H **satisfies the labeled 0-1 law for** Ψ if

$$\ell(\psi|H) = 1 \quad \text{or} \quad \ell(\psi|H) = 0$$

holds for any $\psi \in \Psi$.
- From the preceding claims we obtain:

Theorem

Let H be a nontrivial parametric class. Then H satisfies the labeled 0-1 law for $L^{\omega}_{\infty\omega}$ and, hence, for FO.

Subsection 3

Unlabeled 0-1 Laws

Introduction

- From a probabilistic point of view the definition of $\ell_n(K)$ is quite natural.
- But note that for $\tau := \{P\}$, with unary P, and $i = 1, \ldots, n$, the structures $\mathcal{A}_i := (\{1, \ldots, n\}, P^{A_i})$ with $P^{A_i} := \{i\}$ are counted as n different structures in the definition of $\ell_n(K)$ even though they are isomorphic.
- In this section we study the so-called unlabeled probability $u_n(K)$, which is the proportion of isomorphism types of structures of cardinality n in K.
- Similarly, we define unlabeled conditional probabilities.

Unlabeled Asymptotic Probability

- Fix a vocabulary τ.
- For a class K let $U_n(K)$ be the number of isomorphism types of structures of cardinality n in K.
- Equivalently,

$$U_n(K) \;:=\; \text{number of isomorphism types of structures in } K \text{ with universe } \{1, \ldots, n\}.$$

- If K is the class of all τ-structures, we write $U_n(\tau) := U_n(K)$.
- If K is the class of models of φ, we write $U_n(\varphi) := U_n(K)$.
- For arbitrary K we set

$$u_n(K) := \frac{U_n(K)}{U_n(\tau)}.$$

- We denote (in case it exists) by $u(K) := \lim_{n\to\infty} u_n(K)$ the **unlabeled asymptotic probability**.

Unlabeled Asymptotic Conditional Probability

- If K and H are classes of structures we call
$$u_n(K|H) := \frac{U_n(K \cap H)}{U_n(H)}$$
the **unlabeled probability of K with respect to H.**
- Moreover, we set
$$u(K|H) := \lim_{n\to\infty} u_n(K|H)$$
the **unlabeled asymptotic probability of K with respect to H.**

Example: $u_n(\mathrm{CONN}|\mathrm{GRAPH})$ is the number of isomorphism types of connected n vertex graphs divided by the total number of isomorphism types of n vertex graphs.

The Class of Rigid Structures

- A structure $\mathcal{A}$ is called **rigid** if the identity on A is the only automorphism of $\mathcal{A}$.
- $\mathrm{RIG} = \mathrm{RIG}[\tau]$ denotes the class of rigid τ-structures.

Lemma

Let K be a class of τ-structures.

(a) $L_n(K) \leq U_n(K) \cdot n!$;

(b) If $K \subseteq \mathrm{RIG}$, then $L_n(K) = U_n(K) \cdot n!$;

(c) If $K \subseteq \mathrm{RIG}^c$, then $L_n(K) \leq U_n(K) \cdot \frac{n!}{2}$.

(b) Let $\mathcal{A}$ be a structure with universe $\{1, \ldots, n\}$.

There are $n!$ permutations of $\{1, \ldots, n\}$.

Every permutation π induces a structure $\mathcal{A}_\pi$ on $\{1, \ldots, n\}$ such that $\pi : \mathcal{A} \cong \mathcal{A}_\pi$.

The Class of Rigid Structures

- For permutations π and ρ of $\{1, \ldots, n\}$ we have

$$\mathcal{A}_\pi = \mathcal{A}_\rho \quad \text{iff} \quad \pi^{-1} \circ \rho : \mathcal{A} \cong \mathcal{A}.$$

  Hence, if $\mathcal{A}$ is rigid, we have $\mathcal{A}_\pi = \mathcal{A}_\rho$ iff $\pi = \rho$.
  Thus, each rigid structure leads to $n!$ distinct structures on $\{1, \ldots, n\}$.
  This shows (b).

(c) Suppose $\mathcal{A}$ is not rigid and ρ is a nontrivial automorphism of $\mathcal{A}$.
Then $\mathcal{A} = \mathcal{A}_{\pi \circ \rho}$, for any permutation π.
Hence, for any nonrigid structure, there are at most $\frac{n!}{2}$ distinct structures on $\{1, \ldots, n\}$. This proves (c).

(a) We have

$$\begin{array}{rcl} L_n(K) & = & L_n(K \cap \mathrm{RIG}) + L_n(K \cap \mathrm{RIG}^c) \\ & \leq & U_n(K \cap \mathrm{RIG}) \cdot n! + U_n(K \cap \mathrm{RIG}^c) \cdot \frac{n!}{2} \\ & \leq & (U_n(K \cap \mathrm{RIG}) + U_n(K \cap \mathrm{RIG}^c))n! \\ & = & U_n(K) \cdot n!. \end{array}$$

Asymptotic Probability of Rigid Structures

Lemma

For any class H, $u_n(\mathrm{RIG}|H) \leq \ell_n(\mathrm{RIG}|H)$. In particular,

$$u(\mathrm{RIG}|H) = 1 \quad \text{implies} \quad \ell(\mathrm{RIG}|H) = 1.$$

- We have

$$\begin{aligned} u_n(\mathrm{RIG}|H) &= \frac{U_n(\mathrm{RIG} \cap H) \cdot n!}{U_n(\mathrm{RIG} \cap H) \cdot n! + U_n(\mathrm{RIG}^c \cap H) \cdot n!} \\ &\leq \frac{L_n(\mathrm{RIG} \cap H)}{L_n(\mathrm{RIG} \cap H) + L_n(\mathrm{RIG}^c \cap H)} \\ &= \ell_n(\mathrm{RIG}|H). \end{aligned}$$

Characterization of Almost Certain Rigidity

- We show that almost all structures in a class H are rigid iff $L_n(H) \approx U_n(H) \cdot n!$.

Proposition

Let H he a class of structures. Then

$$u(\mathrm{RIG}|H) = 1 \quad \text{iff} \quad \lim_{n\to\infty} \frac{L_n(H)}{U_n(H)\cdot n!} = 1.$$

- We have

$$\begin{aligned} \frac{L_n(H)}{U_n(H)\cdot n!} &= \frac{L_n(\mathrm{RIG}\cap H)}{U_n(H)\cdot n!} + \frac{L_n(\mathrm{RIG}^c\cap H)}{U_n(H)\cdot n!} \\ &= u_n(\mathrm{RIG}|H) + \frac{L_n(\mathrm{RIG}^c\cap H)}{U_n(H)\cdot n!}. \end{aligned}$$

Characterization of Almost Certain Rigidity (Cont'd)

- We obtained

$$\frac{L_n(H)}{U_n(H)\cdot n!} = u_n(\mathrm{RIG}|H) + \frac{L_n(\mathrm{RIG}^c \cap H)}{U_n(H)\cdot n!}.$$

Using the preceding lemma, we get

$$\begin{aligned} u_n(\mathrm{RIG}|H) &\le \frac{L_n(H)}{U_n(H)\cdot n!} \\ &\le u_n(\mathrm{RIG}|H) + \frac{1}{2}u_n(\mathrm{RIG}^c|H) \\ &= 1 - \frac{1}{2}u_n(\mathrm{RIG}^c|H). \end{aligned}$$

Now we infer the following:

- $u(\mathrm{RIG}|H) = 1$ implies $\lim\limits_{n\to\infty} \frac{L_n(H)}{U_n(H)\cdot n!} = 1$.
- $\lim\limits_{n\to\infty} \frac{L_n(H)}{U_n(H)\cdot n!} = 1$ implies $u(\mathrm{RIG}^c|H) = 0$ implies $u(\mathrm{RIG}|H) = 1$.

Rigidity, Labeled and Unlabeled Asymptotic Probabilities

Theorem

Let H be a class of structures. If almost all structures in H are rigid, i.e., if $u(\mathrm{RIG}|H) = 1$, then, for any class K, the labeled and the unlabeled asymptotic probabilities with respect to H coincide, that is

$$\ell(K|H) = u(K|H) \ (= u(K|\mathrm{RIG} \cap H)).$$

- By assumption, $u(\mathrm{RIG}|H) = 1$. By the preceding lemma, $\ell(\mathrm{RIG}|H) = 1$. By the definitions, $\ell(K|H) = \ell(K|\mathrm{RIG} \cap H)$ and $u(K|H) = u(K|\mathrm{RIG} \cap H)$. Hence,

$$\begin{aligned} \ell_n(K|\mathrm{RIG} \cap H) &= \frac{L_n(K \cap \mathrm{RIG} \cap H)}{L_n(\mathrm{RIG} \cap H)} \\ &= \frac{U_n(K \cap \mathrm{RIG} \cap H) \cdot n!}{U_n(\mathrm{RIG} \cap H) \cdot n!} = u_n(K|\mathrm{RIG} \cap H). \end{aligned}$$

Free Parametric Classes and Rigidity

- Let H_0 be the parametric class consisting of all τ-structures $\mathcal{A}$ such that $R^A = \varnothing$ for all $R \in \tau$.

  Note that:
  - $L_n(H_0) = 1$;
  - $U_n(H_0) = 1$;
  - $u_n(\mathrm{RIG}|H_0) = 0$ (for $n \geq 2$).
- A parametric class H is free if, roughly speaking, for some $r \geq 2$, there is a real choice when fixing the parts of the relations corresponding to r-tuples of distinct elements.
- Note that, in contrast to H_0, almost all structures in a "free" parametric class are rigid.

Free Parametric Classes

- Suppose $H = \mathrm{Mod}(\varphi_0)$ with a parametric φ_0.
- Then H is **free**, if for some $m \geq 2$, there is a relation symbol R, say of arity r, and a surjection $i : \{1, \ldots, r\} \to \{1, \ldots, m\}$, such that

$$\varphi_0 \wedge \exists x_1 \cdots \exists x_m (R x_{i(1)} \ldots x_{i(r)} \wedge \bigwedge_{1 \leq k < \ell \leq m} \neg x_k = x_\ell),$$
$$\varphi_0 \wedge \exists x_1 \cdots \exists x_m (\neg R x_{i(1)} \ldots x_{i(r)} \wedge \bigwedge_{1 \leq k < \ell \leq m} \neg x_k = x_\ell)$$

  are satisfiable.

  Example:
  - The class H_0 introduced in the preceding slide is not free.
  - The class of graphs is free.
  - For any relational τ containing at least one relation symbol of arity ≥ 2, the class of all τ-structures is a free parametric class.

Rigid Structures in a Free Parametric Class

Proposition

Let H be a nontrivial free parametric class. Then almost all structures in H are rigid, that is, $u(\mathrm{RIG}|H) = 1$.

- We later give a proof for the special case in which H is the class of all structures.

Corollary

Let H be a nontrivial free parametric class. Then the labeled and the unlabeled asymptotic probabilities with respect to H coincide.

- This follows from the proposition and the preceding theorem.

A Few Prerequisites

- Suppose that $\tau = \{R_1, \ldots, R_k\}$, with unary relation symbols R_i.
- For $\alpha : \{1, \ldots, k\} \to \{0, 1\}$ and a τ-structure $\mathcal{A}$, denote by A_α the subset $X_1 \cap \cdots \cap X_k$, where:
  - $X_i := R_i^A$, if $\alpha(i) = 1$;
  - $X_i := A \backslash R_i^A$, if $\alpha(i) = 0$.
- For any τ-structures $\mathcal{A}$ and $\mathcal{B}$ and $m \geq 1$,

$$\mathcal{A} \equiv_m \mathcal{B} \quad \text{iff} \quad \min\{\|A_\alpha\|, m\} = \min\{\|B_\alpha\|, m\} \text{ holds for all } \alpha.$$

- Thus, every sentence φ from $\mathrm{FO}[\tau]$ is equivalent to a boolean combination of sentences of the form $\exists^{=\ell} x R^\alpha x$, where

$$R^\alpha x := \varphi_1 \wedge \cdots \wedge \varphi_k,$$

with $\varphi_i = R_i x$, if $\alpha(i) = 1$, and $\varphi_i = \neg R_i x$, if $\alpha(i) = 0$.

A Few Prerequisites (Cont'd)

- For all $s \geq 1$ and all τ-structures $\mathcal{A}$ and $\mathcal{B}$,

$$W_s^s(\mathcal{A}, \mathcal{B}) = W_\infty^s(\mathcal{A}, \mathcal{B}).$$

- It follows that the class K of all finite τ-structures is bounded.
- So FO and $L_{\infty\omega}^{\omega}$ have the same expressive power on K.

Labeled and Unlabeled Probabilities in Parametric Classes

Proposition

Let H be any nontrivial parametric class. Then the labeled and the unlabeled asymptotic probabilities with respect to H coincide.

- By the preceding corollary, we only have to consider the nonfree case.
  First we consider vocabularies with only unary relation symbols.
  Suppose $\tau = \{R_1, \ldots, R_m\}$, with unary $R_1, \ldots, R_m$.
  Let φ_0 be a nontrivial parametric sentence.
  For $\alpha : \{1, \ldots, m\} \to \{0, 1\}$, set

$$R^\alpha x := \varphi_1 \wedge \cdots \wedge \varphi_m,$$

  where:
  - $\varphi_i = R_i x$, if $\alpha(i) = 1$;
  - $\varphi_i = \neg R_i x$, if $\alpha(i) = 0$.

  Recall that any boolean combination of formulas $R_i x$ can be written as a disjunction of formulas $R^\alpha x$.

Labeled and Unlabeled Probabilities (Cont'd)

- It follows that there are distinct $\alpha_1, \ldots, \alpha_t$ such that φ_0 and $\forall x(R^{\alpha_1}x \vee \cdots \vee R^{\alpha_t}x)$ are logically equivalent.

  The case $t = 1$ is trivial. So we assume $t \geq 2$.

  For $i = 1, \ldots, t$ and $k \geq 0$, the sentence $\exists^{>k} x R^{\alpha_i} x$ is a consequence of the set $T_{\mathrm{rand}}(\varphi_0)$ of extension axioms compatible with φ_0. Hence,

$$\ell(\exists^{=k} x R^{\alpha_i} x | \varphi_0) = 0.$$

  The isomorphism type of a model $\mathcal{A}$ of φ_0 is determined by $(n_1, \ldots, n_t)$, where $n_i := \|\{a \in A : \mathcal{A} \vDash R^{\alpha_i} x[a]\}\|$.
  Using induction on n and (nested) on i, we may show:

  The number of i-tuples $(n_1, \ldots, n_t)$, such that $n_1 + \cdots + n_t = n$, equals the binomial coefficient $\binom{n+t-1}{t-1}$, a polynomial in n of degree $t-1$.

  Hence, $U_n(\varphi_0) = \binom{n+t-1}{t-1}$.

Labeled and Unlabeled Probabilities (Cont'd)

- By the prerequisites, any sentence of $L^{\omega}_{\infty\omega}[\tau]$ is equivalent to a finite boolean combination of sentences of the form $\exists^{=k}xR^{\alpha}x$, where $k \geq 0$ and $\alpha : \{1, \ldots, m\} \to \{0, 1\}$. Thus, it suffices to show that

$$u(\exists^{=k}xR^{\alpha}x|\varphi_0) = \ell(\exists^{=k}xR^{\alpha}x|\varphi_0) \in \{0, 1\}.$$

  Suppose, first, that $\alpha \neq \alpha_1, \ldots, \alpha \neq \alpha_t$.

  Then:

  - If $k > 0$, $u(\exists^{=k}xR^{\alpha}x|\varphi_0) = \ell(\exists^{=k}xR^{\alpha}x|\varphi_0) = 0$;
  - If $k = 0$, $u(\exists^{=k}xR^{\alpha}x|\varphi_0) = \ell(\exists^{=k}xR^{\alpha}x|\varphi_0) = 1$.

  Let $\alpha = \alpha_i$. Then, by what was just shown above:

  - $U_n(\exists^{=k}xR^{\alpha}x \wedge \varphi_0) = \binom{n-k+t-2}{t-2}$ is a polynomial in n of degree $t-2$;
  - $U_n(\varphi_0) = \binom{n+t-1}{t-1}$ is a polynomial in n of degree $t-1$.

Labeled and Unlabeled Probabilities (Conclusion)

- Hence,

$$u(\exists^{=k} x R^{\alpha} x | \varphi_0) = \lim_{n \to \infty} \frac{U_n(\exists^{=k} x R^{\alpha} x \wedge \varphi_0)}{U_n(\varphi_0)} = 0.$$

Thus, by our first assertion,

$$u(\exists^{=k} x R^{\alpha} x | \varphi_0) = \ell(\exists^{=k} x R^{\alpha} x | \varphi_0) = 0.$$

Finally, we turn to arbitrary vocabularies.

Let H be a nontrivial parametric class which is not free.

By the definition of freeness, if $\mathcal{A} \in H$, then any bijection of the universe that preserves the induced unary relations $\{a : R^A a \dots a\}$, for $R \in \tau$, is an automorphism of $\mathcal{A}$.

Hence, the counting arguments of the special case remain valid.

Unlabeled 0-1 Law in Parametric Classes

- The class H satisfies the **unlabeled 0-1 law for the set** Φ of sentences if, for all $\varphi \in \Phi$,

$$u(\varphi|H) = 1 \quad \text{or} \quad u(\varphi|H) = 0.$$

Theorem

Let H be a nontrivial parametric class. Then H satisfies the unlabeled 0-1 law for $L^{\omega}_{\infty\omega}$ and hence, for FO.

- By a previous theorem combined with the proposition.

Almost all Structures are Rigid

Proposition

Let H be the class of all τ-structures. Then almost all structures in H are rigid, that is, $u(\mathrm{RIG}|H) = 1$.

- Fix a vocabulary τ and $n \geq 1$.

  Let $\mathcal{A}, \mathcal{B}, \ldots$ range over τ-structures with universe $\{1, \ldots, n\}$.

  Let $\pi, \rho, \ldots$ range over permutations of $\{1, \ldots, n\}$.

  We set:

$$\begin{array}{rcl} \mathrm{Aut}(\mathcal{A}) & := & \{\pi : \pi : \mathcal{A} \cong \mathcal{A}\}; \\ \mathrm{Str}(\pi) & := & \{\mathcal{A} : \pi : \mathcal{A} \cong \mathcal{A}\}. \end{array}$$

Almost all Structures are Rigid (Cont'd)

Lemma

$U_n(\tau) \cdot n! = \sum_\pi \|\mathrm{Str}(\pi)\|$.

- $U_n(\tau)$ is the number of equivalence classes of the relation $\sim$, where

$$\mathcal{A} \sim \mathcal{B} \quad \text{iff} \quad \mathcal{A} \cong \mathcal{B}.$$

Clearly, $\mathcal{A} \cong \mathcal{B}$ implies $\|\mathrm{Aut}(\mathcal{A})\| = \|\mathrm{Aut}(\mathcal{B})\|$.

Given $\mathcal{A}$ and π, let $\mathcal{A}_\pi$ be the structure $\mathcal{A}$ with $\pi : \mathcal{A} \cong \mathcal{A}_\pi$.

We have already remarked that $\mathcal{A}_\pi = \mathcal{A}_\rho$ iff $\pi^{-1} \circ \rho \in \mathrm{Aut}(\mathcal{A})$.

This implies that $\|\{\mathcal{B} : \mathcal{A} \cong \mathcal{B}\}\|$ is the index of the subgroup $\mathrm{Aut}(\mathcal{A})$ in the group of all permutations of $\{1, \ldots, n\}$.

Almost all Structures are Rigid (Cont'd)

- Hence,
$$\|\mathrm{Aut}(\mathcal{A})\| = \frac{n!}{\|\{\mathcal{B} : \mathcal{A} \cong \mathcal{B}\}\|}.$$
Therefore, for fixed $\mathcal{A}$, $\sum_{\mathcal{B},\mathcal{B}\cong\mathcal{A}} \|\mathrm{Aut}(\mathcal{B})\| = \sum_{\mathcal{B},\mathcal{B}\cong\mathcal{A}} \|\mathrm{Aut}(\mathcal{A})\| = n!$.
But there are $U_n(\tau)$ many equivalence classes with respect to $\cong$.
Therefore,
$$\sum_{\mathcal{B}} \|\mathrm{Aut}(\mathcal{B})\| = U_n(\tau) \cdot n!.$$
On the other hand,
$$\sum_{\mathcal{B}} \|\mathrm{Aut}(\mathcal{B})\| = \|\{(\pi, \mathcal{B}) : \pi \in \mathrm{Aut}(\mathcal{B})\}\| = \sum_{\pi} \|\mathrm{Str}(\pi)\|.$$
The last two equations yield the lemma.

Support of a Bijection

- Let M be finite and $f : M \to M$ be a bijection.
- Denote by $c(f)$ the number of f-cycles.
- Denote by $\mathrm{spt}(f)$ the **support** of f,

$$\mathrm{spt}(f) := \{a \in M : f(a) \neq a\}.$$

- Set $s(f) := \|\mathrm{spt}(f)\|$.
- Note that:
  - $a \in M \backslash \mathrm{spt}(f)$ gives rise to the f-cycle $\{a\}$;
  - The f-cycle of any $a \in \mathrm{spt}(f)$ has at least two elements.
- Therefore, we have

$$c(f) \leq \|M\| - s(f) + \frac{s(f)}{2} = \|M\| - \frac{s(f)}{2}.$$

Asymptotics for Rigid Structures

Proposition

Let τ be a relational vocabulary that contains at least one relation symbol of arity ≥ 2. Then $u(\mathrm{RIG}) = 1$.

- Obviously, we can assume $\|\tau\| = 1$.
  For simplicity, we restrict ourselves to $\tau = \{E\}$ with binary E.
  By a previous result, it suffices to show that $\lim_{n\to\infty} \frac{L_n(\tau)}{U_n(\tau)\cdot n!} = 1$.
  Equivalently, we show that $\lim_{n\to\infty} \frac{U_n(\tau)\cdot n!}{L_n(\tau)} = 1$.
  Clearly, $L_n(\tau) = 2^{n^2}$.
  Fix n and recall our conventions:
  - $\pi, \rho, \ldots$ denote permutations of $\{1, \ldots, n\}$;
  - $\mathcal{A}, \mathcal{B}, \ldots$ denote τ-structures with universe $\{1, \ldots, n\}$.

  Each π induces a permutation $\widetilde{\pi}$ of $\{1, \ldots, n\} \times \{1, \ldots, n\}$,
  $$\widetilde{\pi}((i,j)) := (\pi(i), \pi(j)).$$

Asymptotics for Rigid Structures (Cont'd)

- If π is an automorphism of $\mathcal{A}$ and $\{\overline{a}, \widetilde{\pi}(\overline{a}), \widetilde{\pi}(\widetilde{\pi}(\overline{a})), \ldots\}$ a $\widetilde{\pi}$-cycle, then $E^{\mathcal{A}}\overline{a}$ iff $E^{\mathcal{A}}\widetilde{\pi}(\overline{a})$ iff $E^{\mathcal{A}}\widetilde{\pi}(\widetilde{\pi}(\overline{a}))$ iff $\ldots$.
  I.e., we have $E^{\mathcal{A}}\overline{b}$ for all elements $\overline{b}$ of the $\widetilde{\pi}$-cycle of $\overline{a}$, or for none.
  Therefore, π is an automorphism for exactly $2^{c(\widetilde{\pi})}$ many τ-structures,
  $$\|\mathrm{Str}(\pi)\| = 2^{c(\widetilde{\pi})}.$$
  By the preceding lemma, we obtain
  $$\frac{U_n(\tau) \cdot n!}{L_n(\tau)} = \frac{\sum_\pi \|\mathrm{Str}(\pi)\|}{2^{n^2}} = \sum_\pi 2^{c(\widetilde{\pi}) - n^2}.$$
  We must show that $\sum_\pi 2^{c(\widetilde{\pi}) - n^2} \to 1$.
  For π the identity on $\{1, \ldots, n\}$, we have $c(\widetilde{\pi}) = n^2$.
  So we must show that
  $$\sum_{\pi \neq \mathrm{id}} 2^{c(\widetilde{\pi}) - n^2} \to 0.$$

Asymptotics for Rigid Structures (Cont'd)

- For any π, $\mathrm{spt}(\pi) \times \{1, \ldots, n\} \subseteq \mathrm{spt}(\widetilde{\pi})$. Hence, $c(\widetilde{\pi}) < n^2 - \frac{s(\pi) \cdot n}{2}$.

  So we get

  $$\sum_{\pi \neq \mathrm{id}} 2^{c(\widetilde{\pi}) - n^2} \leq \sum_{\pi \neq \mathrm{id}} 2^{\frac{-s(\pi) \cdot n}{2}}.$$

  For $k = 2, \ldots, n$, the number of π, with $s(\pi) = k$, is $\leq \binom{n}{k} k! \leq n^k$.

  Therefore, for $n > 2 \cdot \log n$, we have

  $$\begin{array}{rcl} \sum_{\pi \neq \mathrm{id}} 2^{\frac{-s(\pi) \cdot n}{2}} & \leq & \sum_{k=2}^{n} n^k \cdot 2^{-\frac{k \cdot n}{2}} \\ & = & \sum_{k=2}^{n} 2^{-\frac{1}{2} k (n - 2 \log n)} \\ & \leq & (n-1) \cdot 2^{-(n - 2 \log n)}. \end{array}$$

  For the last inequality, note that $k = 2$ gives the largest summand of the third sum.

  Since $(n-1) \cdot 2^{-(n - 2 \log n)} \to 0$, we obtain the result.

Subsection 4

Examples and Consequences

Summary of Previous Results

- Fix a nontrivial parametric sentence φ_0.
- Then $T_{\mathrm{rand}}(\varphi_0)$ has a uniquely determined countable model $\mathcal{R}(\varphi_0)$, the random model of φ_0.
- Some results of the preceding sections are summarized in:

Proposition

For an $L^{\omega}_{\infty\omega}$-sentence φ, the following are equivalent:

(i) $T_{\mathrm{rand}}(\varphi_0) \models \varphi$;

(ii) $\mathcal{R}(\varphi_0) \models \varphi$;

(iii) $\ell(\varphi|\varphi_0) = 1$;

(iv) $u(\varphi|\varphi_0) = 1$.

Properties of Nontrivial Parametric Sentences

Proposition

(a) Let $\mathcal{B}$ be a finite model of φ_0. Then almost all finite models of φ_0 contain a substructure isomorphic to $\mathcal{B}$.

(b) Let $\mathcal{B}$ be a finite model of φ_0 and π_0 an isomorphism of a substructure $\mathcal{A}$ of $\mathcal{B}$ into $\mathcal{R}(\varphi_0)$. Then π_0 can he extended to an isomorphism of $\mathcal{B}$ into $\mathcal{R}(\varphi_0)$.

(a) Let $\mathcal{B}$ consist of s elements and let φ_0^s be the conjunction of φ_0 and the finitely many r-extension axioms of $T_{\mathrm{rand}}(\varphi_0)$, with $r \leq s$.

Clearly, any model of φ_0^s contains a substructure isomorphic to $\mathcal{B}$.

As $\ell(\varphi_0^s | \varphi_0) = 1$, the claim follows.

Properties of Nontrivial Parametric Sentences (Cont'd)

(b) Let $A = \{\overline{a}\}$ and $B = \{\overline{a}, \overline{b}\}$.

Note that $\varphi^0_{\mathcal{A},\overline{a}} = \varphi^0_{\mathcal{B},\overline{a}}$.

The sentence

$$\forall \overline{v}(\varphi^0_{\mathcal{A},\overline{a}}(\overline{v}) \to \exists \overline{w}\varphi^0_{\mathcal{B},\overline{a}\overline{b}}(\overline{v},\overline{w}))$$

is a consequence of the extension axioms in $T_{\text{rand}}(\varphi_0)$.

Hence, $\mathcal{R}(\varphi_0)$ is a model of it.

Now $\mathcal{A} \vDash \varphi^0_{\mathcal{A},\overline{a}}[\overline{a}]$ and $\varphi^0_{\mathcal{A},\overline{a}}$ is quantifier-free.

Hence, $\mathcal{R}(\varphi_0) \vDash \varphi^0_{\mathcal{A},\overline{a}}[\pi_0(\overline{a})]$.

So there are $\overline{d}$ in $\mathcal{R}(\varphi_0)$ such that $\mathcal{R}(\varphi_0) \vDash \varphi^0_{\mathcal{B},\overline{a}\overline{b}}[\pi_0(\overline{a}),\overline{d}]$.

Thus, $\overline{a}\overline{b} \mapsto \pi_0(\overline{a})\overline{d}$ is the required isomorphism.

Isomorphic Substructures of $\mathcal{R}(\varphi_0)$

Proposition

If $\mathcal{B}$ and $\mathcal{C}$ are isomorphic finite substructures of $\mathcal{R}(\varphi_0)$, then there is an automorphism of $\mathcal{R}(\varphi_0)$ mapping $\mathcal{B}$ onto $\mathcal{C}$.

- Assume that $\mathcal{R}(\varphi_0)$ has universe $\{a_0, a_1, a_2, \ldots\}$ and that $\pi : \mathcal{B} \cong \mathcal{C}$.

  Take the finite substructure $\mathcal{B}_0'$ of $\mathcal{R}(\varphi_0)$ with $B_0' = B \cup \{a_0\}$.

  By part (b) of the preceding proposition, there is an isomorphism $\pi_0' : \mathcal{B}_0' \cong \mathcal{C}_0'$ for a suitable $\mathcal{C}_0' \subseteq \mathcal{R}(\varphi_0)$ that extends π.

  Similarly, we get $\pi_0 : \mathcal{B}_0 \cong \mathcal{C}_0$ with $\pi_0' \subseteq \pi_0$, $\mathcal{B}_0 \subseteq \mathcal{R}(\varphi_0)$, $\mathcal{C}_0 \subseteq \mathcal{R}(\varphi_0)$ and $C_0 = C_0' \cup \{a_0\}$.

  Continuing this way we obtain a sequence $\pi_0 \subseteq \pi_1 \subseteq \pi_2 \subseteq \cdots$ with $a_i \in \mathrm{dom}(\pi_i) \cap \mathrm{ran}(\pi_i)$.

  Thus, $\pi := \bigcup_{i \geq 0} \pi_i$ is the desired automorphism of $\mathcal{R}(\varphi_0)$.

Graphs

- Let φ_G be a parametric sentence axiomatizing the class of graphs.
- $T_{\text{rand}}(\varphi_G)$ is equivalent to

$$T_{\text{rand},G} \quad := \quad \{\varphi_G, \varphi_{\geq 2}\} \cup \{\forall \text{ distinct } x_1 \ldots x_n y_1 \ldots y_n \\ \exists z \left(\bigwedge_{i=1}^n Ex_i z \wedge \bigwedge_{i=1}^{\ell}(\neg Ey_i z \wedge \neg y_i = z)\right) : n + \ell \geq 1\right\}.$$

- In fact:
  - The sentences in $T_{\text{rand},G}$ are implied by the extension axioms in $T_{\text{rand}}(\varphi_G)$. Hence, any model of $T_{\text{rand}}(\varphi_G)$ is a model of $T_{\text{rand},G}$.
  - On the other hand, a back and forth argument using the axioms in $T_{\text{rand},G}$ shows that any two of its models and hence, any model of $T_{\text{rand},G}$ and any model of $T_{\text{rand}}(\varphi_G)$ are partially isomorphic.

Therefore, any model of $T_{\text{rand},G}$ is a model of $T_{\text{rand}}(\varphi_G)$.

Planarity and Connectedness

- It is well known that a graph cannot be planar if it contains the subgraph $\mathcal{K}_5$, a clique with 5 elements.
- Thus, using Part (a) of a preceding proposition, we get:

Proposition

Almost all finite graphs are not planar.

Proposition

$\mathcal{R}(\varphi_G)$, the random graph, and almost all finite graphs $\mathcal{G}$ are connected, the diameter $D(\mathcal{G}) := \max\{d(a,b) : a, b \in G\}$ being 2.

- Note that, by the extension axioms, $\mathcal{R}(\varphi_G)$ and almost all finite graphs are models of $\exists x \exists y \neg Exy \wedge \forall x \forall y \exists z (Exz \wedge Eyz)$.

  Any graph $\mathcal{G}$ satisfying this sentence is connected with $D(\mathcal{G}) = 2$.

Rigidity

- Even though "connectedness" is not expressible in first-order logic, the first-order sentence $\exists x \exists y \neg Exy \wedge \forall x \forall y \exists z (Exz \wedge Eyz)$ is a property of almost all graphs that implies "connectedness".
- The situation for "rigidity" is different:

Proposition

(a) Almost all finite graphs are rigid.

(b) $\mathcal{R}(\varphi_G)$ is not rigid.

(c) No $L^{\omega}_{\infty\omega}$-definable property of almost all graphs implies rigidity.

- For part (a) see previous results.

  Part (b) is an immediate consequence of a preceding proposition.

  There, we take $\mathcal{B}$ and $\mathcal{C}$ to be two substructures of cardinality one.

  Part (c) follows from (b) because any $L^{\omega}_{\infty\omega}$-definable property of almost all graphs holds in $\mathcal{R}(\varphi_G)$.

Using Fragments of Second Order Logic

- Connectedness of graphs can be expressed by a Π_1^1-sentence, for example by

$$\varphi_{\mathrm{CONN}} := \forall X(\forall x X x \vee \forall x \neg X x \vee \exists x \exists y(X x \wedge \neg X y \wedge E x y)).$$

- Nonrigidity is expressible by

$$\begin{aligned}&\exists X \forall x \forall y \forall u \forall v \exists z_1 \exists z_2 \exists w(X z_1 x \wedge X x z_2 \wedge \neg X w w \wedge \\ &\qquad((X x y \wedge X u v) \rightarrow((x=u \leftrightarrow y=v) \wedge(E x u \leftrightarrow E y v)))).\end{aligned}$$

- This is called a $\Sigma_1^1(\forall^* \exists^*)$-sentence, that is, a sentence of the form $\exists X_1 \cdots \exists X_s \forall y_1 \cdots \forall y_m \exists z_1 \cdots \exists z_{\ell} \chi$, with $s, m, \ell \in \mathbb{N}$ and χ quantifier-free.
- A $\Sigma_1^1(\exists^* \forall^*)$-sentence has the form $\exists X_1 \cdots \exists X_s \exists y_1 \cdots \exists y_m \forall z_1 \cdots \forall z_{\ell} \chi$, where χ is quantifier-free.

Parametric Sentences and Second Order Fragments

- Part (a) of the following proposition generalizes the fact that connectedness is implied by a first-order property of almost all graphs.
- Part (b) shows that nonrigidity cannot be expressed by a $\Sigma_1^1(\exists^*\forall^*)$-sentence (otherwise almost all finite graphs would be nonrigid).

Proposition

Suppose that φ_0 is nontrivial parametric.

(a) Let φ be a Π_1^1-sentence. If $\mathcal{R}(\varphi_0) \models \varphi$, then there is a first-order sentence ψ such that $u(\psi|\varphi_0) = 1$ and $\models \psi \rightarrow \varphi$.

(b) Let φ be a $\Sigma_1^1(\exists^*\forall^*)$-sentence. If $\mathcal{R}(\varphi_0) \models \varphi$, then there is a first-order sentence ψ such that $u(\psi|\varphi_0) = 1$ and $\models_{\text{fin}} \psi \rightarrow \varphi$.

Parametric Sentences and Second Order (Part (a))

(a) Assume that $\mathcal{R}(\varphi_0) \vDash \varphi$ holds for the Π^1_1-sentence $\varphi = \forall X_1 \cdots \forall X_s \chi$, where χ contains no second-order quantifiers.

Claim: The set $T_{\mathrm{rand}}(\varphi_0) \cup \{\neg\chi\}$ of $\tau \cup \{X_1, \ldots, X_s\}$-sentences has no model.

Suppose, to the contrary, that $T_{\mathrm{rand}}(\varphi_0) \cup \{\neg\chi\}$ has a model.

By the Löwenheim-Skolem Theorem, it has a countable model.

Its τ-reduct is (isomorphic to) the unique countable model $\mathcal{R}(\varphi_0)$ of $T_{\mathrm{rand}}(\varphi_0)$. Thus, $\mathcal{R}(\varphi_0) \vDash \exists X_1 \cdots \exists X_s \neg\chi$.

This contradicts $\mathcal{R}(\varphi_0) \vDash \forall X_1 \cdots \forall X_s \chi$.

By compactness, there is a finite subset T_0 of $T_{\mathrm{rand}}(\varphi_0)$, such that $T_0 \cup \{\neg\chi\}$ is not satisfiable.

Let ψ be the conjunction of the sentences in T_0.

Then $u(\psi|\varphi_0) = 1$ and $\vDash \psi \to \chi$. Hence, $\vDash \psi \to \forall X_1 \cdots \forall X_s \chi$.

Parametric Sentences and Second Order (Part (b))

(b) Suppose that for the $\Sigma_1^1(\exists^*\forall^*)$-sentence $\varphi = \exists X_1 \cdots \exists X_s \exists \overline{x} \forall \overline{y} \chi$, with quantifier-free χ, we have $\mathcal{R}(\varphi_0) \vDash \varphi$, say,

$$(\mathcal{R}(\varphi_0), X_1, \ldots, X_s) \vDash \exists \overline{x} \forall \overline{y} \chi.$$

Choose $\overline{a}$ in $\mathcal{R}(\varphi_0)$, such that

$$(\mathcal{R}(\varphi_0), X_1, \ldots, X_s) \vDash \forall \overline{y} \chi[\overline{a}].$$

Denote by $\mathcal{A}_0$ the submodel of $\mathcal{R}(\varphi_0)$ with universe $\{\overline{a}\}$.
Now $\exists \overline{x} \varphi_{\overline{a}}^0(\overline{x})$ holds in $\mathcal{R}(\varphi_0)$.
Thus, there is a ψ, that is the conjunction of φ_0 and of finitely many extension axioms compatible with φ_0, such that

$$\vDash \psi \rightarrow \exists \overline{x} \varphi_{\overline{a}}^0(\overline{x}).$$

Obviously, $u(\psi | \varphi_0) = 1$.

Parametric Sentences and Second Order (Part (b) Cont'd)

Claim: $\models_{\text{fin}} \psi \to \varphi$.

Let $\mathcal{B}$ be a finite model of ψ.

Choose $\overline{d}$ in $\mathcal{B}$ such that $\mathcal{B} \models \varphi^0_{\overline{a}}[\overline{d}]$. Then $\overline{d} \mapsto \overline{a}$ is an isomorphism from the substructure of $\mathcal{B}$ with universe $\{\overline{d}\}$ into $\mathcal{R}(\varphi_0)$.

By a previous proposition, there is an extension π of $\overline{d} \mapsto \overline{a}$ that is an isomorphism of $\mathcal{B}$ onto a substructure $\mathcal{B}'$ of $\mathcal{R}(\varphi_0)$.

It suffices to show that $\mathcal{B}'$ is a model of φ.

We have $(\mathcal{R}(\varphi_0), X_1, \ldots, X_s) \models \forall \overline{y}\chi[\overline{a}]$ and $\forall \overline{y}\chi$ is universal.

It follows that $(\mathcal{B}', X_1 \cap B', \ldots, X_s \cap B') \models \forall \overline{y}\chi[\overline{a}]$.

Hence, $(\mathcal{B}', X_1 \cap B', \ldots, X_s \cap B') \models \exists \overline{x} \forall y \chi$.

Thus, $\mathcal{B}' \models \exists X_1 \cdots \exists X_s \exists \overline{x} \forall \overline{y} \chi$.

A Consequence

Theorem

Let φ_0 be nontrivial parametric and let φ be a $\Sigma_1^1(\exists^*\forall^*)$-sentence.

(a) If $\mathcal{R}(\varphi_0) \vDash \varphi$, then $u(\varphi|\varphi_0) = 1$.

(b) If $\mathcal{R}(\varphi_0) \nvDash \varphi$, then $u(\varphi|\varphi_0) = 0$.

(a) If $\mathcal{R}(\varphi_0) \vDash \varphi$, then, by Part (b) of the proposition, there is a first-order sentence ψ, such that $u(\psi|\varphi_0) = 1$ and $\vDash_{\mathrm{fin}} \psi \to \varphi$.

In particular, $u(\varphi|\varphi_0) = 1$.

(b) Assume that $\mathcal{R}(\varphi_0) \vDash \neg\varphi$.

But $\neg\varphi$ is (logically equivalent to) a Π_1^1-sentence.

By Part (a) of the proposition, there exists a first-order sentence ψ, such that $u(\psi|\varphi_0) = 1$ and $\vDash \psi \to \neg\varphi$.

Therefore, $u(\neg\varphi|\varphi_0) = 1$. Hence, $u(\varphi|\varphi_0) = 0$.

Labeled Probabilities

Corollary

$\Sigma_1^1(\exists^*\forall^*)$ satisfies the labeled and the unlabeled 0-1 law with respect to nontrivial parametric classes.

- By a previous proposition, the preceding results remain valid for the labeled probability as well.
- The satisfiability problem for $\exists^*\forall^*$-sentences is decidable.

  We have just seen that $\Sigma_1^1(\exists^*\forall^*)$ has the 0-1 law.
- This is a special case of a general phenomenon.

  The satisfiability problem for a prefix class Φ of first-order logic is decidable just in case the 0-1 law holds for $\Sigma_1^1\Phi := \{\exists\overline{R}\varphi : \varphi \in \Phi\}$.

Example: A Class of Ordered Structures

- Let $\tau = \{<, P\}$, with P unary.

  Let $\mathcal{O} = \mathcal{O}[\tau]$ be the class of τ-structures ordered by $<$.

  Denote by φ a first-order sentence expressing that the first element of the ordering is in P, say,

$$\varphi = \exists x(Px \wedge \forall y \neg y < x).$$

  Then

$$u(\varphi|\mathcal{O}) = \ell(\varphi|\mathcal{O}) = \frac{1}{2}.$$

  By a previous result, $\mathcal{O}$ is not parametric.

Example: Orderings and 0-1 Laws

- Let $\tau = \{<\}$.
  Consider the class ORD of orderings.
  - ORD satisfies the labeled and the unlabeled 0-1 law for FO.
    By a previous result, for any first order sentence φ we have
    $$u(\varphi|\mathrm{ORD}) = \ell(\varphi|\mathrm{ORD}) = 1 \quad \text{iff} \quad (\{0,\ldots,2^k\},<) \vDash \varphi,$$
    where k is the quantifier rank of φ.
  - ORD does not satisfy the (labeled or unlabeled) 0-1 law for $L^2_{\infty\omega}$.
    The probabilities $\ell(\varphi|\mathrm{ORD})$ and $u(\varphi|\mathrm{ORD})$ do not exist for any $L^2_{\infty\omega}$-sentence φ expressing that the ordering has an even number of elements. As φ one can take
    $$\bigvee_{n\geq 1} \chi_{2n},$$
    where χ_{2n} are the FO^2-formulas introduced in a previous example.