Finite Model Theory

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1. Satisfiability in the Finite
   - Finite Model Property of FO^2
   - Finite Model Property of $\forall^2\exists^*$-Sentences

Subsection 1

Finite Model Property of FO^2

Finite Model Property

- A class Φ of sentences has the **finite model property** if every satisfiable sentence of Φ has a finite model.
- We may ask whether a given class Φ has the finite model property.

  Example: Let φ be a first-order sentence expressing that $<$ is a partial ordering without maximal elements.
  - φ is satisfiable.
  - φ has no finite model.

  As φ we can take either of the following:

  (1) $\forall x \neg x < x \wedge \forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z) \wedge \forall x \exists y x < y$
  This sentence uses only three variables.

  (2) $\forall x \forall y \forall z \exists u (\neg x < x \wedge ((x < y \wedge y < z) \rightarrow x < z) \wedge x < u)$
  This sentence is a $\forall^3 \exists$-sentence.
- The sentences in the example are "best" possible:
  - Every satisfiable sentence with at most two variables has a finite model;
  - Every satisfiable $\forall^2 \exists^*$-sentence without equality has a finite model.

Remark on Function Symbols

- We prove that every satisfiable sentence with at most two variables in a relational vocabulary has a finite model.
- We remark that:
  - One can remove the restriction on constants;
  - The result is not valid for "vocabularies" with function symbols.

Example: Consider the sentence

$$\forall x \forall y (f(x) = f(y) \to x = y) \wedge \exists y \forall x \neg f(x) = y$$

expressing that f is injective but not surjective.

This is a sentence with two variables.

It is satisfiable but does not have a finite model.

Normal First Order Formulas

- Fix a relational vocabulary τ.
- Let $x := v_1$ and $y := v_2$.
- A first-order formula (possibly containing second order variables) is **normal**, if it has the form

$$\forall x \forall y \psi \wedge \bigwedge_{i=1}^{r} \forall x \exists y \psi_i,$$

  where $\psi, \psi_i \in FO^2$ are quantifier-free.

A Form Reduction Lemma

Lemma

Every sentence $\exists\overline{X}(\varphi \wedge \forall x \forall y \psi)$, where φ is normal and $\psi \in FO^2$, is equivalent to a sentence of the form $\exists\overline{Y}\chi$, where χ is normal.

- We proceed by induction on the number of quantifiers in ψ.

  If ψ contains no quantifiers, the result is immediate.

  In the induction step we show how to eliminate a quantifier in ψ.

  So let, say, $\forall x \psi_0$ be a subformula of ψ with quantifier-free ψ_0.

  Then, ψ is logically equivalent to

$$\exists X(\forall y(Xy \leftrightarrow \forall x \psi_0) \wedge \forall x \forall y \psi'),$$

  where ψ' results from ψ by replacing $\forall x \psi_0$ by Xy.

A Form Reduction Lemma (Cont'd)

- Hence, it is logically equivalent to

$$\exists X(\forall x \forall y(Xy \to \psi_0) \wedge \forall y \exists x(\neg \psi_0 \vee Xy) \wedge \forall x \forall y \psi').$$

So it also equivalent to

$$\exists X(\forall x \forall y(Xy \to \psi_0) \wedge \forall x \exists y(\neg \psi_0 \binom{yx}{xy} \vee Xx) \wedge \forall x \forall y \psi'),$$

where $\psi_0 \binom{yx}{xy}$ is obtained from ψ_0 by simultaneously replacing *all occurrences* of x and y by y and x, respectively.
Altogether, $\exists \overline{X}(\varphi \wedge \forall x \forall y \psi)$ is equivalent to

$$\exists \overline{X} \exists X(\varphi \wedge \forall x \forall y(Xy \to \psi_0) \wedge \forall x \exists y(\neg \psi_0 \binom{yx}{xy} \vee Xx) \wedge \forall x \forall y \psi'),$$

where the first conjunct is normal and ψ' has less quantifiers than ψ.
By the induction hypothesis, we obtain the claim.

A "Canonical Form"

Corollary

Every sentence of FO^2 is equivalent to a sentence of the form $\exists \overline{Y} \chi$, where χ is normal.

- Given an FO^2-sentence ψ, apply the preceding lemma to

$$\exists \overline{X}(\varphi \wedge \forall x \forall y \psi),$$

where $\overline{X}$ is the empty sequence and $\varphi := \forall x \forall y x = x$.

Finite Model Property of FO^2-Sentences

Theorem

Every satisfiable first-order sentence with at most two variables in a relational vocabulary has a finite model.

- Let φ be such a sentence. We apply the preceding corollary.

  Note that $\exists\overline{Y}\chi$ and χ have models over the same universes.

  So we may assume that φ has the form

$$\varphi = \forall x \forall y \psi \wedge \bigwedge_{i=1}^{r} \forall x \exists y \psi_i,$$

  where $\psi, \psi_i \in FO^2$ are quantifier-free.

  Moreover, note that over structures with at least two elements $\forall x \exists y \psi_i(x, y)$ is equivalent to $\forall x \exists y (x \neq y \wedge (\psi_i(x, y) \vee \psi_i(x, x)))$.

  So we may also suppose that for $i = 1, \ldots, r$, $\psi_i \vDash x \neq y$.

Finite Model Property of FO^2-Sentences (Cont'd)

- Let $\mathcal{A}$ be a model of φ.

  Call an element $a \in A$ a **king** (in $\mathcal{A}$) if there is no other element b of A with the same 0-isomorphism type, i.e., $\varphi^0_{\mathcal{A},a} = \varphi^0_{\mathcal{A},b}$.

  If $\mathcal{A} \models \psi_i[a,b]$, we call b a **child** (or an i-**child**) of a (in $\mathcal{A}$).

  For $a \in A$, and $i \in \{1,\ldots,r\}$, we let a^i be a fixed i-child of a.

  Then, $a \neq a^i$.

  Set
  $$C := \bigcup_{a \in A,\ a \text{ king}} \{a, a^1, \ldots, a^r\}.$$

  Clearly, C is finite.

Finite Model Property of FO^2-Sentences (Cont'd)

- We show that there is a $\mathcal{B}$ such that
  (i) $B = C \cup (\{\varphi^0_{\mathcal{A},a} : a \in A \text{ no king}\} \times \{1, \ldots, r\} \times \{0, 1, 2\})$;
  (ii) Each 0-isomorphism type of a pair of elements of B is realized in $\mathcal{A}$;
  (iii) For $i = 1, \ldots, r$, all elements of B have an i-child in $\mathcal{B}$.

  Then $\mathcal{B}$ is:
  - A model of $\forall x \forall y \psi$, by Clause (ii);
  - A model of $\bigwedge_{i=1}^r \forall x \exists y \psi_i$, by Clause (iii).

  Thus by Clause (i), $\mathcal{B}$ is a finite model of φ.

  To define $\mathcal{B}$, we fix the 0-isomorphism type of all pairs of elements of B in a suitable way to ensure that Clauses (ii) and (iii) hold.

  In case τ contains relation symbols of arity ≥ 3, the rest can be fixed in an arbitrary way.

  Step 1: For $a, b \in C$, $a \neq b$, we set $\varphi^0_{\mathcal{B},a,b} := \varphi^0_{\mathcal{A},a,b}$.

Finite Model Property of FO^2-Sentences (Cont'd)

Step 2: Let $b \in B$. We aim at providing children for b in $\mathcal{B}$.
So let $i \in \{i, \ldots, r\}$.

- Suppose $b \in C$ and b is a king or b has an i-child in $\mathcal{A}$ that lies in C. Then b has an i-child in $\mathcal{B}$ because of Step 1.
- Suppose $b = a^j$ for a king a, but b has no i-child in C. Let $b' := (\varphi^0_{\mathcal{A},(a^j)^i}, i, 0)$ be an i-child of b in $\mathcal{B}$ by setting
$$\varphi^0_{\mathcal{B},b,b'} := \varphi^0_{\mathcal{A},a^j,(a^j)^i}.$$
(In case there are several possibilities for a and j, we fix one choice; and we also do so in similar situations.)
- Suppose $b = (\varphi^0_{\mathcal{A},a}, j, k)$ ($a \in A$ not a king in $\mathcal{A}$) and a^i is a king in $\mathcal{A}$. Then we let a^i be an i-child of b in $\mathcal{B}$ by setting $\varphi^0_{\mathcal{B},b,a^i} := \varphi^0_{\mathcal{A},a,a^i}$.
- Suppose $b = (\varphi^0_{\mathcal{A},a}, j, k)$ and a^i is not a king in $\mathcal{A}$. Let $b' := (\varphi^0_{\mathcal{A},a^i}, i, (k+1) \pmod 3)$ be an i-child of b in $\mathcal{B}$ by setting $\varphi^0_{\mathcal{B},b,b'} := \varphi^0_{\mathcal{A},a,a^i}$.

In all cases, by fixing a type $\varphi^0_{\mathcal{B},a,b}$, we also fix $\varphi^0_{\mathcal{B},b,a}$.

Finite Model Property of FO^2-Sentences (Cont'd)

Step 3: If, e.g., for $d \in C$, $b := (\varphi^0_{\mathcal{A},a}, j, k)$ and $b' := (\varphi^0_{\mathcal{A},a'}, j', k')$, the 0-isomorphism type of (d, b) or of (b, b') has not been fixed in the first two steps, we set

$$\varphi^0_{\mathcal{B},d,b} := \varphi^0_{\mathcal{A},d,a} \quad \text{or} \quad \varphi^0_{\mathcal{B},b,b'} := \varphi^0_{\mathcal{A},a,a'},$$

respectively.

The definitions we gave do not contradict each other, since:

- For $c \in C$, we have $\varphi^0_{\mathcal{B},c} = \varphi^0_{\mathcal{A},c}$;
- For $b = (\varphi^0_{\mathcal{A},a}, j, k)$, we have $\varphi^0_{\mathcal{B},b} = \varphi^0_{\mathcal{A},a}$.

Moreover, by construction, Clauses (ii) and (iii) are satisfied.

Decidability

Corollary

For any relational vocabulary τ, the set Φ of logically valid first order sentences with at most two variables is decidable.

- By the Completeness Theorem for first order logic the set Φ is enumerable.

  Consider its "complement"

  $$\Phi^{\mathrm{nv}} := \{\varphi \ \mathrm{FO}^2[\tau]\text{-sentence} : \varphi \text{ is not logically valid}\}.$$

  By the preceding theorem,

  $$\Phi^{\mathrm{nv}} = \{\varphi \ \mathrm{FO}^2[\tau]\text{-sentence} : \neg\varphi \text{ has a finite model}\}.$$

  Therefore, Φ^{nv} is enumerable too.

  Hence, Φ is decidable.

Subsection 2

Finite Model Property of $\forall^2\exists^*$-Sentences

FO^2-Sentences Reviewed

- We fix a relational vocabulary τ.
- We proved that every FO^2-sentence has models in the same cardinalities as a sentence of the form

$$\forall x \forall y \psi \wedge \bigwedge_{k=1}^{r} \forall x \exists y \psi_i,$$

  with quantifier-free ψ, ψ_i.
- This sentence is equivalent to

$$\forall x \forall y \exists y_1 \cdots \exists y_r \left(\psi(x,y) \wedge \bigwedge_{i=1}^{r} \psi_i(x, y_i) \right).$$

- We then have proved the finite model property for these sentences.

$\forall^2\exists^*$-Sentences

- By a $\forall^2\exists^*$-**sentence** we mean a first-order sentence of the form

$$\forall x_1 \forall x_2 \exists y_1 \cdots \exists y_k \psi',$$

  where $k \geq 0$ and ψ' is quantifier-free.
- We extend the result about the finite model property to sentences of this form, under the proviso that they have models without kings.
- Recall that an element $a \in A$ is a **king** in the structure $\mathcal{A}$ if for no $b \in A$, $b \neq a$, is it the case that

$$\varphi_a^0 = \varphi_b^0.$$

$\forall^2\exists^*$-Sentences with Models Without Kings

Theorem

Suppose that τ is a relational vocabulary. If ψ is a $\forall^2\exists^*$-sentence which has a model without kings, then it has a finite model.

- In models with at least two elements a $\forall^2\exists^*$-sentence $\forall v_1 \forall v_2 \exists v_3 \cdots \exists v_k \psi'(v_1, \ldots, v_k)$ is equivalent to the sentence

$$\forall v_1 \forall v_2 \exists x_3 \cdots \exists x_k \exists z_3 \cdots \exists z_k (\neg v_1 = v_2 \to (\psi'(v_1, v_1, x_3, \ldots, x_k) \wedge \psi'(v_1, v_2, z_3, \ldots, z_k))).$$

  Moreover, a sentence $\forall v_1 \forall v_2 \exists x \exists y \psi'(v_1, v_2, x, y)$, with ψ' is quantifier-free, is equivalent to

$$\forall v_1 \forall v_2 \exists x \exists y ((\psi'(v_1, v_2, x, x) \vee \psi'(v_1, v_2, x, y)) \wedge \neg x = y).$$

$\forall^2\exists^*$-Sentences with Models Without Kings (Types)

- The preceding equivalences allow us to assume that our $\forall^2\exists^*$-sentence ψ is of the form

$$\forall v_1 \forall v_2 \exists v_3 \cdots \exists v_k \left(\neg v_1 = v_2 \to \left(\psi'(v_1, \ldots, v_k) \wedge \bigwedge_{3 \le i < j \le k} \neg v_i = v_j \right) \right),$$

with ψ' quantifier-free.
Choose a model $\mathcal{A}$ of ψ without kings. Let:
  - $S := \{\varphi_a^0 : a \in A\}$ be the 0-isomorphism types of elements of $\mathcal{A}$;
  - $T := \{\varphi_{ab}^0 : a, b \in A, a \neq b\}$ be the 0-isomorphism types of pairs in $\mathcal{A}$.

Let $\rho(v_1, \ldots, v_\ell)$ be a 0-isomorphism type of any ℓ-tuple.
For $1 \le m, n \le \ell$, with $m \neq n$, let:
  - $\rho_m(v_1)$ be the induced 0-isomorphism type of v_m;
  - $\rho_{m,n}(v_1, v_2)$ be the induced 0-isomorphism type of v_m, v_n.

In particular, for any $\mathcal{B}$ and $b_1, \ldots, b_\ell \in B$,

$$\mathcal{B} \vDash \rho[b_1, \ldots, b_\ell] \quad \text{implies} \quad (\varphi_{b_m}^0 = \rho_m \text{ and } \varphi_{b_m b_n}^0 = \rho_{m,n}).$$

$\forall^2\exists^*$-Sentences with Models Without Kings (Conditions)

- As $\mathcal{A}$ has no kings, we get:

  (1) For all $\varphi, \varphi' \in S$ there is a $\chi \in T$ such that $\varphi = \chi_1$ and $\varphi' = \chi_2$.

  Moreover, since $\mathcal{A}$ is a model of ψ, we have:

  (2) For every $\chi \in T$, there is a 0-isomorphism type $\rho(v_1, \ldots, v_k)$ with

  - (a) $\rho_i \in S$, for $i = 1, \ldots, k$;
  - (b) $\rho_{m,n} \in T$, for $1 \leq m < n \leq k$ and $\rho_{1,2} = \chi$;
  - (c) $\vDash \psi'(v_1, \ldots, v_k)$.

  To get the statement of the theorem it suffices to show:

  Suppose that S and T are nonempty sets of 0-isomorphism types of elements and of pairs of elements, respectively, satisfying (1) and (2). Then ψ has a finite model.

$\forall^2\exists^*$-Sentences with Models Without Kings (Outline)

- Let $s := \|S\|$, $t := \|T\|$ and fix an ordering on S.

  We give a method to construct, for every $n \geq k$, structures $\mathcal{B}$ with universe $\{1, 2, \ldots, n \cdot s\}$.

  Subsequently, we will show that with nonvanishing probability these structures are models of ψ.
  - The 0-isomorphism types of elements are fixed by a deterministic algorithm;
  - The 0-isomorphism types of tuples of more than one element are chosen randomly.

$\forall^2\exists^*$-Sentences with Models Without Kings (Details)

- The exact construction of $\mathcal{B}$ reads as follows:
  (i) If $a \in \{1, 2, \ldots, n \cdot s\}$ and $a = i \cdot s + j$, for some i, j such that $0 \le i < n$ and $i \le j < s$, ensure that φ_a^0 is equal to the j-th element in S.
  (ii) If $a, b \in \{1, 2, \ldots, n \cdot s\}$, $1 \le a < b \le n \cdot s$, choose at random a χ in $\{\chi \in T : \chi_1 = \varphi_a^0, \chi_2 = \varphi_b^0\}$ (this set is nonempty by (1)) and ensure that $\varphi_{ab}^0 = \chi$.
  (iii) If R is an m-ary relation symbol in τ, define the truth value of $Ra_1 \ldots a_m$ at random for any $a_1, \ldots, a_m \in \{1, 2, \ldots, n \cdot s\}$ containing at least three and at most k distinct members.
  (iv) If R is an m-ary relation symbol in τ, define the truth value of $Ra_1 \ldots a_m$ to be "false" if $a_1 \ldots a_m$ contains more than k distinct members.

$\forall^2\exists^*$-Sentences with Models Without Kings (Notation)

- Let Str(n) be the collection of possible values of $\mathcal{B}$ with $\{1, 2, \ldots, n \cdot s\}$ as universe.

  Equip Str(n) with the uniform probability distribution μ.

  Let $\overline{a} = a_1 \ldots a_k$ denote pairwise distinct elements of $\{1, 2, \ldots, n \cdot s\}$.

  Let d be the number of formulas

$$Rv_{i_1} \ldots v_{i_m},$$

  where:
  - $R \in \tau$;
  - $\{v_{i_1}, \ldots, v_{i_m}\}$ contains at least three and at most k distinct variables.

$\forall^2\exists^*$-Sentences with Models Without Kings (Claim 1)

Claim 1: Suppose that $\chi \in T$ and that the 0-isomorphism type $\rho(v_1,\ldots,v_k)$ satisfies (2) with respect to χ. Then the conditional probability

$$\mu(\varphi^0_{\overline{a}} = \rho \mid \varphi^0_{a_1a_2} = \chi, \varphi^0_{a_i} = \rho_i \text{ for } i = 3,\ldots,k) \geq \delta,$$

where $\delta = (\frac{1}{t})^{\binom{k}{2}-1} \cdot (\frac{1}{2})^d$. That is,

$$\mu(\{\mathcal{B} : \mathcal{B} \vDash \rho[\overline{a}]\} \mid \{\mathcal{B} : \mathcal{B} \vDash \chi[a_1,a_2], \mathcal{B} \vDash \rho_i[a_i], \text{ for } i = 3,\ldots,k\}) \geq \delta.$$

For the proof note that, once $\varphi^0_{a_i}$, for $i = 1,\ldots,k$, and $\varphi^0_{a_1a_2}$ are fixed:

- We must choose randomly one of the t types in T, for each of the $\binom{k}{2} - 1$ pairs of elements other than a_1, a_2;
- We must choose randomly among the two options, for each of the d formulas $Rv_{i_1}\ldots v_{i_m}$, with at least three and at most k distinct variables.

$\forall^2\exists^*$-Sentences with Models Without Kings (Claim 2)

Claim 2: Fix a_1, a_2. Then

$$\mu\left(\left\{\mathcal{B} : \mathcal{B} \nvDash \exists v_3 \cdots \exists v_k \left(\psi'(a_1, a_2, v_3, \ldots, v_k) \wedge \bigwedge_{1 \le i < j \le k} v_i \ne v_j\right)\right\}\right) \le (1-\delta)^f,$$

where f is the integer part of $\frac{n-2}{k-2}$.

Let $\chi \in T$ and choose a corresponding $\rho(v_1, \ldots, v_k)$ according to (2). It suffices to prove that the conditional probability

$$\mu(\{\mathcal{B} : \mathcal{B} \nvDash \exists v_3 \cdots \exists v_k \rho(a_1, a_2, v_3, \ldots, v_k)\} | \varphi^0_{a_1,a_2} = \chi) \le (1-\delta)^f.$$

By Condition (i), in any $\mathcal{B}$, every 0-isomorphism type in S is realized by n $(\ge 2 + f \cdot (k-2))$ distinct elements.

Therefore, for $i = 3, \ldots, k$ and $j = 1, \ldots, f$, there are pairwise distinct elements $a^j_i \in \{1, 2, \ldots, n \cdot s\} \backslash \{a_1, a_2\}$ with $\varphi^0_{a^j_i} = \rho_i$.

Under the given conditions, the events $\mathcal{B} \vDash \rho(a_1, a_2, a^j_3, \ldots, a^j_k)$, for $1 \le j \le f$ are independent (compare the construction procedure).

$\forall^2\exists^*$-Sentences with Models Without Kings (Conclusion)

- Now,

$$\{\mathcal{B} : \mathcal{B} \not\models \exists v_3 \cdots \exists v_k \rho(a_1, a_2, v_3, \ldots, v_k)\}$$
$$\subseteq \{\mathcal{B} : \text{ for } j = 1, \ldots, f, \mathcal{B} \not\models \rho(a_1, a_2, a_3^j, \ldots, a_k^j)\}.$$

Therefore, by Claim 1, we obtain

$$\mu(\{\mathcal{B} : \mathcal{B} \not\models \exists v_3 \cdots \exists v_k \rho(a_1, a_2, v_3, \ldots, v_k)\} | \varphi^0_{a_1,a_2} = \chi) \leq (1-\delta)^f.$$

Now note that $\{\mathcal{B} : \mathcal{B} \not\models \psi\} =$

$$\bigcup_{\substack{a_1,a_2 \\ a_1 \neq a_2}} \left\{\mathcal{B} : \mathcal{B} \not\models \exists v_3 \cdots \exists v_k \left(\psi'(a_1, a_2, v_3, \ldots, v_k) \wedge \bigwedge_{1 \leq i < j \leq k} v_i \neq v_j\right)\right\}.$$

Hence, by Claim 2, $\mu(\{\mathcal{B} : \mathcal{B} \not\models \psi\}) \leq n \cdot s \cdot (n \cdot s - 1) \cdot (1-\delta)^f$.
As $f = \lfloor \frac{n-2}{k-2} \rfloor$, $n \cdot s \cdot (n \cdot s - 1) \cdot (1-\delta)^f < 1$ for big enough n.
Then the probability that $\mathcal{B}$ satisfies ψ is positive.
Therefore, some member of $\mathrm{Str}(n)$ satisfies ψ.

The Structure $\mathcal{A} \times \ell$

- Let τ be a relational vocabulary.
- For a τ-structure $\mathcal{A}$ and $\ell \geq 2$, denote by $\mathcal{A} \times \ell$ the structure which, for every element of A, contains ℓ duplicates.
- More precisely, $\mathcal{A} \times \ell$ is the τ-structure with universe

$$A \times \{0, \ldots, \ell - 1\},$$

  such that, for any n-ary R in τ,

$$R^{\mathcal{A} \times \ell} := \{((a_1, i_1), \ldots, (a_n, i_n)) : R^A a_1 \ldots a_n, 0 \leq i_1, \ldots, i_n \leq \ell - 1\}.$$

- Observe that:
  - $\mathcal{A} \times \ell$ is a structure without kings;
  - $\mathcal{A} \vDash \psi$ iff $\mathcal{A} \times \ell \vDash \psi$ holds for all sentences ψ without equality. This proof uses structural induction.

Finite Model Property for Equality Free $\forall^2\exists^*$-Sentences

- As a corollary of the above theorem we obtain

Corollary

Suppose that τ is a relational vocabulary and ψ is a $\forall^2\exists^*$-sentence without equality. If ψ is satisfiable then it has a finite model.

- As in a previous corollary, applying to FO^2, we now get:

Corollary

The set of logically valid $\forall^2\exists^*$-sentences without equality in a relational vocabulary is decidable.