Introduction to Fractal Geometry

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Subsection 1

Hausdorff Measure

Covers

- Recall that if U is any non-empty subset of n-dimensional Euclidean space, $\mathbb{R}^n$, the **diameter** of U is defined as

$$|U| = \sup\{|x-y| : x, y \in U\},$$

i.e., as the greatest distance apart of any pair of points in U.

- A δ-**cover** $\{U_i\}$ of F is a countable (or finite) collection of sets of diameter at most δ that cover F.
- This means that

$$F \subseteq \bigcup_{i=1}^{\infty} U_i, \text{ with } 0 \leq |U_i| \leq \delta, \text{ for each } i.$$

Hausdorff Measure

- Let F be a subset of $\mathbb{R}^n$.
- Let s be a non-negative number.
- For any $\delta > 0$, we define

$$\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

- So this process involves:
  - Looking at all covers of F by sets of diameter at most δ;
  - Seeking to minimize the sum of the s-th powers of the diameters.

Hausdorff Measure (Cont'd)

- As δ decreases, the class of permissible covers of F is reduced.
- Therefore, the infimum $\mathcal{H}^s_\delta(F)$ increases.
- So it approaches a limit as $\delta \to 0$.
- We write
$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$
- This limit exists for any subset F of $\mathbb{R}^n$.
- The limiting value can be (and usually is) 0 or ∞.
- We call $\mathcal{H}^s(F)$ the s**-dimensional Hausdorff measure of** F.

Hausdorff Measure is a Measure

- With a certain amount of effort, $\mathcal{H}^s$ may be shown to be a measure.
- It is straightforward to show that:
  - $\mathcal{H}^s(\emptyset) = 0$;
  - If E is contained in F then $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$;
  - If $\{F_i\}$ is any countable collection of sets, then

$$\mathcal{H}^s\left(\bigcup_{i=1}^{\infty} F_i\right) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(F_i).$$

- It is rather harder to show that there is equality in the last relation if the $\{F_i\}$ are disjoint Borel sets.

Hausdorff Measure and Volume

- Hausdorff measures generalize the familiar ideas of length, area, volume, etc.
- It may be shown that, for subsets of $\mathbb{R}^n$, n-dimensional Hausdorff measure is, to within a constant multiple, just n-dimensional Lebesgue measure, i.e., the usual n-dimensional volume.
- Let c_n be the volume of an n-dimensional ball of diameter 1,

$$c_n = \begin{cases} \frac{\pi^{n/2}}{2^n}(\frac{n}{2})!, & \text{if } n \text{ is even,} \\ \pi^{(n-1)/2}\frac{(\frac{n-1}{2})!}{n!}, & \text{if } n \text{ is odd.} \end{cases}$$

- Then, if F is a Borel subset of $\mathbb{R}^n$, then

$$\mathcal{H}^n(F) = \frac{1}{c_n}\text{vol}^n(F).$$

Hausdorff Measure and Volume: Low Dimensions

- Similarly, for "nice" lower-dimensional subsets of $\mathbb{R}^n$, we have that:
  - $\mathcal{H}^0(F)$ is the number of points in F;
  - $\mathcal{H}^1(F)$ gives the length of a smooth curve F;
  - $\mathcal{H}^2(F) = \frac{4}{\pi}\text{area}(F)$ if F is a smooth surface;
  - $\mathcal{H}^3(F) = \frac{6}{\pi}\text{vol}(F)$;
  - $\mathcal{H}^m(F) = \frac{1}{c_m}\text{vol}^m(F)$ if F is a smooth m-dimensional submanifold of $\mathbb{R}^n$ (i.e., an m-dimensional surface in the classical sense).

Introducing the Scaling Property

- On magnification by a factor λ:
  - The length of a curve is multiplied by λ;
  - The area of a plane region is multiplied by λ^2;
  - The volume of a 3-dimensional object is multiplied by λ^3.
- The s-dimensional Hausdorff measure scales with a factor λ^s.

The Scaling Property

Scaling Property

Let S be a similarity transformation of scale factor $\lambda > 0$. If $F \subseteq \mathbb{R}^n$, then

$$\mathcal{H}^s(S(F)) = \lambda^s \mathcal{H}^s(F).$$

- Suppose $\{U_i\}$ is a δ-cover of F. Then we have:
  - $\{S(U_i)\}$ is a $\lambda\delta$-cover of $S(F)$;
  - $\sum |S(U_i)|^s = \lambda^s \sum |U_i|^s$.

  On taking the infimum,

  $$\mathcal{H}^s_{\lambda\delta}(S(F)) \leq \lambda^s \mathcal{H}^s_\delta(F).$$

  Letting $\delta \to 0$ gives that

  $$H^s(S(F)) \leq \lambda^s \mathcal{H}^s(F).$$

  Replacing S by S^{-1}, and so λ by $\frac{1}{\lambda}$, and F by $S(F)$ gives the opposite inequality.

More General Transformations and Hausdorff Measure

Proposition

Let $F \subseteq \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ be a mapping such that

$$|f(x) - f(y)| \leq c|x - y|^\alpha, \quad x, y \in F,$$

for constants $c > 0$ and $\alpha > 0$. Then for each s,

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F).$$

- Suppose $\{U_i\}$ is a δ-cover of F.

  We have

  $$|f(F \cap U_i)| \leq c|F \cap U_i|^\alpha \leq c|U_i|^\alpha.$$

  It follows that $\{f(F \cap U_i)\}$ is an ε-cover of $f(F)$, where $\varepsilon = c\delta^\alpha$.

More General Transformations (Cont'd)

- We got that $\{f(F \cap U_i)\}$ is an ε-cover of $f(F)$, where $\varepsilon = c\delta^\alpha$.

  We also have
  $$\sum_i |f(F \cap U_i)|^{s/\alpha} \leq c^{s/\alpha} \sum_i |U_i|^s.$$
  So
  $$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}^s_\delta(F).$$
  As $\delta \to 0$, we get $\varepsilon \to 0$.

  This shows that
  $$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}^s(F).$$

Hölder and Lipschitz Conditions

- Condition
$$|f(x) - f(y)| \leq c|x - y|^{\alpha}$$
is known as a **Hölder condition of exponent** α.
- Such a condition implies that f is continuous.
- Particularly important is the case $\alpha = 1$, i.e.,
$$|f(x) - f(y)| \leq c|x - y|, \quad x, y \in F.$$
- Then f is called a **Lipschitz mapping**.
- Moreover, we get
$$\mathcal{H}^s(f(F)) \leq c^s \mathcal{H}^s(F).$$

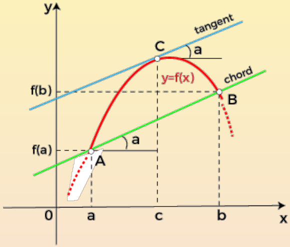
Differentiable Functions With Bounded Derivative

- Let f be a differentiable function with bounded derivative,

$$|f'(x)| \leq c, \quad \text{for all } x.$$

- The Mean Value Theorem asserts that, for all a, b, there exists $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



- Hence, for all x, y, there exists $x < z < y$, such that

$$|f(x) - f(y)| = |f'(z)||x - y| \leq c|x - y|.$$

- So f is a Lipschitz mapping.

Translation and Rotation Invariance of Hausdorff Measure

- Suppose f is an isometry, i.e.,

$$|f(x) - f(y)| = |x - y|.$$

- In particular, both f and f^{-1} are Lipschitz.
- Thus, $\mathcal{H}^s(f(F)) = \mathcal{H}^s(F)$.
- It follows that Hausdorff measures are:
  - Translation invariant, i.e.,

$$\mathcal{H}^s(F + z) = \mathcal{H}^s(F),$$

    where $F + z = \{x + z : x \in F\}$;
  - Rotation invariant.

Subsection 2

Hausdorff Dimension

The Hausdorff Dimension

- We defined

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}.$$

- For any set $F \subseteq \mathbb{R}^n$ and $\delta < 1$, $\mathcal{H}^s_\delta(F)$ is non-increasing with s.
- So

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$$

  is also non-increasing.
- In fact, if $t > s$ and $\{U_i\}$ is a δ-cover of F, we have

$$\sum_i |U_i|^t \leq \sum_i |U_i|^{t-s}|U_i|^s \leq \delta^{t-s} \sum_i |U_i|^s.$$

The Hausdorff Dimension (Cont'd)

- Taking infima,
$$\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F).$$
- Letting $\delta \to 0$ we see that, for $t > s$,
$$\mathcal{H}^s(F) < \infty \quad \text{implies} \quad \mathcal{H}^t(F) = 0.$$
- Thus, a graph of $\mathcal{H}^s(F)$ against s shows that there is a critical value of s at which $\mathcal{H}^s(F)$ "jumps" from ∞ to 0.
- This critical value is called the **Hausdorff dimension** of F, and written
$$\dim_{\mathrm{H}} F.$$

The Hausdorff Dimension and s-Sets

- The **Hausdorff dimension** of a set $F \subseteq \mathbb{R}^n$ is defined formally by

$$\dim_{\mathrm{H}} F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\},$$

  taking the supremum of the empty set to be 0.

- We have

$$\mathcal{H}^s(F) = \begin{cases} \infty, & \text{if } 0 \leq s < \dim_{\mathrm{H}} F, \\ 0, & \text{if } s > \dim_{\mathrm{H}} F. \end{cases}$$

- If $s = \dim_{\mathrm{H}} F$, then $\mathcal{H}^s(F)$ may be zero or infinite, or may satisfy $0 < \mathcal{H}^s(F) < \infty$.
- A Borel set F satisfying $0 < \mathcal{H}^s(F) < \infty$ is called an ***s*-set**.

Example

- Let F be a flat disc of unit radius in $\mathbb{R}^3$.

  From familiar properties of length, area and volume, we have:
  - $\mathcal{H}^1(F) = \text{length}(F) = \infty$;
  - $0 < \mathcal{H}^2(F) = \frac{4}{\pi} \times \text{area}(F) = 4 < \infty$;
  - $\mathcal{H}^3(F) = \frac{6}{\pi} \times \text{vol}(F) = 0$.

  Thus $\dim_{\mathrm{H}} F = 2$ and we have:
  - $\mathcal{H}^s(F) = \infty$, if $s < 2$;
  - $\mathcal{H}^s(F) = 0$, if $s > 2$.

Monotonicity of Hausdorff Dimension

- If $E \subseteq F$, then $\dim_{\rm H} E \leq \dim_{\rm H} F$.
  By the measure property, for all s,
  $$\mathcal{H}^s(E) \leq \mathcal{H}^s(F).$$
  Therefore,
  $$\begin{array}{rcl} \dim_{\rm H} E & = & \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} \\ & \leq & \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} \\ & = & \dim_{\rm H} F. \end{array}$$

Countable Stability of Hausdorff Dimension

- If $F_1, F_2, \ldots$ is a (countable) sequence of sets then

$$\dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i = \sup_{1 \leq i < \infty} \{\dim_{\mathrm{H}} F_i\}.$$

By Monotonicity, for every j, $\dim_{\mathrm{H}} F_j \leq \dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i$.
Therefore, $\sup_{1 \leq i < \infty} \{\dim_{\mathrm{H}} F_i\} \leq \dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i$.
Suppose, for all i,

$$\dim_{\mathrm{H}} F_i \leq s.$$

Then, for all i, $\mathcal{H}^s(F_i) = 0$. So $\mathcal{H}^s(\bigcup_{i=1}^{\infty} F_i) = 0$.
We conclude

$$\begin{array}{rcl} \dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i & = & \inf \{s \geq 0 : \mathcal{H}^s(\bigcup_{i=1}^{\infty} F_i) = 0\} \\ & \leq & \sup_{1 \leq i < \infty} \{\dim_{\mathrm{H}} F_i\}. \end{array}$$

Hausdorff Dimension of Countable Sets

- If F is countable, then

$$\dim_{\mathrm{H}} F = 0.$$

If F_i is a single point, we have:
  - $\mathcal{H}^0(F_i) = 1$;
  - $\dim_{\mathrm{H}} F_i = 0$.

So by countable stability

$$\dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i = 0.$$

Hausdorff Dimension of Open Sets

- If $F \subseteq \mathbb{R}^n$ is open, then

$$\dim_{\mathrm{H}} F = n.$$

Suppose $F \subseteq \mathbb{R}^n$ is open.

Clearly, F contains a ball of positive n-dimensional volume.

Therefore, $\dim_{\mathrm{H}} F \geq n$.

But F is contained in countably many balls.

Therefore, by Monotonicity and Countable Stability,

$$\dim_{\mathrm{H}} F \leq n.$$

We conclude that $\dim_{\mathrm{H}} F = n$.

Hausdorff Dimension of Smooth Sets

- If F is a smooth (i.e., continuously differentiable) m-dimensional submanifold (i.e., m-dimensional surface) of $\mathbb{R}^n$ then

$$\dim_{\mathrm{H}} F = m.$$

- In particular:
  - Smooth curves have dimension 1;
  - Smooth surfaces have dimension 2.
- Essentially, this may be deduced from the relationship between Hausdorff and Lebesgue measures.

Transformations Satisfying a Hölder Condition

Proposition

Let $F \subseteq \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$ satisfies a Hölder condition

$$|f(x) - f(y)| \le c|x - y|^\alpha, \quad x, y \in F.$$

Then $\dim_{\rm H} f(F) \le \frac{1}{\alpha} \dim_{\rm H} F$.

- If $s > \dim_{\rm H} F$, then by the preceding proposition,

$$\mathcal{H}^{s/\alpha}(f(F)) \le c^{s/\alpha} \mathcal{H}^s(F) = 0.$$

Thus, $\dim_{\rm H} f(F) \le \frac{s}{\alpha}$, for all $s > \dim_{\rm H} F$.

Lipschitz Transformations

Corollary

(a) If $f : F \to \mathbb{R}^m$ is Lipschitz, then $\dim_H f(F) \leq \dim_H F$.

(b) If $f : F \to \mathbb{R}^m$ is a bi-Lipschitz transformation, i.e.,

$$c_1|x-y| \leq |f(x)-f(y)| \leq c_2|x-y|, \quad x, y \in F,$$

where $0 < c_1 \leq c_2 < \infty$, then $\dim_H f(F) = \dim_H F$.

- Part (a) follows from the preceding proposition taking $\alpha = 1$. Applying this to $f^{-1} : f(F) \to F$ gives the other inequality for (b).
- Hausdorff dimension is invariant under bi-Lipschitz transformations.
- Thus, if two sets have different dimensions there cannot be a bi-Lipschitz mapping from one onto the other.

Sets of Hausdorff Dimension Less Than 1

Proposition

A set $F \subseteq \mathbb{R}^n$ with $\dim_H F < 1$ is totally disconnected.

- Let x and y be distinct points of F.
  Define a mapping $f : \mathbb{R}^n \to [0, \infty)$ by
  $$f(z) = |z - x|.$$
  f does not increase distances, since
  $$\begin{array}{rcl} |f(z) - f(w)| & = & ||z - x| - |w - x|| \\ & \leq & |(z - x) - (w - x)| \\ & = & |z - w|. \end{array}$$
  By the preceding corollary,
  $$\dim_H f(F) \leq \dim_H F < 1.$$

Sets of Hausdorff Dimension Less Than 1 (Cont'd)

- Thus $f(F)$ is a subset of $\mathbb{R}$ of $\mathcal{H}^1$-measure or length zero.

  So $f(F)$ has a dense complement.

  Choose r with $r \notin f(F)$ and $0 < r < f(y)$.

  Then we have

$$F = \{z \in F : |z - x| < r\} \cup \{z \in F : |z - x| > r\}.$$

  Thus, F is contained in two disjoint open sets with x in one set and y in the other.

  So x and y lie in different connected components of F.

Subsection 3

Calculation of Hausdorff Dimension

Example: The Cantor Dust

- Let F be the Cantor dust constructed from the unit square.

  At each stage of the construction:
  - The squares are divided into 16 squares with a quarter of the side length;
  - The same pattern of four squares is retained.

  Then $1 \leq \mathcal{H}^1(F) \leq \sqrt{2}$. So $\dim_{\mathrm{H}} F = 1$.

  Let E_k be the k-th stage of the construction.

  Observe that E_k consists of 4^k squares of side 4^{-k}.

  Thus, each square is of diameter $4^{-k}\sqrt{2}$.

  Take the squares of E_k as a δ-cover of F, where $\delta = 4^{-k}\sqrt{2}$.

  Then an estimate for the infimum in the definition is

$$\mathcal{H}^1_\delta(F) \leq 4^k 4^{-k}\sqrt{2}.$$

  As $k \to \infty$, $\delta \to 0$, giving $\mathcal{H}^1(F) \leq \sqrt{2}$.

Example: The Cantor Dust (Cont'd)

- We turn to providing a lower estimate.

  Let proj denote orthogonal projection onto the x-axis.

  Orthogonal projection does not increase distances.

  So, if $x, y \in \mathbb{R}^2$,

  $$|\text{proj}x - \text{proj}y| \leq |x - y|.$$

  So proj is a Lipschitz mapping.

  By virtue of the construction of F, the projection or "shadow" of F on the x-axis, proj$F$, is the unit interval $[0, 1]$.

  So we have,

  $$\begin{aligned} 1 &= \text{length}[0,1] \\ &= \mathcal{H}^1([0,1]) \\ &= \mathcal{H}^1(\text{proj}F) \\ &\overset{\text{Lipschitz}}{\leq} \mathcal{H}^1(F). \end{aligned}$$

Remarks

- The same argument and result hold for a set obtained by:
  - Repeated division of squares into m^2 squares of side length $\frac{1}{m}$;
  - Retention of one square in each column.
- This trick of using orthogonal projection to get a lower estimate of Hausdorff measure only works in special circumstances and is not the basis of a more general method.

Example: The Cantor Set

- Let F be the middle third Cantor set. If $s = \frac{\log 2}{\log 3} = 0.6309\ldots$, then $\dim_{\rm H} F = s$ and $\frac{1}{2} \leq \mathcal{H}^s(F) \leq 1$.

  Heuristic Calculation: The Cantor set F splits into a left part $F_{\rm L} = F \cap [0, \frac{1}{3}]$ and a right part $F_{\rm R} = F \cap [\frac{2}{3}, 1]$.
  - Both parts are geometrically similar to F but scaled by a ratio $\frac{1}{3}$;
  - $F = F_{\rm L} \cup F_{\rm R}$, with this union disjoint.

  Thus, for any s, using Scaling of Hausdorff measures,

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_{\rm L}) + \mathcal{H}^s(F_{\rm R}) = \left(\frac{1}{3}\right)^s \mathcal{H}^s(F) + \left(\frac{1}{3}\right)^s \mathcal{H}^s(F).$$

  Suppose at the critical value $s = \dim_{\rm H} F$ we have $0 < \mathcal{H}^s(F) < \infty$.

  Then, we may divide by $\mathcal{H}^s(F)$ to get $1 = 2\left(\frac{1}{3}\right)^s$.

  Thus, $s = \frac{\log 2}{\log 3}$.

Example: The Cantor Set (Cont'd)

- **Rigorous Calculation**: We call the intervals that make up the sets E_k in the construction of F **level-k intervals**.
  E_k consists of 2^k level-k intervals each of length 3^{-k}.
  Take the intervals of E_k as a 3^{-k}-cover of F.
  Then, for $s = \frac{\log 2}{\log 3}$, we get
  $$\mathcal{H}^s_{3^{-k}}(F) \leq 2^k 3^{-ks} = 2^k \left(3^{\frac{\log 2}{\log 3}}\right)^{-k} = 2^k (3^{\log_3 2})^{-k} = 2^k 2^{-k} = 1.$$
  Letting $k \to \infty$ gives $\mathcal{H}^s(F) \leq 1$.
  To prove that $\mathcal{H}^s(F) \geq \frac{1}{2}$, we show that, for any cover $\{U_i\}$ of F,
  $$\sum |U_i|^s \geq \frac{1}{2} = 3^{-s}.$$
  It is enough to assume that the $\{U_i\}$ are intervals.
  By expanding them slightly and using the compactness of F, we need only consider a finite collection $\{U_i\}$ of closed subintervals of $[0, 1]$.

Example: The Cantor Set (Conclusion)

- For each U_i, let k be the integer such that
$$3^{-(k+1)} \leq |U_i| < 3^{-k}.$$
The separation of the level-k intervals is at least 3^{-k}.
So U_i can intersect at most one level-k interval.
If $j \geq k$, then, by construction, U_i intersects at most $2^{j-k} = 2^j 3^{-sk} \leq 2^j 3^s |U_i|^s$ level-j intervals of E_j.
Choose j large enough so that $3^{-(j+1)} \leq |U_i|$, for all U_i.
The $\{U_i\}$ intersect all 2^j basic intervals of length 3^{-j}.
So counting intervals gives
$$2^j \leq \sum_i 2^j 3^s |U_i|^s.$$
This reduces to the desired inequality.

Remarks on the Heuristic Method

- The "heuristic" method of calculation used in the preceding example gives the right answer for the dimension of many self-similar sets.

  Example: The von Koch curve is made up of four copies of itself scaled by a factor $\frac{1}{3}$. Hence it has dimension $\frac{\log 4}{\log 3}$.
- More generally, suppose

$$F = \bigcup_{i=1}^{m} F_i,$$

  where:
  - Each F_i is geometrically similar to F but scaled by a factor c_i;
  - The F_i do not overlap "too much".

  Then the heuristic argument gives $\dim_{\mathrm{H}} F$ as the number s satisfying

$$\sum_{i=1}^{m} c_i^s = 1.$$