Introduction to Fractal Geometry

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1. Alternative Definitions of Dimension
   - Introduction
   - Box-Counting Dimensions
   - Properties and Problems of Box-Counting Dimension

Subsection 1

Introduction

Measurements at Scale δ

- Most definitions of dimension use the fundamental idea of "measurement at scale δ".
- For each δ, we measure a set in a way that ignores irregularities of size less than δ.
- Then we see how these measurements behave as $\delta \to 0$.

Example

- Suppose F is a plane curve.
- Our measurement, $M_\delta(F)$, might be the number of steps required by a pair of dividers set at length δ to traverse F.
- A dimension of F is then determined by the power law (if any) obeyed by $M_\delta(F)$ as $\delta \to 0$.
- Suppose that, for constants c and s,

$$M_\delta(F) \sim c\delta^{-s}.$$

- Then we might say that F has "divider dimension" s, with c regarded as the "s-dimensional length" of F.

Example (Cont'd)

- We assumed
$$M_\delta(F) \sim c\delta^{-s}.$$
- Taking logarithms
$$\log M_\delta(F) \simeq \log c - s \log \delta,$$
in the sense that the difference of the two sides tends to 0 with δ.
- So we get
$$s = \lim_{\delta \to 0} \frac{\log M_\delta(F)}{-\log \delta}.$$

General Guidelines for "Dimensions"

- For the value of s given by $M_\delta(F) \sim c\delta^{-s}$ to behave like a dimension, the method of measurement needs to scale with the set.

  So, doubling the size of F and at the same time doubling the scale at which measurement takes place does not affect the answer, i.e., we require

  $$M_\delta(\delta F) = M_1(F), \quad \text{for all } \delta.$$

- If we modify our example and redefine $M_\delta(F)$ to be the sum of the divider step lengths, then $M_\delta(F)$ is homogeneous of degree 1.

  That is, we have $M_\delta(\delta F) = \delta^1 M_1(F)$ for $\delta > 0$.

  This must be taken into account when defining the dimension.

- In general, suppose $M_\delta(F)$ is homogeneous of degree d.

  That is, we have $M_\delta(\delta F) = \delta^d M_1(F)$.

  Then power law of the form $M_\delta(F) \sim c\delta^{d-s}$ corresponds to a dimension s.

Desirable Properties of "Dimension"

- Here are some desirable properties of a "dimension", possessed by the Hausdorff dimension:
  - **Monotonicity**: If $E \subseteq F$, then $\dim_{\mathrm{H}} E \leq \dim_{\mathrm{H}} F$.
  - **Stability**: $\dim_{\mathrm{H}}(E \cup F) = \max(\dim_{\mathrm{H}} E, \dim_{\mathrm{H}} F)$.
  - **Countable Stability**: $\dim_{\mathrm{H}}(\bigcup_{i=1}^{\infty} F_i) = \sup_{1 \leq i < \infty} \dim_{\mathrm{H}} F_i$.
  - **Geometric Invariance**: $\dim_{\mathrm{H}} f(F) = \dim_{\mathrm{H}} F$, if f is a transformation of $\mathbb{R}^n$, such as a translation, rotation, similarity or affinity.
  - **Lipschitz Invariance**: $\dim_{\mathrm{H}} f(F) = \dim_{\mathrm{H}} F$ if f is a bi-Lipschitz transformation.
  - **Countable Sets**: $\dim_{\mathrm{H}} F = 0$ if F is finite or countable.
  - **Open Sets**: If F is an open subset of $\mathbb{R}^n$, then $\dim_{\mathrm{H}} F = n$.
  - **Smooth Manifolds**: $\dim_{\mathrm{H}} F = m$, if F is a smooth m-dimensional manifold (curve, surface, etc.).

Overview of "Dimension" Properties

- All definitions of dimension are monotonic.
- Most are stable.
- As we shall see, some common definitions fail to exhibit countable stability and may have countable sets of positive dimension.
- All the usual dimensions are Lipschitz invariant, and, therefore, geometrically invariant.
- The "open sets" and "smooth manifolds" properties ensure that the dimension is an extension of the classical definition.
- Finally, we note that different definitions of dimension can provide different information about which sets are Lipschitz equivalent.

Subsection 2

Box-Counting Dimensions

Box-Counting Dimension

- Let F be any non-empty bounded subset of $\mathbb{R}^n$.
- Let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F.
- The **lower** and **upper box-counting dimensions** of F, respectively, are defined as

$$\underline{\dim}_B F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}, \quad \overline{\dim}_B F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

- If these are equal we refer to the common value as the **box-counting dimension** or **box dimension** of F,

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

Remarks

- The box-counting dimension of F is given by

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

- We always assume that $\delta > 0$ is sufficiently small to ensure that $-\log \delta$ and similar quantities are strictly positive.
- To avoid problems with "$\log 0$" or "$\log \infty$" we generally consider box dimension only for non-empty bounded sets.

An Equivalent Definition of Box Dimension

- Consider the collection of cubes in the δ-coordinate mesh of $\mathbb{R}^n$, i.e. cubes of the form $[m_1\delta, (m_1+1)\delta] \times \cdots \times [m_n\delta, (m_n+1)\delta]$, where $m_1, \ldots, m_n$ are integers.
- Recall that a "cube" is an interval in $\mathbb{R}^1$ and a square in $\mathbb{R}^2$.
- Let $N'_\delta(F)$ be the number of δ-mesh cubes that intersect F.
- They provide a collection of $N'_\delta(F)$ sets of diameter $\delta\sqrt{n}$ that cover F.
- So $N_{\delta\sqrt{n}}(F) \leq N'_\delta(F)$.
- If $\delta\sqrt{n} < 1$, then $\frac{\log N_{\delta\sqrt{n}}(F)}{-\log(\delta\sqrt{n})} \leq \frac{\log N'_\delta(F)}{-\log\sqrt{n} - \log\delta}$.
- So taking limits as $\delta \to 0$,

$$\underline{\dim}_B F \leq \varliminf_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta}, \quad \overline{\dim}_B F \leq \varlimsup_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta}.$$

Equivalent Definition of Box Dimension (Cont'd)

- On the other hand, consider any set of diameter at most δ.
- Choose a cube of side δ containing some point of the set together with its neighboring cubes of side δ.
- So the set is contained in 3^n mesh cubes of side δ.
- Thus,
$$N'_\delta(F) \leq 3^n N_\delta(F).$$
- Taking logarithms and limits as $\delta \to 0$, leads to the inequalities
$$\underline{\dim}_B F \geq \varliminf_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log \delta}, \quad \overline{\dim}_B F \geq \varlimsup_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log \delta}.$$
- Hence to find the box dimensions, we can equally well take $N_\delta(F)$ to be the number of mesh cubes of side δ that intersect F.

Using Arbitrary Cubes or Closed Balls

- Another frequently used definition of box dimension is obtained by:
  - Taking $N_\delta(F)$ to be the smallest number of arbitrary cubes of side δ required to cover F;
  - Defining
$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$
whenever the limit exists.
- The equivalence of this definition follows as in the mesh cube case, noting that:
  - Any cube of side δ has diameter $\delta\sqrt{n}$;
  - Any set of diameter of at most δ is contained in a cube of side δ.
- Similarly, we get exactly the same values if we take $N_\delta(F)$ as the smallest number of closed balls of radius δ that cover F.

Another Formulation Using Disjoint Balls

- Let $N'_\delta(F)$ be the largest number of disjoint balls of radius δ with centers in F.
- Let $B_1, \ldots, B_{N'_\delta(F)}$ be disjoint balls centered in F and of radius δ.
- If x belongs to F then x must be within distance δ of one of the B_i.

  Otherwise the ball of center x and radius δ can be added to form a larger collection of disjoint balls.
- Thus, the $N'_\delta(F)$ balls concentric with the B_i but of radius 2δ (diameter 4δ) cover F.
- This gives $N_{4\delta}(F) \leq N'_\delta(F)$.

Another Formulation Using Disjoint Balls (Cont'd)

- Suppose also that $B_1, \ldots, B_{N'_\delta(F)}$ are disjoint balls of radii δ with centers in F.
- Let $U_1, \ldots, U_k$ be any collection of sets of diameter at most δ which cover F.
- The U_j must cover the centers of the B_i.
- So each B_i must contain at least one of the U_j.
- As the B_i are disjoint, there are at least as many U_j as B_i.
- Hence, $N'_\delta(F) \leq N_\delta(F)$.
- Taking logarithms and limits of these inequalities shows that the value of the box dimension remains unaltered if $N_\delta(F)$ is replaced by this $N'_\delta(F)$.

Summary of Equivalent Definitions

- The lower and upper box-counting dimensions of a subset F of $\mathbb{R}^n$ are given by

$$\underline{\dim}_B F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}, \quad \overline{\dim}_B F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

and the box-counting dimension of F by $\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$ (if the limit exists), where $N_\delta(F)$ is any of the following:

(i) The smallest number of closed balls of radius δ that cover F;
(ii) The smallest number of cubes of side δ that cover F;
(iii) The number of δ-mesh cubes that intersect F;
(iv) The smallest number of sets of diameter at most δ that cover F;
(v) The largest number of disjoint balls of radius δ with centers in F.

The Way δ Approaches Zero

- In the limits defining the upper, lower and box dimension, it is enough to consider limits as δ tends to 0 through any decreasing sequence δ_k, such that $\delta_{k+1} \geq c\delta_k$, for some constant $0 < c < 1$; in particular for $\delta_k = c^k$.

  Suppose $\delta_{k+1} \leq \delta < \delta_k$.

  Then, with $N_\delta(F)$ the least number of sets in a δ-cover of F,

$$\begin{aligned} \frac{\log N_\delta(F)}{-\log \delta} &\leq \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_k} = \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_{k+1} + \log \frac{\delta_{k+1}}{\delta_k}} \\ &\leq \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_{k+1} + \log c}. \end{aligned}$$

  So we get

$$\overline{\lim_{\delta \to 0}} \frac{\log N_\delta(F)}{-\log \delta} \leq \overline{\lim_{k \to \infty}} \frac{\log N_{\delta_k}(F)}{-\log \delta_k}.$$

  The opposite inequality is trivial.

  The case of lower limits may be dealt with in the same way.

The Minkowski Content

- Recall that the δ-**neighborhood** F_δ of a subset F of $\mathbb{R}^n$ is

$$F_\delta = \{x \in \mathbb{R}^n : |x - y| \leq \delta, \text{ for some } y \in F\},$$

- This is the set of points within distance δ of F.
- We consider the rate at which the n-dimensional volume of F_δ shrinks as $\delta \to 0$.

  Example: In $\mathbb{R}^3$, we have the following:
  - If F is a single point then F_δ is a ball with $\text{vol}(F_\delta) = \frac{4}{3}\pi\delta^3$;
  - If F is a segment of length ℓ, then F_δ is "sausage-like" with $\text{vol}(F_\delta) \sim \pi\ell\delta^2$;
  - If F is a flat set of area a, then F_δ is essentially a thickening of F with $\text{vol}(F_\delta) \sim 2a\delta$.
- In each case, $\text{vol}(F_\delta) \sim c\delta^{3-s}$, where s is the dimension of F.
- So the exponent of δ is indicative of the dimension.
- The coefficient c of δ^{3-s}, known as the **Minkowski content** of F, is a measure of the length, area or volume of the set as appropriate.

The s-Dimensional Content

- Let F is a subset of $\mathbb{R}^n$.
- Denote by vol^n the n-dimensional volume.
- Suppose, for some s, $\frac{\text{vol}^n(F_\delta)}{\delta^{n-s}}$ tends to a positive finite limit as $\delta \to 0$.
- Then it makes sense to regard F as s-dimensional.
- The limiting value is called the ***s*-dimensional content** of F.
- This concept is of slightly restricted use since it is not necessarily additive on disjoint subsets, i.e., is not a measure.
- Even if this limit does not exist, we may be able to extract the critical exponent of δ.
- This exponent turns out to be related to the box dimension.

Box Dimension and Volume of δ-Neighborhoods

Proposition

If F is a subset of $\mathbb{R}^n$, then

$$\underline{\dim}_B F = n - \overline{\lim_{\delta\to 0}} \frac{\log \mathrm{vol}^n(F_\delta)}{\log \delta}, \quad \overline{\dim}_B F = n - \underline{\lim_{\delta\to 0}} \frac{\log \mathrm{vol}^n(F_\delta)}{\log \delta},$$

where F_δ is the δ-neighborhood of F.

- Suppose F can be covered by $N_\delta(F)$ balls of radius $\delta < 1$.
  Then F_δ can be covered by the concentric balls of radius 2δ.
  Hence, $\mathrm{vol}^n(F_\delta) \leq N_\delta(F) c_n (2\delta)^n$, where c_n is the volume of the unit ball in $\mathbb{R}^n$.
  Taking logarithms,

$$\frac{\log \mathrm{vol}^n(F_\delta)}{-\log \delta} \leq \frac{\log 2^n c_n + n \log \delta + \log N_\delta(F)}{-\log \delta}.$$

Box Dimension and Volume of δ-Neighborhoods (Cont'd)

- We obtained $\frac{\log \text{vol}^n(F_\delta)}{-\log \delta} \le \frac{\log 2^n c_n + n \log \delta + \log N_\delta(F)}{-\log \delta}$.

  So
  $$\underline{\lim}_{\delta \to 0} \frac{\log \text{vol}^n(F_\delta)}{-\log \delta} \le -n + \underline{\dim}_B F.$$

  A similar inequality holds for the upper limits.

  Suppose there are $N_\delta(F)$ disjoint balls of radius δ with centers in F.

  Then, by adding their volumes,
  $$N_\delta(F) c_n \delta^n \le \text{vol}^n(F_\delta).$$

  Taking logarithms and letting $\delta \to 0$ gives the opposite inequality.
- In this context, box dimension is sometimes referred to as **Minkowski dimension** or **Minkowski-Bouligand dimension**.

Box-Counting Dimension and Hausdorff Dimension

- Suppose F can be covered by $N_\delta(F)$ sets of diameter δ.
- Then, by the definition of $\mathcal{H}_\delta^s(F)$,

$$\mathcal{H}_\delta^s(F) \leq N_\delta(F)\delta^s.$$

- Suppose $1 < \mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}_\delta^s(F)$.
- Then $\log N_\delta(F) + s\log\delta > 0$, if δ is sufficiently small.
- Thus, $s \leq \underline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}$.
- So, for every $F \subseteq \mathbb{R}^n$,

$$\dim_{\mathrm{H}} F \leq \underline{\dim}_B F \leq \overline{\dim}_B F.$$

- We do not in general get equality here.

Box Dimension and Hausdorff Dimension (Cont'd)

- Roughly speaking

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

  says that $N_\delta(F) \simeq \delta^{-s}$, for small δ, where $s = \dim_B F$.

- More precisely, it says that

$$\begin{array}{ll} N_\delta(F)\delta^s \to \infty, & \text{if } s < \dim_B F, \\ N_\delta(F)\delta^s \to 0, & \text{if } s > \dim_B F. \end{array}$$

Box Dimension and Hausdorff Dimension (Cont'd)

- Now consider

$$\begin{aligned} N_\delta(F)\delta^s &= \inf\{\textstyle\sum_i \delta^s : \{U_i\} \text{ is a (finite) } \delta\text{-cover of } F\}; \\ \mathcal{H}^s_\delta(F) &= \inf\{\textstyle\sum_i |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\}. \end{aligned}$$

- In calculating:
  - Hausdorff dimension, we assign different weights $|U_i|^s$ to the covering sets U_i;
  - Box dimension, we use the same weight δ^s for each covering set.
- This difference implies the following:
  - Box dimensions indicate the efficiency with which a set may be covered by small sets of equal size;
  - Hausdorff dimension involves coverings by sets of small but perhaps widely varying size.

Example: Middle Third Cantor Set

- Let F be the middle third Cantor set. Then $\underline{\dim}_B F = \overline{\dim}_B F = \frac{\log 2}{\log 3}$.

  We have a covering by the 2^k level-k intervals of E_k of length 3^{-k}.

  This gives that, for $3^{-k} < \delta \leq 3^{-k+1}$, $N_\delta(F) \leq 2^k$.

  Thus, we get

$$\overline{\dim}_B F = \varlimsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \leq \varlimsup_{k \to \infty} \frac{\log 2^k}{\log 3^{k-1}} = \frac{\log 2}{\log 3}.$$

  On the other hand, any interval of length δ with $3^{-k-1} \leq \delta < 3^{-k}$ intersects at most one of the level-k intervals of length 3^{-k} used in the construction of F. There are 2^k such intervals. So at least 2^k intervals of length δ are required to cover F. Hence $N_\delta(F) \geq 2^k$.

  So $\underline{\dim}_B F \geq \frac{\log 2}{\log 3}$.

  Thus, at least for the Cantor set, $\dim_H F = \dim_B F$.

Subsection 3

Properties and Problems of Box-Counting Dimension

Properties of Box-Counting Dimension

- The following elementary properties of box dimension mirror those of Hausdorff dimension, and may be verified in much the same way.
  (i) A smooth m-dimensional submanifold of $\mathbb{R}^n$ has $\dim_B F = m$.
  (ii) $\underline{\dim}_B$ and $\overline{\dim}_B$ are monotonic.
  (iii) $\overline{\dim}_B$ is finitely stable, i.e.,
  $$\overline{\dim}_B(E \cup F) = \max\{\overline{\dim}_B E, \overline{\dim}_B F\}.$$
  The corresponding identity does not hold for $\underline{\dim}_B$.
  (iv) $\underline{\dim}_B$ and $\overline{\dim}_B$ are bi-Lipschitz invariant.
  Suppose that:
    - $|f(x) - f(y)| \leq c|x - y|$;
    - F can be covered by $N_\delta(F)$ sets of diameter at most δ.

  Then the $N_\delta(F)$ images of these sets under f form a cover of $f(F)$ by sets of diameter at most $c\delta$.
  This shows that $\dim_B f(F) \leq \dim_B F$.
  Similarly, box dimensions behave just like Hausdorff dimensions under bi-Lipschitz and Hölder transformations.

Box Dimension and Closures

Proposition

Let $\overline{F}$ denote the closure of F, i.e., the smallest closed subset of $\mathbb{R}^n$ containing F. Then

$$\underline{\dim}_B \overline{F} = \underline{\dim}_B F \quad \text{and} \quad \overline{\dim}_B \overline{F} = \overline{\dim}_B F.$$

- Let $B_1, \ldots, B_k$ be a finite collection of closed balls of radii δ.

  If the closed set $\bigcup_{i=1}^k B_i$ contains F, it also contains $\overline{F}$.

  Hence the smallest number of closed balls of radius δ that cover F equals the smallest number required to cover the larger set $\overline{F}$.

  This yields the result.

Negative Consequences

- An immediate consequence of the proposition is that if F is a dense subset of an open region of $\mathbb{R}^n$ then $\underline{\dim}_B F = \overline{\dim}_B F = n$.
  Example: Let F be the (countable) set of rational numbers in $(0, 1)$.
  Then $\overline{F}$ is the entire interval $[0, 1]$.
  So $\underline{\dim}_B F = \overline{\dim}_B F = 1$.
- Thus, countable sets can have non-zero box dimension.
  Example (Cont'd): The box-counting dimension of each rational number regarded as a one-point set is clearly zero
  However, the countable union of these singleton sets has dimension 1.
- Consequently, it is not generally true that

$$\dim_B \bigcup_{i=1}^{\infty} F_i = \sup_i \dim_B F_i.$$

- This severely limits the usefulness of box dimension, since introducing a small, i.e., countable, set of points can change the dimension.

Example

- $F = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is a compact set with $\dim_B F = \frac{1}{2}$.
  Let $0 < \delta < \frac{1}{2}$ and let k be the integer satisfying $\frac{1}{k(k-1)} > \delta \geq \frac{1}{k(k+1)}$.
  Note that $\frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)} > \delta$.
  So, if $|U| \leq \delta$, then U can cover at most one of $1, \frac{1}{2}, \ldots, \frac{1}{k}$.
  Thus, at least k sets of diameter δ are required to cover F.
  So $N_\delta(F) \geq k$. This gives $\frac{\log N_\delta(F)}{-\log \delta} \geq \frac{\log k}{\log k(k+1)}$.
  Letting $\delta \to 0$, so $k \to \infty$, gives $\underline{\dim}_B F \geq \frac{1}{2}$.
  Conversely, if $\frac{1}{2} > \delta > 0$, take k such that $\frac{1}{k(k-1)} > \delta \geq \frac{1}{k(k+1)}$.
  Then $k + 1$ intervals of length δ cover $[0, \frac{1}{k}]$.
  This remining $k - 1$ points can be covered by another $k - 1$ intervals.
  Thus $N_\delta(F) \leq 2k$. So $\frac{\log N_\delta(F)}{-\log \delta} \leq \frac{\log (2k)}{\log k(k-1)}$.
  This gives $\overline{\dim}_B F \leq \frac{1}{2}$.