Introduction to Fractal Geometry

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LSSU Math 500

1. Techniques for Calculating Dimensions
   - Basic Methods
   - Subsets of Finite Measure
   - Potential Theoretic Methods

Subsection 1

Basic Methods

Bounding Hausdorff Dimension Using Box Dimension

Proposition

Suppose F can be covered by n_k sets of diameter at most δ_k, with $\delta_k \to 0$ as $k \to \infty$. Then

$$\dim_H F \leq \underline{\dim}_B F \leq \varliminf_{k\to\infty} \frac{\log n_k}{-\log \delta_k}.$$

Moreover, if $n_k \delta_k^s$ remains bounded as $k \to \infty$, then $\mathcal{H}^s(F) < \infty$. If $\delta_k \to 0$ but $\delta_{k+1} \geq c\delta_k$, for some $0 < c < 1$, then

$$\overline{\dim}_B F \leq \varlimsup_{k\to\infty} \frac{\log n_k}{-\log \delta_k}.$$

- The inequalities for the box-counting dimension are immediate from the definitions and the remark regarding sequences δ_k.

  That $\dim_H F \leq \underline{\dim}_B F$ was shown previously.

Bounding Hausdorff Dimension (Cont'd)

- Suppose $n_k \delta_k^s$ is bounded. Then $\mathcal{H}^s_{\delta_k}(F) \leq n_k \delta_k^s$.

  So $\mathcal{H}^s_{\delta_k}(F)$ tends to a finite limit $\mathcal{H}^s(F)$ as $k \to \infty$.

  Example: Consider the middle third Cantor set.

  We know that the natural coverings by 2^k intervals of length 3^{-k} give

$$\dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F \leq \frac{\log 2}{\log 3}.$$

Difficulties in Obtaining Lower Bounds

- Surprisingly often, the "obvious" upper bound for the Hausdorff dimension of a set turns out to be the actual value.
- However, demonstrating this can be difficult.
- To obtain an upper bound it is enough to evaluate sums of the form $\sum |U_i|^s$ for specific coverings $\{U_i\}$ of F.
- For a lower bound we must show that $\sum |U_i|^s$ is greater than some positive constant for all δ-coverings of F.
- Clearly an enormous number of such coverings are available.
- In particular, when working with Hausdorff dimension as opposed to box dimension, consideration must be given to covers where some of the U_i are very small and others have relatively large diameter.
- This prohibits sweeping estimates for $\sum |U_i|^s$ such as those available for upper bounds.

Overcoming the Difficulties for Lower Bounds

- One way of getting around these difficulties is to show that no individual set U can cover too much of F compared with its size measured as $|U|^s$.
- Then if $\{U_i\}$ covers the whole of F, the sum $\sum |U_i|^s$ cannot be too small.
- The usual way to do this is to:
  - Concentrate a suitable mass distribution μ on F;
  - Compare the mass $\mu(U)$ covered by U with $|U|^s$, for each U.
- Recall that a mass distribution on F is a measure with support contained in F such that $0 < \mu(F) < \infty$.

Mass Distribution Principle

Mass Distribution Principle

Let μ be a mass distribution on F. Suppose that, for some s, there are numbers $c > 0$ and $\varepsilon > 0$, such that $\mu(U) \leq c|U|^s$, for all sets U with $|U| \leq \varepsilon$. Then $\mathcal{H}^s(F) \geq \frac{\mu(F)}{c}$ and

$$s \leq \dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F.$$

- If $\{U_i\}$ is any cover of F then

$$0 < \mu(F) \leq \mu\left(\bigcup_i U_i\right) \leq \sum_i \mu(U_i) \leq c \sum_i |U_i|^s.$$

Taking infima, $\mathcal{H}^s_\delta(F) \geq \frac{\mu(F)}{c}$, if δ is small enough.
So $\mathcal{H}^s(F) \geq \frac{\mu(F)}{c}$.
Since $\mu(F) > 0$, we get $\dim_H F \geq s$.

Example

- The Mass Distribution Principle gives a quick lower estimate for the Hausdorff dimension of the middle third Cantor set F.
  Let μ be the natural mass distribution on F.
  Each of the 2^k k-th level intervals of length 3^{-k} in E_k in the construction of F carry a mass 2^{-k}.
  We imagine that:
  - We start with unit mass on E_0;
  - Repeatedly divide the mass on each interval of E_k between its two subintervals in E_{k+1}.

  Let U be a set with $|U| < 1$.
  Let k be the integer such that $3^{-(k+1)} \leq |U| < 3^{-k}$.
  Then U can intersect at most one of the intervals of E_k.

$$\mu(U) \leq 2^{-k} = (3^{\log 2/\log 3})^{-k} = (3^{-k})^{\log 2/\log 3} \leq (3|U|)^{\log 2/\log 3}.$$

  Hence, by the principle, $\mathcal{H}^{\log 2/\log 3}(F) \geq 3^{-\log 2/\log 3} = \frac{1}{2}$.
  This gives $\dim_H F \geq \frac{\log 2}{\log 3}$.

Example

- Let $F_1 = F \times [0,1] \subseteq \mathbb{R}^2$ be the product of:
  - The middle third Cantor set F;
  - The unit interval.

For $s = 1 + \frac{\log 2}{\log 3}$, $\dim_B F_1 = \dim_H F_1 = s$, with $0 < \mathcal{H}^s(F_1) < \infty$.

For each k, there is a covering of F by 2^k intervals of length 3^{-k}.

A column of 3^k squares of side 3^{-k} (diameter $3^{-k}\sqrt{2}$) covers the part of F_1 above each such interval.

So all together, F_1 may be covered by $2^k 3^k$ squares of side 3^{-k}.

So we get

$$\begin{aligned}\mathcal{H}^s_{3^{-k}\sqrt{2}}(F_1) &\leq 3^k 2^k (3^{-k}\sqrt{2})^s \\ &= (3 \cdot 2 \cdot 3^{-1-\log 2/\log 3})^k 2^{s/2} \\ &= 2^{s/2}.\end{aligned}$$

So $\mathcal{H}^s(F_1) \leq 2^{s/2}$. Thus, $\dim_H F_1 \leq \underline{\dim}_B F_1 \leq \overline{\dim}_B F_1 \leq s$.

Example (Cont'd)

- We define a mass distribution μ on F_1 by:
  - Taking the natural mass distribution on F described above (each k-th level interval of F of side 3^{-k} having mass 2^{-k});
  - "Spreading it" uniformly along the intervals above F.

  So, if U is a rectangle, with sides parallel to the coordinate axes, of height $h \leq 1$, above a k-th level interval of F, then $\mu(U) = h2^{-k}$.

  Any set U is contained in a square of side $|U|$ with sides parallel to the coordinate axes.

  If $3^{-(k+1)} \leq |U| < 3^{-k}$, then U lies above at most one k-th level interval of F of side 3^{-k}.

Example (Cont'd)

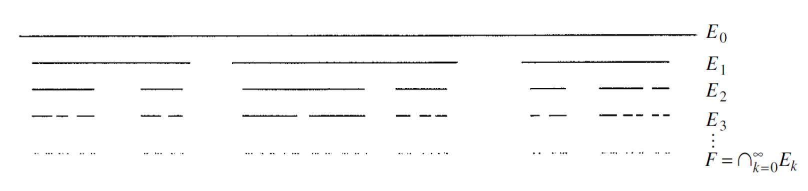
- It follows that

$$\begin{array}{rcl}\mu(U) & \leq & |U|2^{-k} \\ & \leq & |U|3^{-k\log 2/\log 3} \\ & \leq & |U|(3|U|)^{\log 2/\log 3} \\ & = & 3^{\log 2/\log 3}|U|^s \\ & = & 2|U|^s.\end{array}$$

By the Mass Distribution Principle, $\mathcal{H}^s(F_1) > \frac{1}{2}$.

Generalization of the Cantor Construction

- Let $[0,1] = E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$ be a decreasing sequence of sets, with each E_k a union of a finite number of disjoint closed intervals, called k**-th level basic intervals**, such that:
  - Each interval of E_k contains at least two intervals of E_{k+1};
  - The maximum length of k-th level intervals tending to 0 as $k \to \infty$.
- Then the set $F = \bigcap_{k=0}^{\infty} E_k$ is a totally disconnected subset of $[0,1]$ which is generally a fractal.



Computing the Dimension of F

- Obvious upper bounds for the dimension of F are available by taking the intervals of E_k as covering intervals, for each k.
- As usual, lower bounds are harder to find.
- In the following examples:
  - The upper estimates for $\dim_H F$ depend on the number and size of the basic intervals;
  - The lower estimates depend on their spacing.

For these to be equal, the $(k+1)$-th level intervals must be "nearly uniformly distributed" inside the k-th level intervals.

Example

- Let s be a number strictly between 0 and 1.
- We repeat the same general construction.
- We assume that, for each k-th level interval I, the $(k+1)$-st level intervals $I_1, \ldots, I_m$, $m \geq 2$, contained in I are:
  - Of equal length and equally spaced;
  - The lengths are given by

$$|I_i|^s = \frac{1}{m}|I|^s, \quad 1 \leq i \leq m;$$

  - The left-hand ends of I_1 and I coincide;
  - The right-hand ends of I_m and I coincide.
- Then $\dim_H F = s$ and $0 < \mathcal{H}^s(F) < \infty$.
- In general, m may be different for different intervals I.
- So the k-th level intervals may have different lengths.

Example (Upper Bound)

- Let I, I_i be as above.
  Then
  $$|I|^s = \sum_{i=1}^{m} |I_i|^s.$$
  Apply this inductively to the k-th level intervals for successive k.
  We obtain, for each k,
  $$1 = \sum |I_i|^s,$$
  where the sum is over all the k-th level intervals I_i.
  The k-th level intervals cover F.
  The maximum interval length tends to 0 as $k \to \infty$.
  So we get
  $$\mathcal{H}^s_\delta(F) \leq 1,$$
  for sufficiently small δ.
  Thus, $\mathcal{H}^s(F) \leq 1$.

Example (Lower Bound)

- Distribute a mass μ on F in such a way that $\mu(I) = |I|^s$ whenever I is any level k interval.
  - Start with unit mass on $[0, 1]$;
  - Divide this equally between each level 1 interval;
  - The mass on each level 1 interval is divided equally between each level 2 subinterval;

    $\vdots$

  The equation $|I|^s = \sum_{i=1}^{m} |I_i|^s$ ensures that we get a mass distribution on F with $\mu(I) = |I|^s$, for every basic interval.

  We estimate $\mu(U)$ for an interval U with endpoints in F.

  Let I be the smallest basic interval that contains U.

  Suppose it is a k-th level interval.

  Let $I_1, \ldots, I_m$ be the $(k+1)$-st level intervals contained in I.

  Then U intersects a number $j \geq 2$ of the I_i.

  Otherwise, U would be contained in a smaller basic interval.

Example (Lower Bound Cont'd)

- The spacing between consecutive I_i is

$$\begin{array}{rcl}\frac{|I|-m|I_i|}{m-1} & = & |I|\frac{1-m\frac{|I_i|}{|I|}}{m-1} \\ & = & |I|\frac{1-m^{1-1/s}}{m-1} \\ & \stackrel{m\geq 2,\, 0<s<1}{\geq} & |I|\frac{1-2^{1-1/s}}{m} \\ & = & c_s\frac{|I|}{m},\end{array}$$

where $c_s = 1 - 2^{1-1/s}$.

Thus,

$$|U| \geq \frac{j-1}{m}c_s|I| \geq \frac{j}{2m}c_s|I|.$$

Example (Lower Bound Cont'd)

- We know that $|I_i|^s = \frac{1}{m}|I|^s$.

  So we have

$$\begin{array}{rcl} \mu(U) & \leq & j\mu(I_i) \\ & = & j|I_i|^s \\ & = & \frac{j}{m}|I|^s \\ & \leq & 2^s c_s^{-s}(\frac{j}{m})^{1-s}|U|^s \quad (|U| \geq \frac{j}{2m}c_s|I|) \\ & \leq & 2^s c_s^{-s}|U|^s. \end{array}$$

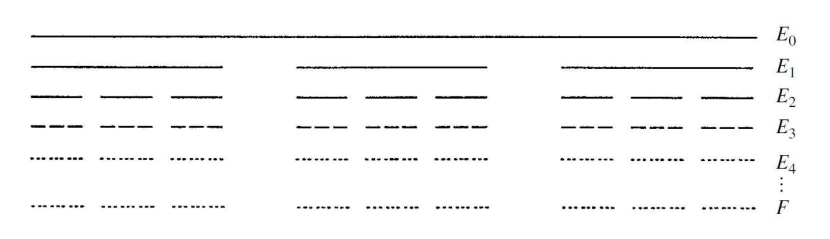
  This is true for any interval U with endpoints in F.

  So it holds also for any set U (by applying the last inequality to the smallest interval containing $U \cap F$).

  By the Mass Distribution Principle, $\mathcal{H}^s(F) > 0$.

Uniform Cantor Sets

- We call the sets obtained when m is kept constant throughout the construction of the preceding example **uniform Cantor sets**.



- They provide a natural generalization of the middle third Cantor set.

Example: Uniform Cantor Sets

- Let $m \geq 2$ be an integer and $0 < r < \frac{1}{m}$.

  Let F be the set obtained by the construction in which:
  - Each basic interval I is replaced by m equally spaced subintervals of lengths $r|I|$;
  - The ends of I coinciding with the ends of the extreme subintervals.

  Then $\dim_H F = \dim_B F = \frac{\log m}{-\log r}$ and $0 < \mathcal{H}^{\log m/-\log r}(F) < \infty$.

  The set F is obtained by taking in the preceding example:
  - m constant;
  - $s = \frac{\log m}{-\log r}$.

  The equation $|I_i|^s = \frac{1}{m}|I|^s$ becomes $(r|I|)^s = \frac{1}{m}|I|^s$.

  This equation is satisfied identically.

  So $\dim_H F = s$.

Example: Uniform Cantor Sets (Cont'd)

- We now turn to the box dimension.

  For each k, F is covered by m^k k-th level intervals of length r^{-k}.

  This gives
  $$\overline{\dim}_B F \leq \frac{\log m}{-\log r}.$$

- The **middle λ Cantor set** is obtained by repeatedly removing a proportion $0 < \lambda < 1$ from the middle of intervals, starting with $[0, 1]$.

  This is a special case of a uniform Cantor set, with:
  - $m = 2$;
  - $r = \frac{1}{2}(1 - \lambda)$.

  Thus, it has Hausdorff and box dimensions
  $$\frac{\log 2}{\log (2/(1 - \lambda))}.$$

Example

- Suppose in the general construction each $(k-1)$-st level interval contains at least $m_k \geq 2$ k-th level intervals, $k = 1, 2, \ldots$ which are separated by gaps of at least ε_k, where $0 < \varepsilon_{k+1} < \varepsilon_k$, for each k. Then

$$\dim_H F \geq \varliminf_{k \to \infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}.$$

We may assume that the right hand side of the inequality is positive, since otherwise the inequality is obvious.

We may also assume that each $(k-1)$-st level interval contains exactly m_k k-th level intervals.

Otherwise, we may throw out excess intervals to get smaller sets E_k and F for which the condition holds.

Example (Cont'd)

- We define a mass distribution μ on F by assigning mass $(m_1 \cdots m_k)^{-1}$ to each of the $m_1 \ldots m_k$ k-th level intervals.

  Let U be an interval with $0 < |U| < \varepsilon_1$.

  We estimate $\mu(U)$.

  Let k be the integer such that $\varepsilon_k \leq |U| < \varepsilon_{k-1}$.

  The number of k-th level intervals that intersect U is:

  (i) At most m_k, since U intersects at most one $(k-1)$-st level interval;
  (ii) At most $\frac{|U|}{\varepsilon_k} + 1 \leq \frac{2|U|}{\varepsilon_k}$, since the k-th level intervals have gaps of at least ε_k between them.

Example (Cont'd)

- Each k-th level interval supports mass $(m_1 \cdots m_k)^{-1}$. So, for all $0 \leq s \leq 1$,

$$\begin{aligned} \mu(U) &\leq (m_1 \cdots m_k)^{-1} \min\left\{\frac{2|U|}{\varepsilon_k}, m_k\right\} \\ &\leq (m_1 \cdots m_k)^{-1} \left(\frac{2|U|}{\varepsilon_k}\right)^s m_k^{1-s}. \end{aligned}$$

Hence,

$$\frac{\mu(U)}{|U|^s} \leq \frac{2^s}{(m_1 \cdots m_{k-1}) m_k^s \varepsilon_k^s}.$$

If $s < \underline{\lim}_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}$, then, for large k,

$$(m_1 \cdots m_{k-1}) m_k^s \varepsilon_k^s > 1 \Rightarrow \mu(U) \leq 2^s |U|^s.$$

Thus, by the Mass Distribution Principle, $\dim_H F \geq s$.

The Case of "Well-Spaced" Intervals

- Suppose that in the preceding example:
  - The k-th level intervals are all of length ε_k;
  - Each $(k-1)$-st level interval contains exactly m_k k-th level intervals, which are "roughly equally spaced" in the sense that $m_k \varepsilon_k \geq c\delta_{k-1}$, where $c > 0$ is a constant.

Then $\dim_H F \geq \underline{\lim}_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}$ becomes

$$\dim_H F \geq \underline{\lim}_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log c - \log \delta_{k-1}} = \underline{\lim}_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log \delta_{k-1}}.$$

But E_{k-1} comprises $m_1 \cdots m_{k-1}$ intervals of length δ_{k-1}.

So this expression equals the upper bound for $\dim_H F$ given by a previous proposition.

Thus, in the situation where the intervals are well spaced, we get equality instead of inequality.

Example

- Fix $0 < s < 1$ and let $n_0, n_1, n_2, \ldots$ be a rapidly increasing sequence of integers, say $n_{k+1} \geq \max\{n_k^k, 4n_k^{1/s}\}$, for each k.
  For each k, let $H_k \subseteq \mathbb{R}$ consist of equally spaced equal intervals:
  - Each has length $n_k^{-1/s}$;
  - The midpoints of consecutive intervals are at distance n_k^{-1} apart.

  Then writing $F = \bigcap_{k=1}^{\infty} H_k$, we have $\dim_H F = s$.
  Since $F \subseteq H_k$ for each k, the set $F \cap [0,1]$ is contained in at most $n_k + 1$ intervals of length $n_k^{-1/s}$. By a previous proposition,

$$\dim_H(F \cap [0,1]) \leq \varliminf_{k\to\infty} \frac{\log(n_k+1)}{-\log n_k^{-1/s}} = s.$$

  Similarly, $\dim_H(F \cap [n, n+1]) \leq s$, for all $n \in \mathbb{Z}$.
  So $\dim_H F \leq s$ as a countable union of such sets.

Example (Cont'd)

- Now let $E_0 = [0,1]$ and, for $k \geq 1$, let E_k consist of the intervals of H_k that are completely contained in E_{k-1}.
  Then each interval I of E_{k-1} contains:
  - At least $n_k|I| - 2 \geq n_k n_{k-1}^{-1/s} - 2 \geq 2$ intervals of E_k;
  - For k large, they are separated by gaps of at least $n_k^{-1} - n_k^{-1/s} \geq \frac{1}{2} n_k^{-1}$.

  Using the preceding example, and noting that setting $m_k = n_k n_{k-1}^{-1/s}$ rather than $m_k = n_k n_{k-1}^{-1/s} - 2$ does not affect the limit,

$$\begin{aligned} \dim_H(F \cap [0,1]) &\geq \dim_H \bigcap_{k=1}^{\infty} E_k \\ &\geq \underline{\lim}_{k\to\infty} \frac{\log((n_1 \cdots n_{k-2})^{1-1/s} n_{k-1})}{-\log(n_k n_{k-1}^{-1/s} \frac{1}{2} n_k^{-1})} \\ &= \underline{\lim}_{k\to\infty} \frac{\log(n_1 \cdots n_{k-2})^{1-1/s} + \log n_{k-1}}{\log 2 + (\log n_{k-1})/s}. \end{aligned}$$

  Provided that n_k is sufficiently rapidly increasing, the terms in $\log n_{k-1}$ in the numerator and denominator dominate.
  So $\dim_H F \geq \dim_H(F \cap [0,1]) \geq s$, as required.

Covering Lemma

Covering Lemma

Let $\mathcal{C}$ be a family of balls contained in some bounded region of $\mathbb{R}^n$. Then there is a (finite or countable) disjoint subcollection $\{B_i\}$, such that

$$\bigcup_{B \in \mathcal{C}} B \subseteq \bigcup_i \widetilde{B}_i,$$

where $\widetilde{B}_i$ is the closed ball concentric with B_i and of four times the radius.

- For simplicity, we give the proof when $\mathcal{C}$ is a finite family.
  The basic idea is the same in the general case.
  We select the $\{B_i\}$ inductively.
  Let B_1 be a ball in $\mathcal{C}$ of maximum radius.
  Suppose that $B_1, \ldots, B_{k-1}$ have been chosen.
  Take B_k to be a largest ball in $\mathcal{C}$ disjoint from $B_1, \ldots, B_{k-1}$.
  The process terminates when no such ball remains.

Covering Lemma (Cont'd)

- Clearly the balls selected are disjoint.
  If $B \in \mathcal{C}$, then one of the following holds:
  - $B = B_i$, for some i;
  - B intersects one of the B_i, with $|B_i| \geq |B|$.

  If this were not the case, then B would have been chosen instead of the first ball B_k with $|B_k| < |B|$.

  Either way, $B \subseteq \widetilde{B}_i$.

  So the required inclusion holds.
- It is easy to see that the result remains true taking $\widetilde{B}_i$ as the ball concentric with B_i and of $3 + \varepsilon$ times the radius, for any $\varepsilon > 0$.
- If $\mathcal{C}$ is finite we may actually take $\varepsilon = 0$.

Hausdorff Bounds Using Balls

Proposition

Let μ be a mass distribution on $\mathbb{R}^n$, let $F \subseteq \mathbb{R}^n$ be a Borel set and let $0 < c < \infty$ be a constant.

(a) If $\overline{\lim}_{r\to 0}\frac{\mu(B(x,r))}{r^s} < c$, for all $x \in F$, then $\mathcal{H}^s(F) \geq \frac{\mu(F)}{c}$.

(b) If $\overline{\lim}_{r\to 0}\frac{\mu(B(x,r))}{r^s} > c$, for all $x \in F$, then $\mathcal{H}^s(F) \leq \frac{2^s\mu(\mathbb{R}^n)}{c}$.

(a) For each $\delta > 0$, let

$$F_\delta = \{x \in F : \mu(B(x,r)) < cr^s \text{ for all } 0 < r \leq \delta\}.$$

Let $\{U_i\}$ be a δ-cover of F.

Then, by hypothesis, it is also a δ-cover of F_δ.

For each U_i containing a point x of F_δ, the ball B with center x and radius $|U_i|$ certainly contains U_i.

Hausdorff Bounds Using Balls (Part (a) Cont'd)

- By definition of F_δ,

$$\mu(U_i) \leq \mu(B) < c|U_i|^s.$$

So

$$\mu(F_\delta) \leq \sum_i \{\mu(U_i) : U_i \text{ intersects } F_\delta\} \leq c \sum_i |U_i|^s.$$

$\{U_i\}$ was an arbitrary δ-cover of F.

So

$$\mu(F_\delta) \leq c\mathcal{H}^s_\delta(F) \leq c\mathcal{H}^s(F).$$

But F_δ increases to F as δ decreases to 0.

So $\mu(F) \leq c\mathcal{H}^s(F)$.

Hausdorff Bounds Using Balls (Part (b))

(b) We prove a weaker version of Part (b) with 2^s replaced by 8^s.

The basic idea of the proof is similar.

Suppose first that F is bounded.

Fix $\delta > 0$ and let $\mathcal{C}$ be the collection of balls

$$\{B(x,r) : x \in F, 0 < r \leq \delta \text{ and } \mu(B(x,r)) > cr^s\}.$$

Then, by hypothesis, $F \subseteq \bigcup_{B \in \mathcal{C}} B$.

Applying the Covering Lemma to the collection $\mathcal{C}$, there is a sequence of disjoint balls $\widetilde{B}_i \in \mathcal{C}$, such that

$$\bigcup_{B \in \mathcal{C}} B \subseteq \bigcup_i \widetilde{B}_i,$$

where $\widetilde{B}_i$ is the ball concentric with B_i but of four times the radius.

Hausdorff Bounds Using Balls (Part (b) Cont'd)

- Thus $\{\widetilde{B}_i\}$ is an 8δ-cover of F.

  It follows that

$$\begin{array}{rcl} \mathcal{H}^s_{8\delta}(F) & \leq & \sum_i |\widetilde{B}_i|^s \\ & \leq & 4^s \sum_i |B_i|^s \\ & \leq & 8^s c^{-1} \sum_i \mu(B_i) \\ & \leq & 8^s c^{-1} \mu(\mathbb{R}^n). \end{array}$$

  Letting $\delta \to 0$, we get

$$\mathcal{H}^s(F) \leq 8^s c^{-1} \mu(\mathbb{R}^n) < \infty.$$

  Finally, suppose F is unbounded and $\mathcal{H}^s(F) > 8^s c^{-1} \mu(\mathbb{R}^n)$.

  Then the $\mathcal{H}^s$-measure of some bounded subset of F will also exceed this value. But this contradicts what was just shown.

Consequences

- It is immediate from the preceding proposition that if

$$\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}=s,\quad \text{for all } x\in F,$$

then $\dim_H F = s$.

- Often the calculations involved can be used in conjunction with the basic properties of dimensions discussed previously.

Example: The function $f(x)=x^2$ is:

  - Lipschitz on $[0,1]$;
  - Bi-Lipschitz on $[\frac{2}{3},1]$.

It follows that, if C is the middle third Cantor set,

$$\dim_H\{x^2 : x\in C\}=\dim_H f(C)=\frac{\log 2}{\log 3}.$$

Subsection 2

Subsets of Finite Measure

Introducing Finitization

- The following theorem guarantees that any (Borel) set F with $\mathcal{H}^s(F) = \infty$ contains a subset E with $0 < \mathcal{H}^s(E) < \infty$, i.e., with E an s-set.
- At first, this might seem obvious - just shave pieces off F until what remains has positive finite measure.
- Unfortunately it is not quite this simple, since it is possible to jump from infinite measure to zero measure without passing through any intermediate value.

Introducing Finitization (Cont'd)

- It is possible to have a decreasing sequence of sets

$$E_1 \supseteq E_2 \supseteq \cdots .$$

such that:
  - $\mathcal{H}^s(E_k) = \infty$, for all k;
  - $\mathcal{H}^s(\bigcap_{k=1}^{\infty} E_k) = 0$.

Example: Consider the sequence

$$E_k = \left[0, \frac{1}{k}\right] \subseteq \mathbb{R}, \quad k = 1, 2, \ldots .$$

Clearly, $E_1 \supset E_2 \supset E_3 \cdots$ and $\bigcap_{k=1}^{\infty} E_k = \{0\}$.
Take $0 < s < 1$. Then we have:
  - $\mathcal{H}^s(E_k) = \infty$, for all k;
  - $\mathcal{H}^s(\bigcap_{k=1}^{\infty} E_k) = 0$.

Finitization

Theorem

Let F be a Borel subset of $\mathbb{R}^n$, with $0 < \mathcal{H}^s(F) \leq \infty$. Then there is a compact set $E \subseteq F$, such that $0 < \mathcal{H}^s(E) < \infty$.

- The complete proof of this is complicated.

  We indicate the ideas involved in the case where:
  - F is a compact subset of $[0,1) \subseteq \mathbb{R}$;
  - $0 < s < 1$.

  We work with the net measures $\mathcal{M}^s$ which are:
  - Defined using the binary intervals $[r2^{-k}, (r+1)2^{-k})$;
  - Related to Hausdorff measure by $\mathcal{H}^s(F) \leq \mathcal{M}^s(F) \leq 2^{s+1}\mathcal{H}^s(F)$.

  We define inductively a decreasing sequence $E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$ of compact subsets of F.

  Let $E_0 = F$.

Finitization (Cont'd)

- For $k \geq 0$ we define E_{k+1} by specifying its intersection with each binary interval I of length 2^{-k}.
  - If $\mathcal{M}^s_{2^{-(k+1)}}(E_k \cap I) \leq 2^{-sk}$, we let $E_{k+1} \cap I = E_k \cap I$.
    Then, using I itself as a covering interval in calculating $\mathcal{M}^s_{2^{-k}}$, gives an estimate at least as large as using shorter binary intervals.
    So we have $\mathcal{M}^s_{2^{-(k+1)}}(E_{k+1} \cap I) = \mathcal{M}^s_{2^{-k}}(E_k \cap I)$.
  - If $\mathcal{M}^s_{2^{-(k+1)}}(E_{k+1} \cap I) > 2^{-sk}$, we take $E_{k+1} \cap I$ to be a compact subset of $E_k \cap I$ with $\mathcal{M}^s_{2^{-(k+1)}}(E_{k+1} \cap I) = 2^{-sk}$. Such a subset exists, since $\mathcal{M}^s_{2^{-(k+1)}}(E_k \cap I \cap [0, u])$ is finite and continuous in u. (This is why we need to work with the $\mathcal{M}^s_\delta$ rather than $\mathcal{M}^s$.)
    Now we have $\mathcal{M}^s_{2^{-k}}(E_k \cap I) = 2^{-sk}$.
    So $\mathcal{M}^s_{2^{-(k+1)}}(E_{k+1} \cap I) = \mathcal{M}^s_{2^{-k}}(E_k \cap I)$ holds.

Summing this relation over all binary intervals of length 2^{-k} we get

$$\mathcal{M}^s_{2^{-(k+1)}}(E_{k+1}) = \mathcal{M}^s_{2^{-k}}(E_k).$$

Finitization (Cont'd)

- We obtained $\mathcal{M}^s_{2^{-(k+1)}}(E_{k+1}) = \mathcal{M}^s_{2^{-k}}(E_k)$.
  Repeated application of this gives $\mathcal{M}^s_{2^{-k}}(E_k) = \mathcal{M}^s_1(E_0)$, for all k.
  Let E be the compact set $\bigcap_{k=0}^{\infty} E_k$.
  Taking the limit as $k \to \infty$ gives $\mathcal{M}^s(E) = \mathcal{M}^s_1(E_0)$.
  $E_0 = F$ is covered by the single interval $[0, 1)$.
  So we have $\mathcal{M}^s(E) = \mathcal{M}^s_1(E_0) \leq 1$.
  Now $\mathcal{M}^s(E_0) \geq \mathcal{H}^s(E_0) > 0$.
  So, for k large enough, we have $\mathcal{M}^s_{2^{-k}}(E_0) > 0$
  Thus, one of the following holds:
  - $\mathcal{M}^s(E) = \mathcal{M}^s_1(E_0) \geq 2^{-ks}$;
  - $\mathcal{M}^s_1(E_0) < 2^{-ks}$.
    So $\mathcal{M}^s(E) = \mathcal{M}^s_1(E_0) = \mathcal{M}^s_{2^{-k}}(E_0) > 0$.

  Thus, $0 < \mathcal{M}^s(E) < \infty$.
  The theorem follows from $\mathcal{H}^s(F) \leq \mathcal{M}^s(F) \leq 2^{s+1}\mathcal{H}^s(F)$.

Compact Subset with Nice Hausdorff Measures

Proposition

Let F be a Borel set satisfying $0 < \mathcal{H}^s(F) < \infty$. There is a constant b and a compact set $E \subseteq F$, with $\mathcal{H}^s(E) > 0$, such that

$$\mathcal{H}^s(E \cap B(x,r)) \leq br^s,$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

- In a previous proposition, it was shown that, for a mass distribution μ on $\mathbb{R}^n$, a Borel set $F \subseteq \mathbb{R}^n$ and a constant $0 < c < \infty$, if $\overline{\lim}_{r\to 0}\frac{\mu(B(x,r))}{r^s} > c$, for all $x \in F$, then $\mathcal{H}^s(F) \leq \frac{2^s\mu(\mathbb{R}^n)}{c}$.

  Take μ as the restriction of $\mathcal{H}^s$ to F, i.e., $\mu(A) = \mathcal{H}^s(F \cap A)$.

  Let

$$F_1 = \left\{x \in \mathbb{R}^n : \overline{\lim}_{r\to 0}\frac{\mathcal{H}^s(F \cap B(x,r))}{r^s} > 2^{1+s}\right\}.$$

Compact Subset with Nice Hausdorff Measures (Cont'd)

- Then, we have

$$\mathcal{H}^s(F_1) \leq 2^s 2^{-(1+s)} \mu(F) = \frac{1}{2}\mathcal{H}^s(F).$$

Thus, $\mathcal{H}^s(F \backslash F_1) \leq \frac{1}{2}\mathcal{H}^s(F) > 0$.
So, if $E_1 = F \backslash F_1$, then:
  - $\mathcal{H}^s(E_1) > 0$;
  - $\overline{\lim}_{r \to 0} \frac{\mathcal{H}^s(F \cap B(x,r))}{r^s} \leq 2^{1+s}$ for $x \in E_1$.

By Egoroff's theorem, there is a compact set $E \subseteq E_1$ with $\mathcal{H}^s(E) > 0$ and a number $r_0 > 0$, such that

$$\frac{\mathcal{H}^s(F \cap B(x,r))}{r^s} \leq 2^{2+s},$$

for all $x \in E$ and all $0 < r \leq r_0$.
But, if $r \geq r_0$, we have $\frac{\mathcal{H}^s(F \cap B(x,r))}{r^s} \leq \frac{\mathcal{H}^s(F)}{r_0^s}$.
So the inequality in the statement holds for all $r > 0$.

Frostman's Lemma

Corollary (Frostman's Lemma)

Let F be a Borel subset of $\mathbb{R}^n$ with $0 < \mathcal{H}^s(F) \leq \infty$. Then there is a compact set $E \subseteq F$, such that $0 < \mathcal{H}^s(E) < \infty$ and a constant b, such that

$$\mathcal{H}^s(E \cap B(x, r)) \leq br^s,$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

- The preceding theorem gives $F_1 \subseteq F$ of positive finite measure.

  Applying the preceding proposition to F_1 gives the result.
- This corollary may be regarded as a converse of the Mass Distribution Principle.

Subsection 3

Potential Theoretic Methods

Potential and Energy Due to Mass Distribution

- For $s \geq 0$, the *s*-**potential** at a point x of $\mathbb{R}^n$ due to the mass distribution μ on $\mathbb{R}^n$ is defined as

$$\phi_s(x) = \int \frac{d\mu(y)}{|x-y|^s}.$$

- If we are working in $\mathbb{R}^3$ and $s = 1$, then this is essentially the familiar Newtonian gravitational potential.
- The *s*-**energy** of μ is

$$I_s(\mu) = \int \phi_s(x) d\mu(x) = \iint \frac{d\mu(x) d\mu(y)}{|x-y|^s}.$$

Energy of Mass Distributions and Hausdorff Measure

Theorem

Let F be a subset of $\mathbb{R}^n$.

(a) If there is a mass distribution μ on F with $I_s(\mu) < \infty$, then $\mathcal{H}^s(F) = \infty$ and $\dim_H F \geq s$.

(b) If F is a Borel set with $\mathcal{H}^s(F) > 0$, then there exists a mass distribution μ on F with $I_t(\mu) < \infty$, for all $0 < t < s$.

(a) Suppose that $I_s(\mu) < \infty$ for some mass distribution μ with support contained in F.

Define

$$F_1 = \left\{ x \in F : \overline{\lim_{r \to 0}} \frac{\mu(B(x,r))}{r^s} > 0 \right\}.$$

Energy and Hausdorff Measure (Part (a) Cont'd)

- Suppose $x \in F_1$.

  Then we may find $\varepsilon > 0$ and a sequence of numbers $\{r_i\}$ decreasing to 0, such that
  $$\mu(B(x, r_i)) \geq \varepsilon r_i^s.$$
  Note that $\mu(\{x\}) = 0$, since, otherwise, $I_s(\mu) = \infty$.

  By the continuity of μ, taking q_i, $0 < q_i < r_i$, small enough, we get
  $$\mu(A_i) \geq \frac{1}{4}\varepsilon r_i^s, \quad i = 1, 2, \ldots,$$
  where A_i is the annulus $B(x, r_i) \backslash B(x, q_i)$.

  Taking subsequences if necessary, assume that $r_{i+1} < q_i$, for all i.

  Then the A_i are disjoint annuli centered on x.

Energy and Hausdorff Measure (Part (a) Cont'd)

- Now we have, for all $x \in F_1$,

$$\phi_s(x) = \int \frac{d\mu(y)}{|x-y|^s} \geq \sum_{i=1}^{\infty} \int_{A_i} \frac{d\mu(y)}{|x-y|^s} \geq \sum_{i=1}^{\infty} \frac{1}{4} \varepsilon r_i^s r_i^{-s} = \infty,$$

since $|x-y|^{-s} \geq r_i^{-s}$ on A_i.

But $I_s(\mu) = \int \phi_s(x) d\mu(x) < \infty$.

So $\phi_s(x) < \infty$ for μ-almost all x.

We conclude that $\mu(F_1) = 0$.

Now, if $x \in F \setminus F_1$, $\overline{\lim}_{r \to 0} \frac{\mu(B(x,r))}{r^s} = 0$.

So by a previous proposition, for all $c \geq 0$, we have

$$\mathcal{H}^s(F) \geq \mathcal{H}^s(F \setminus F_1) \geq \frac{\mu(F \setminus F_1)}{c} \geq \frac{\mu(F) - \mu(F_1)}{c} = \frac{\mu(F)}{c}.$$

Hence, $\mathcal{H}^s(F) = \infty$.

Energy and Hausdorff Measure (Part (b))

(b) Suppose that $\mathcal{H}^s(F) > 0$. We use $\mathcal{H}^s$ to construct a mass distribution μ on F with $I_t(\mu) < \infty$, for every $t < s$.

By the preceding corollary, there exist a compact set $E \subseteq F$, with $0 < \mathcal{H}^s(E) < \infty$ and a constant b, such that

$$\mathcal{H}^s(E \cap B(x,r)) \leq br^s,$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

Let μ be the restriction of $\mathcal{H}^s$ to E, $\mu(A) = \mathcal{H}^s(E \cap A)$.

Then μ is a mass distribution on F.

Fix $x \in \mathbb{R}^n$ and write

$$m(r) = \mu(B(x,r)) = \mathcal{H}^s(E \cap B(x,r)) \leq br^s.$$

Energy and Hausdorff Measure (Part (b) Cont'd)

- Then, if $0 < t < s$,

$$\begin{array}{rcl} \phi_t(x) & = & \int_{|x-y|\leq 1} \frac{d\mu(y)}{|x-y|^t} + \int_{|x-y|>1} \frac{d\mu(y)}{|x-y|^t} \\ & \leq & \int_0^1 r^{-t} dm(r) + \mu(\mathbb{R}^n) \\ & = & [r^{-t} m(r)]_0^1 + t \int_0^1 r^{-(t+1)} m(r) dr + \mu(\mathbb{R}^n) \\ & \leq & b + bt \int_0^1 r^{s-t-1} dr + \mu(\mathbb{R}^n) \\ & = & b(1 + \frac{t}{s-t}) + \mathcal{H}^s(F) = c, \end{array}$$

after integrating by parts and using the definition of $m(r)$.

Thus, $\phi_t(x) \leq c$, for all $x \in \mathbb{R}^n$.

$$I_t(\mu) = \int \phi_t(x) d\mu(x) \leq c\mu(\mathbb{R}^n) < \infty.$$

Using the Energy Theorem

- The theorem is often used to find the dimension of fractals F_θ which depend on a parameter θ.
- There may be a natural way to define a mass distribution μ_θ on F_θ, for each θ.
- Suppose we can show that, for some s,

$$\int I_s(\mu_\theta)d\theta = \iiint \frac{d\mu_\theta(x)d\mu_\theta(y)d\theta}{|x-y|^s} < \infty.$$

- Then $I_s(\mu_\theta) < \infty$, for almost all θ.
- So we may conclude that

$$\dim_H F_\theta \geq s, \quad \text{for almost all } \theta.$$