Introduction to Fractal Geometry

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

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1. Local Structure of Fractals
   - Introduction
   - Densities
   - Structure of 1-Sets
   - Tangent to s-Sets

Subsection 1

Introduction

Restriction to s-Sets

- Our goal os to study the local structure of fractals using the power of Hausdorff measures.
- For this, it is necessary to restrict attention to s-sets.
- These are Borel sets of Hausdorff dimension s with positive finite s-dimensional Hausdorff measure.
- This is not so restrictive as it first appears, since many fractals encountered in practice are s-sets.
- Even if $\mathcal{H}^s(F) = \infty$, by a previous theorem, F has subsets that are s-sets to which this theory can be applied.
- Alternatively, it sometimes happens that a set F of dimension s is a countable union of s-sets, and the properties of these component sets can often be transferred to F.

Difficulty of Proofs

- The material outlined in this set lies at the heart of geometric measure theory.
- In this area, rigorous proofs are often intricate and difficult.
- We include some proofs to give a flavor of the subject
- But we omit the harder proofs.
- We generally restrict attention to subsets of the plane.
- Higher-dimensional analogues, though valid, are often harder.

Subsection 2

Densities

Density of a Set

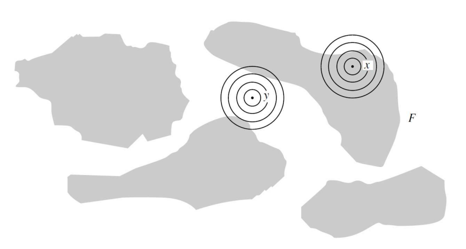
- Let F be a subset of the plane.
- Let $B(x,r)$ be the closed disc of radius r and center x
- The **density** of F at x is

$$\lim_{r\to 0}\frac{\text{area}(F\cap B(x,r))}{\text{area}(B(x,r))}=\lim_{r\to 0}\frac{\text{area}(F\cap B(x,r))}{\pi r^2}.$$

- The classical Lebesgue Density Theorem tells us that, for a Borel set F, except for a set of x of area 0, the density limit exists and:
  - Equals 1 when $x\in F$;
  - Equals 0 when $x\notin F$.

Density of a Set (Illustration)

- For a typical point x of F, small discs centered at x are almost entirely filled by F.



- On the other hand, if x is outside F, then small discs centered at x generally contain very little of F.

Density of Smooth Curves

- Suppose F is a smooth curve in the plane.
- If x is a point of F (other than an endpoint), then $F \cap B(x, r)$ is close to a diameter of $B(x, r)$ for small r and

$$\lim_{r \to 0} \frac{\text{length}(F \cap B(x, r))}{2r} = 1.$$

- If $x \notin F$, then this limit is clearly 0.
- Density theorems such as these tell us how much of the set F, in the sense of area or length, is concentrated near x.

Density of an s-Set

- Suppose F is an s-set in $\mathbb{R}^2$ with $0 < s < 2$.
  - 0-sets are just finite sets of points;
  - $\mathcal{H}^2$ is essentially area, so if $s = 2$ we are in the Lebesgue density situation.
- We would like to know how the s-dimensional Hausdorff measure of $F \cap B(x, r)$ behaves as $r \to 0$.
- The **lower density** of an s-set F at a point $x \in \mathbb{R}^n$ is

$$\underline{D}^s(F, x) = \varliminf_{r \to 0} \frac{\mathcal{H}^s(F \cap B(x, r))}{(2r)^s}.$$

- The **upper density** of an s-set F at a point $x \in \mathbb{R}^n$ is

$$\overline{D}^s(F, x) = \varlimsup_{r \to 0} \frac{\mathcal{H}^s(F \cap B(x, r))}{(2r)^s}.$$

- If $\underline{D}^s(F, x) = \overline{D}^s(F, x)$ we say that the **density** of F at x exists and we write $D^s(F, x)$ for the common value.

Regular and Irregular Points and Sets

- A point x at which $\underline{D}^s(F,x) = \overline{D}^s(F,x) = 1$ is called a **regular point** of F.
- Otherwise, x is an **irregular point**.
- An s-set is termed **regular** if $\mathcal{H}^s$-almost all of its points (i.e., all of its points except for a set of $\mathcal{H}^s$-measure 0) are regular.
- It is called **irregular** if $\mathcal{H}^s$-almost all of its points are irregular.
- Not that, in this sense, "irregular" does not mean "not regular"!
- As we will see, a fundamental result is that an s-set F must be irregular unless s is an integer.
- However, if s is integral an s-set decomposes into a regular and an irregular part. Roughly speaking:
  - A regular 1-set consists of portions of rectifiable curves of finite length;
  - An irregular 1-set is totally disconnected and dust-like, and typically of fractal form.

Behavior of Irregular Sets

Proposition

Let F be an s-set in $\mathbb{R}^n$. Then:

(a) $\underline{D}^s(F,x) = \overline{D}^s(F,x) = 0$, for $\mathcal{H}^s$-almost all $x \notin F$;

(b) $2^{-s} \leq \overline{D}^s(F,x) \leq 1$, for $\mathcal{H}^s$-almost all $x \in F$.

Partial Proof:

(a) Suppose, first, F is closed and $x \notin F$.

Then, for r small enough, $B(x,r) \cap F = \emptyset$.

Hence,

$$\lim_{r \to 0} \frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s} = 0.$$

If F is not closed the proof is a little more involved and we omit it.

Behavior of Irregular Sets (Part (b))

(b) This follows quickly from a previous proposition.

Take μ as the restriction of $\mathcal{H}^s$ to F, i.e. $\mu(A) = \mathcal{H}^s(F \cap A)$.

Let

$$F_1 = \left\{ x \in F : \overline{D}^s(F, x) = \overline{\lim_{r \to 0}} \frac{\mathcal{H}^s(F \cap B(x, r))}{(2r)^s} < 2^{-s} c \right\}.$$

Then

$$\mathcal{H}^s(F_1) \geq \frac{\mathcal{H}^s(F)}{c} \geq \frac{\mathcal{H}^s(F_1)}{c}.$$

If $0 < c < 1$, this is only possible if $\mathcal{H}^s(F_1) = 0$.

Thus, for almost all $x \in F$, we have $\overline{D}^s(F, x) \geq 2^{-s}$.

The upper bound follows similarly, using the same proposition.

- Note that an immediate consequence of Part (b) is that an irregular set has a lower density which is strictly less than 1 almost everywhere.

Density of a Set and Densities of Subsets

- Let F be an s-set and let E be a Borel subset of F. Then

$$\frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s} = \frac{\mathcal{H}^s(E \cap B(x,r))}{(2r)^s} + \frac{\mathcal{H}^s((F \backslash E) \cap B(x,r))}{(2r)^s}.$$

For almost all x in E, we have $\frac{\mathcal{H}^s((F \backslash E) \cap B(x,r))}{(2r)^s} \to 0$ as $r \to 0$, by the preceding proposition. So letting $r \to 0$ gives

$$\underline{D}^s(F,x) = \underline{D}^s(E,x), \quad \overline{D}^s(F,x) = \overline{D}^s(E,x),$$

for $\mathcal{H}^s$-almost all x in E.
- Thus, if E is a subset of an s-set F with $\mathcal{H}^s(E) > 0$, then:
  - E is regular if F is regular;
  - E is irregular if F is irregular.
- In particular, the intersection of a regular and an irregular set, being a subset of both, has measure zero.

Non-Integrality and Irregularity

Theorem

Let F be an s-set in $\mathbb{R}^2$. Then F is irregular unless s is an integer.

Partial Proof: We show that F is irregular if $0 < s < 1$ by showing that the density $D^s(F, x)$ fails to exist almost everywhere in F.

Suppose, to the contrary, that there is a set $F_1 \subseteq F$ of positive measure where the density exists.

By the preceding proposition, $\frac{1}{2} < 2^{-s} \leq D^s(F, x)$.

By Egoroff's Theorem, we may find $r_0 > 0$ and a Borel set $E \subseteq F_1 \subseteq F$ with $\mathcal{H}^s(E) > 0$, such that, for all $x \in E$ and $r < r_0$,

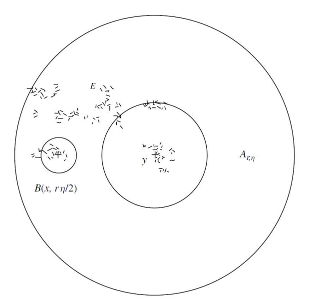
$$\mathcal{H}^s(F \cap B(x, r)) > \frac{1}{2}(2r)^s.$$

Let $y \in E$ be a cluster point of E, i.e., a point y with other points of E arbitrarily close.

Non-Integrality and Irregularity (Cont'd)

- Let η be a number with $0 < \eta < 1$.
  Let $A_{r,\eta}$ be the annulus

  $$A_{r,\eta} = B(y, r(1+\eta)) \backslash B(y, r(1-\eta)).$$



Then,

$$\begin{array}{rcl} (2r)^{-s}\mathcal{H}^s(F \cap A_{r,\eta}) & = & (2r)^{-s}\mathcal{H}^s(F \cap B(y, r(1+\eta))) \\ & & -(2r)^{-s}\mathcal{H}^s(F \cap B(y, r(1-\eta))) \\ & \stackrel{r \to 0}{\to} & D^s(F,y)((1+\eta)^s - (1-\eta)^s). \end{array}$$

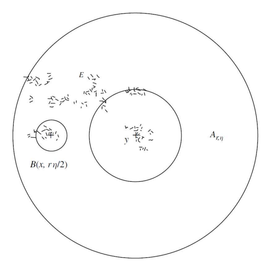
Non-Integrality and Irregularity (Cont'd)

- For a sequence of values of r tending to 0, we may find $x \in E$ with $|x - y| = r$.

  Then $B(x, \frac{1}{2}r\eta) \subseteq A_{r,\eta}$.

  So we get

$$\begin{array}{rcl} \frac{1}{2}r^s\eta^s & < & \mathcal{H}^s\left(F \cap B\left(x, \frac{1}{2}r\eta\right)\right) \\ & \leq & \mathcal{H}^s(F \cap A_{r,\eta}). \end{array}$$



Therefore,

$$\begin{array}{rcl} 2^{-s-1}\eta^s & = & \frac{1}{2}r^s\eta^s\frac{1}{(2r)^s} \\ & = & \frac{1}{(2r)^s}\mathcal{H}^s(F \cap A_{r,\eta}) \\ & \leq & D^s(F, y)((1+\eta)^s - (1-\eta)^s) \\ & = & D^s(F, y)(2s\eta + \text{terms in } \eta^2 \text{ or higher}). \end{array}$$

Letting $\eta \to 0$, we see that this is impossible when $0 < s < 1$.

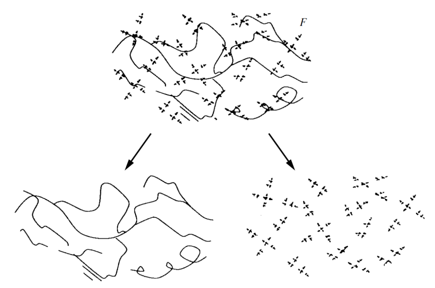
Subsection 3

Structure of 1-Sets

Decomposition Theorem

Decomposition Theorem

Let F be a 1-set. The set of regular points of F forms a regular set, the set of irregular points forms an irregular set.



- Consider the sets of regular and of irregular points of F.

  Then take into account the relation between the dimensions of a set and of its subsets.

Example

- Examples of regular and irregular 1-sets abound.
- Smooth curves are regular, and provide us with the shapes of classical geometry such as the perimeters of circles or ellipses.
- On the other hand the iterated construction of the "Cantor dust" gives an irregular 1-set which is a totally disconnected fractal.
- This is typical, since as we will see:
  - Regular 1-sets are made up from pieces of curve;
  - Irregular 1-sets are dust-like and "curve-free",
    i.e., intersect any (finite length) curve in length zero.

Jordan Curves

- For our purposes a **curve** or **Jordan curve** C is the image of a continuous injection (one-to-one function)

$$\psi : [a, b] \to \mathbb{R}^2,$$

  where $[a, b] \subseteq \mathbb{R}$ is a proper closed interval.
- According to our definition:
  - Curves are not self-intersecting;
  - They have two ends;
  - They are compact connected subsets of the plane.

Rectifiable Curves

- The **length** $\mathcal{L}(C)$ of the curve C is given by polygonal approximation:

$$\mathcal{L}(C) = \sup \sum_{i=1}^{m} |x_i - x_{i-1}|,$$

  where the supremum is taken over all dissections of C by points $x_0, x_1, \ldots, x_m$ in that order along the curve.
- If the length $\mathcal{L}(C)$ is positive and finite, we call C a **rectifiable curve**.

Length and Hausdorff Measure

Lemma

If C is a rectifiable curve then $\mathcal{H}^1(C) = \mathcal{L}(C)$.

- For $x, y \in C$, let $C_{x,y}$ denote that part of C between x and y. Denote by $[x, y]$ the straight-line segment joining x to y. Orthogonal projection onto the line through x and y does not increase distances.
  So we get by the definition of $\mathcal{H}^1$,
  $$\mathcal{H}^1(C_{x,y}) \geq \mathcal{H}^1[x, y] = |x - y|.$$
  Hence, for any dissection $x_0, x_1, \ldots, x_m$ of C,
  $$\sum_{i=1}^{m} |x_i - x_{i-1}| \leq \sum_{i=1}^{m} \mathcal{H}^1(C_{x_i, x_{i-1}}) \leq \mathcal{H}^1(C).$$
  So $\mathcal{L}(C) \leq \mathcal{H}^1(C)$.

Length and Hausdorff Measure (Cont'd)

- On the other hand, let $f : [0, \mathcal{L}(C)] \to C$ be the mapping that takes t to the point on C at distance t along the curve from one of its ends.

  Clearly, for $0 \le t, u \le \mathcal{L}(C)$,

  $$|f(t) - f(u)| \le |t - u|.$$

  This shows that f is Lipschitz.

  Therefore,

  $$\mathcal{H}^1(C) \le \mathcal{H}^1[0, \mathcal{L}(C)] = \mathcal{L}(C).$$

Regularity of Rectifiable Curves

Lemma

A rectifiable curve is a regular 1-set.

- If C is rectifiable, $\mathcal{L}(C) < \infty$.

  C has distinct endpoints p and q.

  So $\mathcal{L}(C) \geq |p - q| > 0$.

  By the preceding lemma, $0 < \mathcal{H}^1(C) < \infty$.

  So C is a 1-set.

  A point x of C, not an endpoint, divides C into $C_{p,x}$ and $C_{x,q}$.

  If r is sufficiently small, then, moving away from x along the curve $C_{x,q}$, we reach a first point y on C with $|x - y| = r$.

  Then $C_{x,y} \subseteq B(x,r)$ and

$$r = |x - y| \leq \mathcal{L}(C_{x,y}) = \mathcal{H}^1(C_{x,y}) \leq \mathcal{H}^1(C_{x,q} \cap B(x,r)).$$

Regularity of Rectifiable Curves (Cont'd)

- We got $r \leq \mathcal{H}^1(C_{x,q} \cap B(x,r))$.
  Similarly, $r \leq \mathcal{H}^1(C_{p,x} \cap B(x,r))$.
  So, for r small enough,
  $$2r \leq \mathcal{H}^1(C \cap B(x,r)).$$
  Thus,
  $$\underline{D}^1(C,x) = \varliminf_{r \to 0} \frac{\mathcal{H}^1(C \cap B(x,r))}{2r} \geq 1.$$
  By the first proposition of the section,
  $$\underline{D}^1(C,x) \leq \overline{D}^1(C,x) \leq 1.$$
  So $D^1(C,x)$ exists and equals 1, for all $x \in C$ other than the two endpoints. This proves that C is regular.

Curve-like 1-Sets

- The relation between dimensions of sets and their subsets gives:
  - Subsets of regular sets are regular;
  - Unions of regular are regular.
- We define a 1-set to be **curve-like** if it is contained in a countable union of rectifiable curves.

Curve-like 1-Sets and Regularity

Proposition

A curve-like 1-set is a regular 1-set.

- Let F be a curve-like 1-set.
  Then $F \subseteq \bigcup_{i=1}^{\infty} C_i$, where the C_i are rectifiable curves.
  For each i and $\mathcal{H}^1$-almost all $x \in F \cap C_i$ we have

$$\begin{array}{rcll} 1 & = & \underline{D}^1(C_i, x) & \text{(preceding lemma)} \\ & = & \underline{D}^1(F \cap C_i, x) & \text{(relation between dimensions)} \\ & \leq & \underline{D}^1(F, x). & \end{array}$$

  Hence $1 \leq \underline{D}^1(F, x)$, for almost all $x \in F$.
  But, by a previous proposition, for almost all $x \in F$ we have

$$\underline{D}^1(F, x) \leq \overline{D}^1(F, x) \leq 1.$$

  So $D^1(F, x) = 1$ almost everywhere, and F is regular.

Curve-Free 1-Sets

- A 1-set is called **curve-free** if its intersection with every rectifiable curve has $\mathcal{H}^1$-measure-zero.

Proposition

An irregular 1-set is curve-free.

- Let F be irregular and C be a rectifiable curve.

  Then $F \cap C$ is a subset of both a regular and an irregular set.

  So $F \cap C$ has zero $\mathcal{H}^1$-measure.

A Bound for the Dimension of Curve-Free 1-Sets

- The two preceding propositions begin to suggest that regular and irregular sets might be characterized as curve-like and curve-free, respectively.
- This is indeed the case, but it is far from easy to prove.
- The proof relies on a lower density estimate.
- This is given in the following proposition, whose omitted proof involves:
  - An intricate use of the properties of curves and connected sets;
  - Some ingenious geometrical arguments.

Proposition

Let F be a curve-free 1-set in $\mathbb{R}^2$. Then

$$\underline{D}^1(F,x) \leq \frac{3}{4} \text{ at almost all } x \in F.$$

Characterizations of Irregular and Regular 1-Sets

Theorem

(a) A 1-set in $\mathbb{R}^2$ is irregular if and only if it is curve-free.

(b) A 1-set in $\mathbb{R}^2$ is regular if and only if it is the union of a curve-like set and a set of $\mathcal{H}^1$-measure zero.

(a) A curve-free set must be irregular by the preceding proposition.
The proposition before the last provides the converse implication.

(b) By a previous proposition, a curve-like set is regular.
Adding in a set of measure zero does not affect densities.
Therefore, regularity also remains unaffected.
Suppose F is regular. Then any Borel subset E of positive measure is regular with $\underline{D}^1(E,x) = 1$, for almost all $x \in E$.
By the preceding proposition, the set E cannot be curve-free.
So some rectifiable curve intersects E in a set of positive length.
We use this to define inductively a sequence of rectifiable curves $\{C_i\}$.

Characterizations of Irregular and Regular 1-Sets (Cont'd)

- We choose C_1 to cover a reasonably large part of F, say

$$\mathcal{H}^1(F \cap C_1) \geq \frac{1}{2} \sup \{\mathcal{H}^1(F \cap C) : C \text{ is rectifiable}\} > 0.$$

Suppose $C_1, \ldots, C_k$ have been selected.

Assume, first, $F_k = F \setminus \bigcup_{i=1}^k C_i$ has positive measure.

Let C_{k+1} be a rectifiable curve for which

$$\mathcal{H}^1(F_k \cap C_{k+1}) \geq \frac{1}{2} \sup \{\mathcal{H}^1(F_k \cap C) : C \text{ is rectifiable}\} > 0.$$

If $F_k = F \setminus \bigcup_{i=1}^k C_i$ has measure zero, the process terminates.

If the process terminates, then for some k the curves $C_1, \ldots, C_k$ cover almost all of F and F is curve-like.

We argue that, if the process does not terminate, then F is the union of a curve-like set and a set of $\mathcal{H}^1$-measure zero.

Characterizations of Irregular and Regular 1-Sets (Cont'd)

- Suppose the process does not terminate.
  The $F_k \cap C_{k+1}$ are disjoint.
  So we have $\sum_k \mathcal{H}^1(F_k \cap C_{k+1}) \leq \mathcal{H}^1(F) < \infty$.
  So $\mathcal{H}^1(F_k \cap C_{k+1}) \to 0$ as $k \to \infty$.
  Suppose $\mathcal{H}^1(F \setminus \sum_{i=1}^{\infty} C_i) > 0$.
  Then, there is a rectifiable curve C, such that, for some $d > 0$,
  $$\mathcal{H}^1\left(\left(F \setminus \sum_{i=1}^{\infty} C_i\right) \cap C\right) = d.$$
  But $\mathcal{H}^1(F_k \cap C_{k+1}) < \frac{1}{2}d$, for some k. So, according to the definition of C_{k+1}, C would have been selected in preference to C_{k+1}.
  This shows that $\mathcal{H}^1(F \setminus \sum_{i=1}^{\infty} C_i) = 0$.
  Hence, F consists of:
  - The curve-like set $F \cap \bigcup_{i=1}^{\infty} C_i$;
  - $F \setminus \bigcup_{i=1}^{\infty} C_i$, which is of measure zero.

Differences Between Regular and Irregular 1-Sets

- In terns of curves:
  - Regular 1-sets are essentially unions of subsets of rectifiable curves;
  - Irregular 1-sets contain no pieces of rectifiable curves at all.

  This dichotomy is remarkable in that the definition of regularity is purely in terms of densities and makes no reference to curves.
- In terms of densities:
  - Almost everywhere, a regular set has lower density 1;
  - An irregular set has lower density at most $\frac{3}{4}$.

  Thus, in any 1-set F the set of points for which $\frac{3}{4} < \underline{D}^1(F, x) < 1$ has $\mathcal{H}^1$-measure zero.
- In terms of connectedness:
  - Regular 1-sets may be connected;
  - Like sets of dimension less than 1, irregular 1-sets must be totally disconnected.
- Further differences between regular and irregular sets include the existence of tangents and projection properties.

Subsection 4

Tangent to s-Sets

Tangent of s-Set at a Point

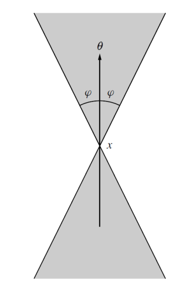
- Any generalization of the definition of tangents should reflect the local directional distribution of sets of positive measure.
- An s-set F in $\mathbb{R}^n$ has a **tangent at x in direction $\boldsymbol{\theta}$** ($\boldsymbol{\theta}$ a unit vector) if

$$\overline{D}^s(F,x) > 0$$

and, for every angle $\varphi > 0$,

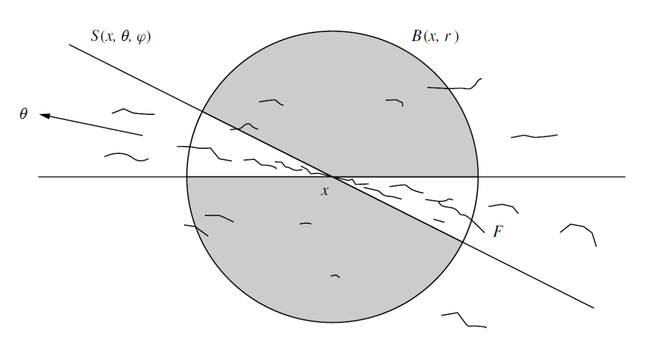
$$\lim_{r\to 0} \frac{\mathcal{H}^s(F \cap (B(x,r) \backslash S(x,\boldsymbol{\theta},\varphi)))}{r^s} = 0,$$

where $S(x,\boldsymbol{\theta},\varphi)$ is the double sector with vertex x, consisting of those y such that the line segment $[x,y]$ makes an angle at most φ with $\boldsymbol{\theta}$ or $-\boldsymbol{\theta}$.



Tangent of s-Set at a Point (Cont'd)

- Thus, for a tangent in direction $\boldsymbol{\theta}$, we require that:
  - A significant part of F lies near x;
  - Of the part lying near x a negligible amount lies outside any double sector $S(x, \boldsymbol{\theta}, \varphi)$.



Tangents of Rectifiable Curves

Proposition

A rectifiable curve C has a tangent at almost all of its points.

- By a previous lemma, the upper density

$$\overline{D}^1(C,x) = 1 > 0, \text{ for almost all } x \in C.$$

We may reparametrize the defining function of C by arc length.
So the function

$$\psi : [0, \mathcal{L}(C)] \to \mathbb{R}^2$$

gives $\psi(t)$ as the point distance t along C from the endpoint $\psi(0)$.
Now $\mathcal{L}(C) < \infty$ means that ψ has bounded variation.
I.e., we have

$$\sup \sum_{i=1}^{m} |\psi(t_i) - \psi(t_{i-1})| < \infty,$$

where the supremum is over all $0 = t_0 < t_1 < \cdots < t_m = \mathcal{L}(C)$.

Tangents of Rectifiable Curves (Cont'd)

- A standard result from the theory of functions asserts that functions of bounded variation are differentiable almost everywhere.
  So $\psi'(t)$ exists as a vector for almost all t.
  Because of the arc length parametrization, $|\psi'(t)| = 1$ for such t.
  So at almost all $\psi(t)$ on C, there exists a unit vector $\boldsymbol{\theta}$, such that
  $$\lim_{u \to t} \frac{\psi(u) - \psi(t)}{u - t} = \boldsymbol{\theta}.$$
  Thus, given $\varphi > 0$, there is a number $\varepsilon > 0$, such that
  $$|u - t| < \varepsilon \quad \text{implies} \quad \psi(u) \in S(\psi(t), \boldsymbol{\theta}, \varphi).$$
  But C has no double points. So we may find r, such that
  $$|u - t| \geq \varepsilon \quad \text{implies} \quad \psi(u) \notin B(\psi(t), r).$$
  So $C \cap (B(\psi(t), r) \backslash S(\psi(t), \boldsymbol{\theta}, \varphi))$ is empty.
  By the definitions, the curve C has a tangent at $\psi(t)$.
  Such points account for almost all points on C.

Tangents of Curve-Like Sets

Proposition

A regular 1-set F in $\mathbb{R}^2$ has a tangent at almost all of its points.

- By definition of regularity, $\overline{D}^1(F,x) = 1 > 0$ at almost all $x \in F$.

  If C is any rectifiable curve, then, by the preceding proposition, for almost all x in C, there exists $\boldsymbol{\theta}$, such that, if $\varphi > 0$,

$$\lim_{r\to 0} \frac{\mathcal{H}^1((F\cap C)\cap(B(x,r)\setminus S(x,\boldsymbol{\theta},\varphi)))}{r} \leq \lim_{r\to 0} \frac{\mathcal{H}^1(C\cap(B(x,r)\setminus S(x,\boldsymbol{\theta},\varphi)))}{r} = 0,$$

Tangents of Curve-Like Sets (Cont'd)

- Moreover, by a previous property, for almost all $x \in C$,

$$\lim_{r\to 0} \frac{\mathcal{H}^1((F\setminus C)\cap(B(x,r)\setminus S(x,\boldsymbol{\theta},\varphi)))}{r} \leq \lim_{r\to 0} \frac{\mathcal{H}^1((F\setminus C)\cap B(x,r))}{r} = 0.$$

Adding these inequalities, we get for almost all $x \in C$,

$$\lim_{r\to 0} \frac{\mathcal{H}^1(F\cap(B(x,r)\setminus S(x,\boldsymbol{\theta},\varphi)))}{r} = 0.$$

A fortiori, this holds also for almost all $x \in F \cap C$.

But a countable collection of such curves covers almost all of F.

We conclude that a regular 1-set F in $\mathbb{R}^2$ has a tangent at almost all of its points.

Irregular Sets and Tangents

Proposition

At almost all points of an irregular 1-set, no tangent exists.

- The proof is difficult and it is omitted.
- It depends on the characterization of irregular sets as curve-free sets.

s-Sets for $0 < s < 1$

- We saw that s-sets in $\mathbb{R}^2$ for non-integral s are necessarily irregular.
- For $0 < s < 1$ tangency questions are not particularly interesting.
  Suppose a set is contained in a smooth curve.
- Then, it will automatically satisfy the second tangent condition with $\boldsymbol{\theta}$ the direction of the tangent to the curve at x.

Example

- Consider the middle third Cantor set F regarded as a subset of the plane.

  It is a $\frac{\log 2}{\log 3}$-set.

  Moreover, it satisfies both conditions, for all x in F and $\varphi > 0$, where $\boldsymbol{\theta}$ is a vector pointing along the set.
- Consider a Cartesian product F of two uniform Cantor sets, each formed by repeated removal of a proportion $\alpha > \frac{1}{2}$ from the center of intervals.

  A little calculation shows that F is an s-set with $s = \frac{2\log 2}{\log \frac{2}{1-\alpha}} < 1$.

  This set has no tangents at any of its points.

s-Sets for $1 < s < 2$

- It is plausible that s-sets in $\mathbb{R}^2$, with $1 < s < 2$, do not have tangents.
  - Such sets are so large that they radiate in many directions from a typical point;
  - So the second tangent condition cannot hold.

Proposition

If F is an s-set in $\mathbb{R}^2$, with $1 < s < 2$, then at almost all points of F, no tangent exists.

- For $r_0 > 0$, let

$$E = \{y \in F : \mathcal{H}^s(F \cap B(y, r)) < 2(2r)^s \text{ for all } r < r_0\}.$$

For each $x \in E$, each unit vector $\boldsymbol{\theta}$ and each angle φ, with $0 < \varphi < \frac{1}{2}\pi$, we estimate how much of E lies in $B(x, r) \cap S(x, \boldsymbol{\theta}, \varphi)$.

s-Sets for $1 < s < 2$ (Cont'd)

- For $r < \frac{r_0}{20}$ and $i = 1, 2, \ldots$, let A_i be the intersection of the annulus and the double sector given by

$$A_i = (B(x, ir\varphi) \backslash B(x, (i-1)r\varphi)) \cap S(x, \boldsymbol{\theta}, \varphi).$$

Then $B(x, r) \cap S(x, \boldsymbol{\theta}, \varphi) \subseteq \bigcup_{i=1}^{m} A_i \cup \{x\}$, for some integer $m < \frac{2}{\varphi}$. Each A_i comprises two parts, both of diameter at most $10r\varphi < r_0$. Applying the definition of E to the parts that contain points of E, and summing,

$$\mathcal{H}^s(E \cap B(x, r) \cap S(x, \boldsymbol{\theta}, \varphi)) \leq 2m2(20r\varphi)^s \leq (4\varphi^{-1})2(20r\varphi)^s.$$

So, if $r < \frac{r_0}{20}$,

$$(2r)^{-s}\mathcal{H}^s(E \cap B(x, r) \cap S(x, \boldsymbol{\theta}, \varphi)) \leq 8 \cdot 10^s \varphi^{s-1}.$$

s-Sets for $1 < s < 2$ (Cont'd)

- By a previous proposition, for almost all $x \in E$, $\overline{D}^s(F \backslash E, x) = 0$.
  Decomposing $F \cap B(x, r)$ into three parts we get,

$$\begin{array}{rcl} \mathcal{H}^s(F \cap B(x,r)) & = & \mathcal{H}^s((F \backslash E) \cap B(x,r)) \\ & & + \mathcal{H}^s(E \cap B(x,r) \cap S(x, \boldsymbol{\theta}, \varphi)) \\ & & + \mathcal{H}^s(E \cap (B(x,r) \backslash S(x, \boldsymbol{\theta}, \varphi))). \end{array}$$

  Divide by $(2r)^s$ and take upper limits as $r \to 0$. For almost all $x \in E$,

$$\overline{D}^s(F, x) \leq 0 + 8 \cdot 10^s \varphi^{s-1} + \overline{\lim_{r \to 0}} (2r)^{-s} \mathcal{H}^s(F \cap (B(x,r) \backslash S(x, \boldsymbol{\theta}, \varphi))).$$

  Now choose φ sufficiently small.

  It follows that, for all $\boldsymbol{\theta}$, the tangent conditions cannot both hold.

  So no tangent exists at such x.

  Finally, note that, by a previous proposition, almost all $x \in F$ belong to the set E for some $r_0 > 0$.