Introduction to Fractal Geometry

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LSSU Math 500

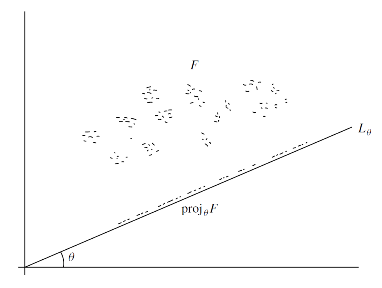
1. Projections, Products and Intersections
   - Projections of Arbitrary Sets
   - Projections of s-Sets of Integral Dimension
   - Projections of Arbitrary Sets of Integral Dimension
   - Product Formulae
   - Intersection of Fractals

Subsection 1

Projections of Arbitrary Sets

Projection of a Set Onto a Line

- Let L_θ be the line through the origin of $\mathbb{R}^2$ that makes an angle θ with the horizontal axis.



- We denote orthogonal projection onto L_θ by

$$\text{proj}_\theta .$$

- If F is a subset of $\mathbb{R}^2$, then $\text{proj}_\theta F$ is the projection of F onto L_θ.

Projection of a Set Onto a Line

- We have, for all $x, y \in \mathbb{R}^2$,

$$|\text{proj}_\theta x - \text{proj}_\theta y| \leq |x - y|.$$

- I.e. proj_θ is a Lipschitz mapping.
- Thus, by a previous corollary, for any F and θ,

$$\dim_H(\text{proj}_\theta F) \leq \min\{\dim_H F, 1\}$$

(as $\text{proj}_\theta F \subseteq L_\theta$, its dimension cannot be more than 1).
- The interesting question is whether the opposite inequality is valid.

The Projection Theorem

Projection Theorem

Let $F \subseteq \mathbb{R}^2$ be a Borel set.

(a) If $\dim_H F \leq 1$, then $\dim_H(\text{proj}_\theta F) = \dim_H F$ for almost all $\theta \in [0, \pi)$.

(b) If $\dim_H F > 1$, then $\text{proj}_\theta F$ has positive length (as a subset of L_θ) and so has dimension 1, for almost all $\theta \in [0, \pi)$.

- We give a proof that uses the potential theoretic characterization of Hausdorff dimension in a very effective way.

  Suppose $s < \dim_H F \leq 1$.

  By a previous theorem, there exists a mass distribution μ on (a compact subset of) F with:
  - $0 < \mu(F) < \infty$;
  - $\int_F \int_F \frac{d\mu(x)d\mu(y)}{|x-y|^s} < \infty$.

  For each θ, we "project" the mass distribution μ onto the line L_θ.

  We get a mass distribution μ_θ on $\text{proj}_\theta F$.

The Projection Theorem (Cont'd)

- Suppose:
  - $\boldsymbol{\theta}$ is the unit vector in the direction θ;
  - x is identified with its position vector;
  - $x \cdot \boldsymbol{\theta}$ is the usual scalar product.

  Then μ_θ is defined by the requirement that, for each interval $[a, b]$,

$$\mu_\theta([a, b]) = \mu\{x : a \le x \cdot \boldsymbol{\theta} \le b\}.$$

  Equivalently, for each non-negative function f,

$$\int_{-\infty}^{\infty} f(t) d\mu_\theta(t) = \int_F f(x \cdot \boldsymbol{\theta}) d\mu(x).$$

The Projection Theorem (Cont'd)

- Now we have

$$\begin{aligned}\int_0^\pi \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\mu_\theta(u)d\mu_\theta(v)}{|u-v|^s}\right]d\theta &= \int_0^\pi \left[\int_F\int_F \frac{d\mu(x)d\mu(y)}{|x\cdot\boldsymbol{\theta}-y\cdot\boldsymbol{\theta}|^s}\right]d\theta \\ &= \int_0^\pi \left[\int_F\int_F \frac{d\mu(x)d\mu(y)}{|(x-y)\cdot\boldsymbol{\theta}|^s}\right]d\theta \\ &= \int_0^\pi \frac{d\theta}{|\boldsymbol{\tau}\cdot\boldsymbol{\theta}|^s}\int_F\int_F \frac{d\mu(x)d\mu(y)}{|x-y|^s},\end{aligned}$$

for any fixed unit vector $\boldsymbol{\tau}$ (the integral of $|(x-y)\cdot\boldsymbol{\theta}|^{-s}$ with respect to θ depends only on $|x-y|$).

The Projection Theorem (Conclusion)

- By hypothesis, the second factor is finite.
  Moreover, if $s < 1$,
  $$\int_0^\pi \frac{d\theta}{|\boldsymbol{\tau} \cdot \boldsymbol{\theta}|^s} = \int_0^\pi \frac{d\theta}{|\cos(\tau - \theta)|^s} < \infty.$$
  So the integral is finite.
  Hence, for almost all $\theta \in [0, \pi)$,
  $$\int_F \int_F \frac{d\mu_\theta(u) d\mu_\theta(v)}{|u - v|^s} < \infty.$$
  By a previous theorem, the existence of such a mass distribution μ_θ on $\text{proj}_\theta F$ implies that $\dim_H(\text{proj}_\theta F) \geq s$.
  This is true for all $s < \dim_H F$.
  So part (a) of the result follows.
  The proof of (b) follows similar lines, though Fourier transforms need to be introduced to show that the projections have positive length.

Higher-Dimensional Projections

- Let $G_{n,k}$ be the set of k-dimensional subspaces or "k-planes through the origin" in $\mathbb{R}^n$.
- These subspaces are naturally parametrized by $k(n-k)$ coordinates ("generalized direction cosines").
- So we may refer to "almost all" subspaces in a consistent way in terms of $k(n-k)$-dimensional Lebesgue measure.
- We write proj_Π for orthogonal projection onto the k-plane Π.

Higher-Dimensional Projection Theorem

Theorem (Higher-Dimensional Projection Theorem)

Let $F \subseteq \mathbb{R}^n$ be a Borel set.

(a) If $\dim_H F \leq k$, then $\dim_H(\text{proj}_\Pi F) = \dim_H F$, for almost all $\Pi \in G_{n,k}$,

(b) If $\dim_H F > k$, then $\text{proj}_\Pi F$ has positive k-dimensional measure and so has dimension k, for almost all $\Pi \in G_{n,k}$.

- The proof of the preceding theorem extends to higher dimensions without difficulty.

Practical Applications

- If F is a subset of $\mathbb{R}^3$, the plane projections of F are, in general, of dimension $\min\{2, \dim_H F\}$.
- In practice we can estimate the dimension of an object in space by estimating the dimension of a photograph taken from a random direction.

  Provided this is less than 2, it may be assumed to equal the dimension of the object.
- Such a reduction can make dimension estimates of spatial objects tractable, since box-counting methods are difficult to apply in 3 dimensions but can be applied with reasonable success in the plane.

Subsection 2

Projections of s-Sets of Integral Dimension

Introduction

- If a subset F of $\mathbb{R}^2$ has Hausdorff dimension exactly 1, then we saw that the projections of F onto almost every L_θ have dimension 1.
- However, in this critical case, no information is given as to whether these projections have zero or positive length.
- In the special case where F is a 1-set, i.e., with $0 < \mathcal{H}^1(F) < \infty$, an analysis is possible.
- Recall from a previous theorem that a 1-set may be decomposed into a regular curve-like part and an irregular dust-like part.

Regular 1-Sets

Theorem

Let F be a regular 1-set in $\mathbb{R}^2$. Then $\text{proj}_\theta F$ has positive length except for at most one $\theta \in [0, \pi)$.

Sketch of Proof: By a previous theorem, it is enough to prove the result if F is a subset of positive length of a rectifiable curve C.

By the Lebesgue Density Theorem, we may approximate such an F by short continuous subcurves of C.

So essentially all we need to consider is the case when F is itself a rectifiable curve C_1 joining distinct points x and y.

Clearly, the projection onto L_θ of such a curve is an interval of positive length, except possibly for the one value of θ for which L_θ is perpendicular to the straight line through x and y.

Regular 1-Sets Comments

- Suppose F is a regular 1-set in $\mathbb{R}^2$.
- In general, $\text{proj}_\theta F$ will have positive length for all θ
- There is an exceptional value of θ only if F is contained in a set of parallel line segments.

Irregular 1-Sets

Theorem

Let F be an irregular 1-set in $\mathbb{R}^2$. Then $\text{proj}_\theta F$ has length zero for almost all $\theta \in [0, \pi)$.

- The proof is complicated, depending on the intricate density and angular density structure of irregular sets.

Corollary

Let F be a 1-set in $\mathbb{R}^2$. If the regular part of F has $\mathcal{H}^1$-measure zero, then $\text{proj}_\theta F$ has length zero for almost all θ. Otherwise, it has positive length for all but at most one value of θ.

Corollary

A 1-set in $\mathbb{R}^2$ is irregular if and only if it has projections of zero length in at least two directions.

Example: Cantor Dust

Claim: The Cantor dust F is an irregular 1-set.

In a preceding example we showed that F is a 1-set.

It is easy to see that the projections of F onto lines L_θ with $\tan\theta = \frac{1}{2}$ and $\tan\theta = -2$ have zero length (look at the first few iterations).

So F is irregular by the preceding corollary.

Widening the Application of the Theorems

- Suppose F is a set that intersects some rectifiable curve in a set of positive length.

  Then F contains a regular subset.

  It follows that $\text{proj}_\theta F$ has positive length for almost all θ.

- Suppose F is a σ-finite irregular set.

  By definition, it may be expressed as a countable union of irregular 1-sets each of finite measure.

  Then $\text{proj}_\theta F$ has zero length for almost all θ.

  This follows by taking countable unions of the projections of these component 1-sets.

Higher-Dimensional Analogs

- We state the higher-dimensional analog of the preceding theorems.
- The proofs are even more complicated than in the plane case.

Theorem

Let F be a k-set in $\mathbb{R}^n$, where k is an integer.

(a) If F is regular then $\text{proj}_\theta F$ has positive k-dimensional measure for almost all $\Pi \in G_{n,k}$.

(b) If F is irregular then $\text{proj}_\theta F$ has zero k-dimensional measure for almost all $\Pi \in G_{n,k}$.

Subsection 3

Projections of Arbitrary Sets of Integral Dimension

Introduction

- The theorems of the last subsection do not provide a complete answer to the question of whether projections of plane sets onto lines have zero or positive length.
  - A subset F of $\mathbb{R}^2$ of Hausdorff dimension 1 need not be a 1-set or even be of σ-finite $\mathcal{H}^1$-measure, i.e., a countable union of sets of finite $\mathcal{H}^1$-measure.
  - Moreover there need not be any dimension function h, for which $0 < \mathcal{H}^h(F) < \infty$, in which case mathematical analysis is extremely difficult.

Introduction (Cont'd)

- We consider sets of Hausdorff dimension 1 but of non-σ-finite $\mathcal{H}^1$-measure.
- We can construct sets with projections more or less what we please.
- E.g., there is a set F in $\mathbb{R}^2$, such that:
    - $\text{proj}_\theta F$ contains an interval of length 1 for almost all θ with $0 \leq \theta < \frac{1}{2}\pi$;
    - $\text{proj}_\theta F$ is of length zero for $\frac{1}{2}\pi \leq \theta < \pi$.

Existence of Sets with Prescribed Projections

Theorem

Let G_θ be a subset of L_θ for each $\theta \in [0, \pi)$ [such that the set $\bigcup_{0 \leq \theta < \pi} G_\theta$ is plane Lebesgue measurable]. Then there exists a Borel set $F \subseteq \mathbb{R}^2$, such that:

(a) $\text{proj}_\theta F \supseteq G_\theta$, for all θ;

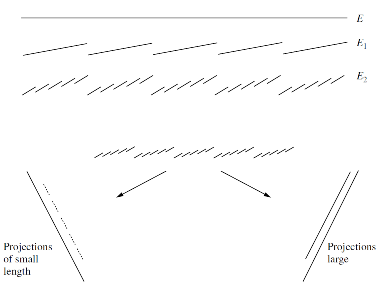
(b) $\text{length}(\text{proj}_\theta F \setminus G_\theta) = 0$, for almost all θ.

In particular, for almost all θ, the set of points of L_θ belonging to either G_θ or $\text{proj}_\theta F$, but not both, has zero length.

Existence of Sets with Prescribed Projections (Cont'd)

- We discuss only the idea behind the proof.

  The basic building block for such sets has been termed the "iterated Venetian blind" construction.

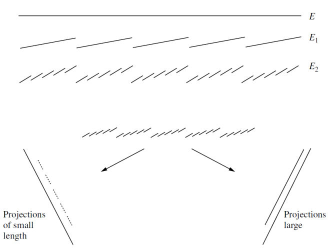


  Let E be a line segment of length λ.

  Let ε be a small angle.

  Let k be a large number.

Existence of Sets with Prescribed Projections (Cont'd)



- Replace E by k line segments of lengths roughly $\frac{\lambda}{k}$, each at an angle ε to E and with endpoints equally spaced along E to form E_1.
- Repeat this process with each segment of E_1 to form a set E_2, with:
  - k^2 line segments all of lengths about $\frac{\lambda}{k^2}$;
  - All of them at angle ε to E.
- Continuing, E_r consists of k^r segments all of lengths about $\frac{\lambda}{k^r}$ and at angle $r\varepsilon$ to E.
- We stop when r is such that $r\varepsilon$ is, say, about $\frac{1}{4}\pi$.

Existence of Sets with Prescribed Projections (Cont'd)

- Comparing the projections of E_r with that of E, we see that:
  - If $0 \leq \theta < \frac{1}{2}\pi$, then $\text{proj}_\theta E$ and $\text{proj}_\theta E_r$ are nearly the same (lines perpendicular to L_θ that cut E also cut E_r).
  - If $-\frac{1}{4}\pi < \theta < 0$, then $\text{proj}_\theta E_r$ will have very small length (most lines perpendicular to L_θ will pass straight between appropriately angled "slats" of the construction).

  Thus the projections of E_r are very similar to those of E in certain directions, but are almost negligible in other directions.

  This idea may be adapted to obtain sets with projections:
  - Very close to G_θ in a narrow band of directions;
  - Almost null in other directions.

  Taking unions of such sets for various small bands of directions gives a set with approximately the required property.

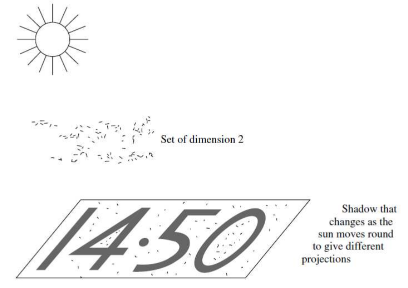
  Taking a limit of a sequence of sets which give increasingly accurate approximations leads to a set with the properties stated.

Higher Dimensions

- This construction may be extended to higher dimensions.
- There exists a set F in $\mathbb{R}^n$, such that almost all projections of F onto k-dimensional subspaces differ from prescribed sets by zero k-dimensional measure.
- In particular, there exists a set in 3-dimensional space with almost all of its plane shadows anything we care to prescribe to within zero area.

Digital Sundial

- By specifying the shadows to be the thickened digits of the time when the sun is shining from a perpendicular direction, we obtain, at least in theory, a digital sundial.



- As the sun moves across the sky we get different projections of the 2-dimensional set.
- This notion was introduced to provide an intuitive view of the result.

Subsection 4

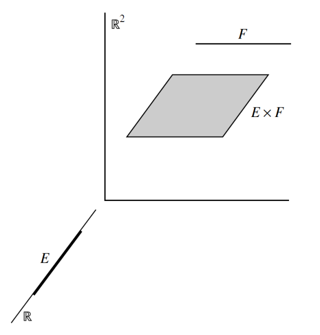
Product Formulae

Cartesian Products

- Let E be a subset of $\mathbb{R}^n$.
- Let F be a subset of $\mathbb{R}^m$.
- The **Cartesian product**, or just **product**, $E \times F$ is defined as the set of points with first coordinate in E and second coordinate in F.
- That is,
$$E \times F = \{(x, y) \in \mathbb{R}^{n+m} : x \in E, y \in F\}.$$

Example

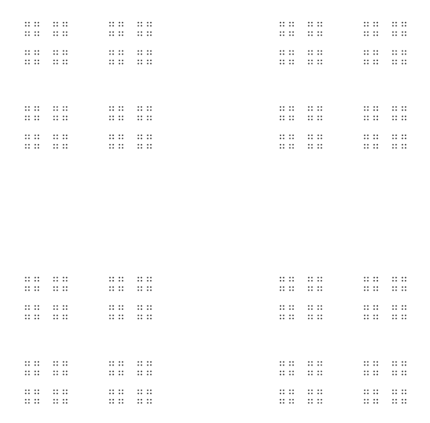
- Let E be a unit interval in $\mathbb{R}$.
- Let F be a unit interval in $\mathbb{R}^2$.



- Then $E \times F$ is a unit square in $\mathbb{R}^3$.

Example

- Let F be the middle third Cantor set.



- Then $F \times F$ is the "Cantor product", consisting of those points in the plane with both coordinates in F.

Dimension of a Product

- In the example involving the unit intervals above, it is obvious that

$$\dim(E \times F) = \dim E + \dim F,$$

using the classical definition of dimension.

- This holds more generally, in the "smooth" situation, where E and F are smooth curves, surfaces or higher-dimensional manifolds.
- Unfortunately, this is not always valid for "fractal" dimensions.
- For Hausdorff dimensions the best general result possible is an inequality

$$\dim_H(E \times F) \geq \dim_H E + \dim_H F.$$

- Nevertheless, in many situations equality does hold.

Dimension and Hausdorff Measure

Proposition

If $E \subseteq \mathbb{R}^n$, $F \subseteq \mathbb{R}^m$ are Borel sets with $\mathcal{H}^s(E), \mathcal{H}^t(F) < \infty$, then

$$\mathcal{H}^{s+t}(E \times F) \geq c\mathcal{H}^s(E)\mathcal{H}^t(F),$$

where $c > 0$ depends only on s and t.

- For simplicity we assume that $E, F \subseteq \mathbb{R}$, so that $E \times F \subseteq \mathbb{R}^2$.

  The general proof is almost identical.

  If either $\mathcal{H}^s(E)$ or $\mathcal{H}^t(F)$ is zero, then the result is trivial.

  Let $0 < \mathcal{H}^s(E), \mathcal{H}^t(F) < \infty$, i.e., E is an s-set and F is a t-set.

  We may define a mass distribution μ on $E \times F$ by utilizing the "product measure" of $\mathcal{H}^s$ and $\mathcal{H}^t$.

Dimension and Hausdorff Measure (Cont'd)

- If $I, J \subseteq \mathbb{R}$, we define μ on the "rectangle" $I \times J$ by

$$\mu(I \times J) = \mathcal{H}^s(E \cap I)\mathcal{H}^t(F \cap J).$$

It may be shown that this defines a mass distribution μ on $E \times F$ with

$$\mu(\mathbb{R}^2) = \mathcal{H}^s(E)\mathcal{H}^t(F).$$

Here, we are concerned with subsets of $\mathbb{R}$.

So the "ball" $B(x, r)$ is just the interval of length $2r$ with midpoint x.

By the density estimate proposition, we have:

  - For $\mathcal{H}^s$-almost all $x \in E$, $\overline{\lim}_{r\to 0} \frac{\mathcal{H}^s(E\cap B(x,r))}{(2r)^s} \leq 1$;
  - For $\mathcal{H}^t$-almost all $y \in F$, $\overline{\lim}_{r\to 0} \frac{\mathcal{H}^t(F\cap B(y,r))}{(2r)^t} \leq 1$.

From the definition of μ, both inequalities hold for μ-almost all (x, y) in $E \times F$.

Dimension and Hausdorff Measure (Cont'd)

- The disc $B((x,y),r)$ is contained in the square $B(x,r)\times B(y,r)$. We have that

$$\begin{array}{rcl}\mu(B((x,y),r)) & \leq & \mu(B(x,r)\times B(y,r)) \\ & = & \mathcal{H}^s(E\cap B(x,r))\mathcal{H}^t(F\cap B(y,r)).\end{array}$$

So

$$\frac{\mu(B((x,y),r))}{(2r)^{s+t}}\leq\frac{\mathcal{H}^s(E\cap B(x,r))}{(2r)^s}\frac{\mathcal{H}^t(F\cap B(y,r))}{(2r)^t}.$$

It follows, using the inequalities above, that, for μ-almost all $(x,y)\in E\times F$,

$$\overline{\lim}_{r\to 0}\frac{\mu(B((x,y),r))}{(2r)^{s+t}}\leq 1.$$

By a previous proposition,

$$\mathcal{H}^s(E\times F)\geq 2^{-(s+t)}\mu(E\times F)=2^{-(s+t)}\mathcal{H}^s(E)\mathcal{H}^t(F).$$

Product Formula

Product Formula

If $E \subseteq \mathbb{R}^n$, $F \subseteq \mathbb{R}^m$ are Borel sets then

$$\dim_H(E \times F) \geq \dim_H E + \dim_H F.$$

- Let s, t be any numbers with $s < \dim_H E$ and $t < \dim_H F$.

  Then $\mathcal{H}^s(E) = \mathcal{H}^t(F) = \infty$.

  By a previous theorem, there are Borel sets $E_0 \subseteq E$ and $F_0 \subseteq F$, with $0 < \mathcal{H}^s(E_0), \mathcal{H}^t(F_0) < \infty$.

  By the preceding proposition,

$$\mathcal{H}^{s+t}(E \times F) \geq \mathcal{H}^{s+t}(E_0 \times F_0) \geq c\mathcal{H}^s(E_0)\mathcal{H}^t(F_0) > 0.$$

  Hence, $\dim_H(E \times F) \geq s + t$. By choosing s and t arbitrarily close to $\dim_H E$ and $\dim_H F$, we get the conclusion.

A Partial Reverse

- In general, the product inequality cannot be reversed.
- If either E or F is "reasonably regular", in the sense of having equal Hausdorff and upper box dimensions, then we do get equality.

Product Formula

For any sets $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$,

$$\dim_H(E \times F) \leq \dim_H E + \overline{\dim}_B F.$$

- For simplicity take $E \subseteq \mathbb{R}$ and $F \subseteq \mathbb{R}$.

  Choose numbers $s > \dim_H E$ and $t > \dim_B F$.

  Then there is a number $\delta_0 > 0$, such that F may be covered by $N_\delta(F) \leq \delta^{-t}$ intervals of length δ, for all $\delta \leq \delta_0$.

  Let $\{U_i\}$ be any δ-cover of E by intervals with $\sum_i |U_i|^s < 1$.

A Partial Reverse (Cont'd)

- For each i, let $U_{i,j}$ be a cover of F by $N_{|U_i|}(F)$ intervals of length $|U_i|$. Then $U_i \times F$ is covered by $N_{|U_i|}(F)$ squares $\{U_i \times U_{i,j}\}$ of side $|U_i|$. It follows that
$$E \times F \subseteq \bigcup_i \bigcup_j (U_i \times U_{i,j}).$$
Now we have
$$\begin{array}{rcl} \mathcal{H}^{s+t}_{\delta\sqrt{2}}(E \times F) & \leq & \sum_i \sum_j |U_i \times U_{i,j}|^{s+t} \\ & \leq & \sum_i N_{|U_i|}(F) 2^{(s+t)/2} |U_i|^{s+t} \\ & \leq & 2^{(s+t)/2} \sum_i |U_i|^{-t} |U_i|^{s+t} \\ & < & 2^{(s+t)/2}. \end{array}$$
Letting $\delta \to 0$ gives, for all $s > \dim_H E$ and $t > \overline{\dim}_B F$,
$$\mathcal{H}^{s+t}(E \times F) < \infty.$$
So $\dim_H(E \times F) \leq s + t$.

A Case when Equality is Attained

Corollary

If $\dim_H F = \overline{\dim}_B F$, then

$$\dim_H(E \times F) = \dim_H E + \dim_H F.$$

- Combining the preceding product formulas gives

$$\dim_H E + \dim_H F \leq \dim_H(E \times F) \leq \dim_H E + \overline{\dim}_B F.$$

Product Formula for Box Dimension

- It is worth noting that the basic product inequality for upper box dimensions is opposite to that for Hausdorff dimensions.

Product Formula for Box Dimension

For any sets $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$,

$$\overline{\dim}_B(E \times F) \leq \overline{\dim}_B E + \overline{\dim}_B F.$$

- The idea is as in the first inequality above.

  Suppose that:
  - E can be covered by $N_\delta(E)$ intervals of side δ;
  - F can be covered by $N_\delta(F)$ intervals of side δ.

  Then $E \times F$ is covered by the $N_\delta(E)N_\delta(F)$ squares formed by products of these intervals.

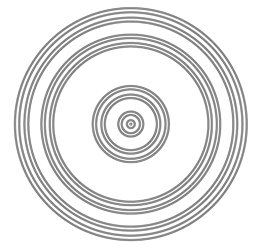
Examples

- Let E, F be subsets of $\mathbb{R}$ with F a uniform Cantor set.

  Then
  $$\dim_H(E \times F) = \dim_H E + \dim_H F.$$
  A previous example showed that uniform Cantor sets have equal Hausdorff and upper box dimensions.

  So the result follows from the preceding corollary.
- The "Cantor product" of the middle third Cantor set with itself has Hausdorff and box dimensions exactly $2\frac{\log 2}{\log 3}$.
- If E is a subset of $\mathbb{R}$ and F is a straight line segment, then
  $$\dim_H(E \times F) = \dim_H E + 1.$$

Example: The Cantor Target



- The "Cantor target" is the plane set given in polar coordinates by

$$F' = \{(r,\theta) : r \in F, 0 \leq \theta \leq 2\pi\},$$

where F is the middle third Cantor set.
Then $\dim_H F' = 1 + \frac{\log 2}{\log 3}$.
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (x\cos y, x\sin y)$.
Then f is a Lipschitz mapping and $F' = f(F \times [0,2\pi])$.

$$\begin{aligned} \dim_H F' &= \dim_H f(F \times [0,2\pi]) \\ &\leq \dim_H (F \times [0,2\pi]) \\ &= \dim_H F + \dim_H [0,2\pi] \\ &= \frac{\log 2}{\log 3} + 1, \end{aligned}$$

by a previous corollary and the preceding example.

Example: The Cantor Target (Cont'd)

- Suppose we restrict f to $[\frac{2}{3}, 1] \times [0, \pi]$,

$$f : \left[\frac{2}{3}, 1\right] \times [0, \pi] \to \mathbb{R}^2; \quad f(x, y) = (x \cos y, x \sin y).$$

Then f is a bi-Lipschitz function on this domain.
But $F' \supseteq f((F \cap [\frac{2}{3}, 1]) \times [0, \pi])$. So we have

$$\begin{aligned} \dim_H F' &\geq \dim_H f((F \cap [\tfrac{2}{3}, 1]) \times [0, \pi]) \\ &= \dim_H ((F \cap [\tfrac{2}{3}, 1]) \times [0, \pi]) \\ &= \dim_H (F \cap [\tfrac{2}{3}, 1]) + \dim_H [0, \pi] \\ &= \tfrac{\log 2}{\log 3} + 1, \end{aligned}$$

by a previous corollary and the preceding example.
It can be similarly shown that F' is an s-set for $s = 1 + \frac{\log 2}{\log 3}$.

Equality Does Not Hold in General

Claim: There exist sets $E, F \subseteq \mathbb{R}$ with $\dim_H E = \dim_H F = 0$ and $\dim_H(E \times F) \geq 1$.

Let $0 = m_0 < m_1 < \cdots$ be a rapidly increasing sequence of integers satisfying a condition to be specified below.

- Let E consist of those numbers in $[0, 1]$, with a zero in the r-th decimal place whenever $m_k + 1 \leq r \leq m_{k+1}$ and k is even.
- Let F consist of those numbers in $[0, 1]$, with zero in the r-th decimal place if $m_k + 1 \leq r \leq m_{k+1}$ and k is odd.

Look at the first m_{k+1} decimal places for even k.

There is an obvious cover of E by 10^{j_k} intervals of length $10^{-m_{k+1}}$, where $j_k = (m_2 - m_1) + (m_4 - m_3) + \cdots + (m_k - m_{k-1})$.

Then we have $\frac{\log 10^{j_k}}{-\log 10^{-m_{k+1}}} = \frac{j_k}{m_{k+1}}$.

Provided that the m_k increase sufficiently rapidly, $\frac{j_k}{m_{k+1}} \stackrel{k\to\infty}{\longrightarrow} 0$.

By a previous proposition, $\dim_H E \leq \underline{\dim}_B E = 0$.

Equality Does Not Hold in General (Cont'd)

- We showed $\dim_H E \leq \underline{\dim}_B E = 0$.

  Similarly, $\dim_H F = 0$.

  If $0 < w < 1$, then we can write

  $$w = x + y,$$

  where $x \in E$ and $y \in F$.

  Just take the r-th decimal digit of w:
  - From E, if $m_k + 1 \leq r \leq m_{k+1}$ and k is odd;
  - From F, if $m_k + 1 \leq r \leq m_{k+1}$ and k is even.

  The mapping $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x + y$ is Lipschitz.

  So, by a previous corollary,

  $$\dim_H(E \times F) \geq \dim_H f(E \times F) \geq \dim_H(0, 1) = 1.$$

Intersection with a Vertical Line

- We work in the (x, y)-plane.
- Let L_x be the line parallel to the y-axis through the point $(x, 0)$.

Proposition

Let F be a Borel subset of $\mathbb{R}^2$. If $1 \leq s \leq 2$, then

$$\int_{-\infty}^{\infty} \mathcal{H}^{s-1}(F \cap L_x) dx \leq \mathcal{H}^s(F).$$

- Given $\varepsilon > 0$, let $\{U_i\}$ be a δ-cover of F, with $\sum_i |U_i|^s \leq \mathcal{H}^s_\delta(F) + \varepsilon$. Each U_i is contained in a square S_i of side $|U_i|$ with sides parallel to the coordinate axes. Let χ_i be the indicator function of S_i, i.e.,

$$\chi_i(x, y) = \begin{cases} 1, & \text{if } (x, y) \in S_i, \\ 0, & \text{if } (x, y) \notin S_i. \end{cases}$$

For each x, the sets $\{S_i \cap L_x\}$ form a δ-cover of $F \cap L_x$.

Intersection with a Vertical Line (Cont'd)

- For each x, the sets $\{S_i \cap L_x\}$ form a δ-cover of $F \cap L_x$. So we get

$$\begin{array}{rcl} \mathcal{H}_\delta^{s-1}(F \cap L_x) & \leq & \sum_i |S_i \cap L_x|^{s-1} \\ & = & \sum_i |U_i|^{s-2}|S_i \cap L_x| \\ & = & \sum_i |U_i|^{s-2} \int \chi_i(x,y)dy. \end{array}$$

Hence,

$$\begin{array}{rcl} \int \mathcal{H}_\delta^{s-1}(F \cap L_x)dx & \leq & \sum_i |U_i|^{s-2} \iint \chi_i(x,y)dxdy \\ & = & \sum_i |U_i|^s \\ & \leq & \mathcal{H}_\delta^s(F) + \varepsilon. \end{array}$$

Since $\varepsilon > 0$ is arbitrary,

$$\int \mathcal{H}_\delta^{s-1}(F \cap L_x)dx \leq \mathcal{H}_\delta^s(F).$$

Letting $\delta \to 0$ gives the conclusion.

Implication for Hausdorff Dimensions

Corollary

Let F be a Borel subset of $\mathbb{R}^2$. Then, for almost all x (in the sense of 1-dimensional Lebesgue measure),

$$\dim_H(F \cap L_x) \le \max\{0, \dim_H F - 1\}.$$

- Take $s > \dim_H F$, so that $\mathcal{H}^s(F) = 0$.
  If $s > 1$, the proposition gives $\mathcal{H}^{s-1}(F \cap L_x) = 0$.
  So $\dim_H(F \cap L_x) \le s - 1$, for almost all x.

A Generalization

Proposition

Let F be any subset of $\mathbb{R}^2$, and let E be any subset of the x-axis. Suppose that there is a constant c, such that $\mathcal{H}^t(F \cap L_x) \geq c$, for all $x \in E$. Then

$$\mathcal{H}^{s+t}(F) \geq bc\mathcal{H}^s(E),$$

where $b > 0$ depends only on s and t.

- We omit the proof.

Corollary

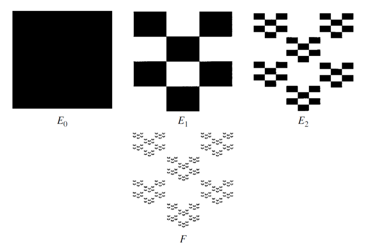
Let F be any subset of $\mathbb{R}^2$, and let E be a subset of the x-axis. If $\dim_H(F \cap L_x) \geq t$, for all $x \in E$, then

$$\dim_H F \geq t + \dim_H E.$$

- The obvious higher-dimensional analogs of these results are all valid.

Example: A Self-Affine Set

- Let F be the set with iterated construction indicated in the figure.



At the k-th stage each rectangle of E_k is replaced with an affine copy of the rectangles in E_1.

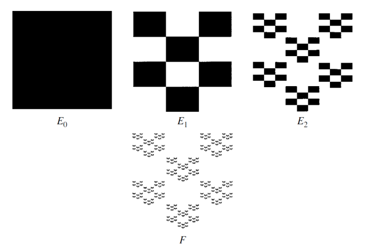
The contraction is greater in the "y" than in the "x" direction.

The width to height ratio of the rectangles in E_k tends to infinity.

In this case,

$$\dim_H F = \dim_B F = \frac{3}{2}.$$

Example: A Self-Affine Set (Cont'd)



- E_k consists of 6^k rectangles of size $3^{-k} \times 4^{-k}$.

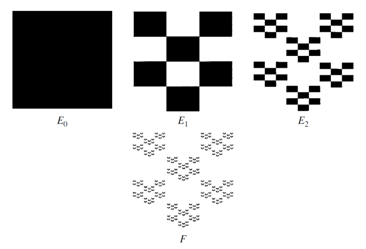
  Each of these rectangles may be covered by at most $(\frac{4}{3})^k + 1$ squares of side 4^{-k}, by dividing the rectangles using a series of vertical cuts.

  Hence E_k may be covered by:
  - $6^k \times 2 \times 4^k \times 3^{-k} = 2 \times 8^k$ squares;
  - Each of diameter $4^{-k}\sqrt{2}$.

  In the usual way, this gives $\dim_H F \leq \overline{\dim}_B F \leq \frac{3}{2}$.

Example: A Self-Affine Set (Cont'd)

- Except for x of the form $j3^{-k}$, where j and k are integers, we have that $E_k \cap L_x$ consists of:
  - 2^k intervals;
  - Each of length 4^{-k}.



A standard application of the mass distribution method shows that, for each such x,

$$\mathcal{H}^{1/2}(E_k \cap L_x) \geq \frac{1}{2}.$$

By a previous proposition,

$$\mathcal{H}^{3/2}(F) \geq \frac{1}{2}.$$

Hence $\dim_H F = \dim_B F = \frac{3}{2}$.

Subsection 5

Intersection of Fractals

Introducing Intersection

- The intersection of two fractals is often a fractal.
- In general, the dimension of the intersection is not related to that of the original sets.

  Example: Suppose F is bounded.

  There is a congruent copy F_1 of F, such that

$$\dim_H(F \cap F_1) = \dim_H F.$$

  We may take $F_1 = F$.

  There is another congruent copy with

$$\dim_H(F \cap F_1) = 0.$$

  We may take F and F_1 disjoint.

More on Intersection

- We can say more provided we consider the intersection of F and a congruent copy in a "typical" relative position.

  Example: Let F and F_1 be unit line segments in the plane.

  Then $F \cap F_1$ can be a line segment, but only in the exceptional situation when F and F_1 are collinear.

  If F and F_1 cross at an angle, then $F \cap F_1$ is a single point.

  Now $F \cap F_1$ remains a single point if F_1 is replaced by a nearby congruent copy.

  Thus, whilst "in general" $F \cap F_1$ contains at most one point, this situation occurs "frequently".

Measuring Sets of Transformations

- Recall that a *rigid motion* or *direct congruence* σ of the plane transforms any set E to a congruent copy $\sigma(E)$ without reflection.
- The rigid motions may be parametrized by three coordinates (x, y, θ):
  - The origin is transformed to (x, y);
  - θ is the angle of rotation.
- This provides a natural measure on the space of rigid motions.
- The measure of a set A of rigid motions is given by the 3-dimensional Lebesgue measure of the (x, y, θ) parametrizing the motions in A.

Examples

- Consider the set of all rigid motions which map the origin to a point of the rectangle $[1,2] \times [0,3]$.

  This set has measure $1 \times 3 \times 2\pi$.
- Let F be unit line segment.
  - Consider the set of transformations σ for which $F \cap \sigma(F)$ is a line segment.
    This has measure 0.
  - Consider the set of transformations σ for which $F \cap \sigma(F)$ is a single point.
    This is a set of transformations of measure 4.

Higher Dimensions

- In $\mathbb{R}^3$, "typically":
  - Two surfaces intersect in a curve;
  - A surface and a curve intersect in a point;
  - Two curves are disjoint.
- In $\mathbb{R}^n$, if smooth manifolds E and F intersect at all, then "in general" they intersect in a submanifold of dimension

$$\max\{0, \dim E + \dim F - n\}.$$

- Suppose $\dim E + \dim F - n > 0$.
  - For a set of rigid motions σ of positive measure,

$$\dim(E \cap \sigma(F)) = \dim E + \dim F - n;$$

  - For almost all other σ, $\dim(E \cap \sigma(F)) = 0$.
- Note that σ is measured using the $\frac{1}{2}n(n+1)$ parameters required to specify a rigid transformation of $\mathbb{R}^n$.

Goal of Investigation

- We would like to find out whether it is true that, as σ ranges over a group G of transformations, such as translations, congruences or similarities:
  - "In general", i.e., "for almost all σ",
  $$\dim_H(E \cap \sigma(F)) \leq \max\{0, \dim_H E + \dim_H F - n\};$$
  - "Often", i.e., "for a set of σ of positive measure",
  $$\dim_H(E \cap \sigma(F)) \geq \dim_H E + \dim_H F - n.$$
- The measurements are supposed to be with respect to a natural measure on the transformations in G.
  - Generally, G can be parametrized by m coordinates in a straightforward way for some integer m;
  - We can use Lebesgue measure on the parameter space $\mathbb{R}^m$.

Upper Bound for Translations

- Recall that

$$F + x = \{x + y : y \in F\}$$

denotes the translation of F by the vector x.

Theorem

If E, F are Borel subsets of $\mathbb{R}^n$, then

$$\dim_H(E \cap (F + x)) \leq \max\{0, \dim_H(E \times F) - n\},$$

for almost all $x \in \mathbb{R}^n$.

- We prove this when $n = 1$.

  The proof for $n > 1$ is similar.

  Denote by L_c be the line in the (x, y)-plane with equation

$$x = y + c.$$

Upper Bound for Translations (Cont'd)

- Suppose that $\dim_H(E \times F) > 1$.

  By a previous corollary (rotating the lines through 45° and changing notation slightly), for almost all $c \in \mathbb{R}$,

$$\dim_H((E \times F) \cap L_c) \leq \dim_H(E \times F) - 1.$$

  But a point $(x, x - c) \in (E \times F) \cap L_c$ if and only if $x \in E \cap (F + c)$.

  Thus, for each c, the projection onto the x-axis of $(E \times F) \cap L_c$ is the set $E \cap (F + c)$.

  In particular,

$$\dim_H(E \cap (F + c)) = \dim_H((E \times F) \cap L_c).$$

  So the result follows from the inequality above.

Lower Bounds

Theorem

Let $E, F \subseteq \mathbb{R}^n$ be Borel sets, and let G be a group of transformations on $\mathbb{R}^n$. Then $\dim_H(E \cap \sigma(F)) \geq \dim_H E + \dim_H F - n$ for a set of motions $\sigma \in G$ of positive measure in the following cases:

(a) G is the group of similarities and E and F are arbitrary sets;

(b) G is the group of rigid motions, E is arbitrary and F is a rectifiable curve, surface, or manifold;

(c) G is the group of rigid motions and E and F are arbitrary, with either $\dim_H E > \frac{1}{2}(n+1)$ or $\dim_H F > \frac{1}{2}(n+1)$.

- The proof, which uses potential theoretic methods, is omitted.

Example

- Let $F \subseteq \mathbb{R}$ be the middle third Cantor set.

  For $\lambda, x \in \mathbb{R}$, write

  $$\lambda F + x = \{\lambda y + x : y \in F\}.$$

  - For almost all $x \in \mathbb{R}$,

    $$\dim_H(F \cap (F + x)) \leq 2\frac{\log 2}{\log 3} - 1;$$

  - For a set of $(x, \lambda) \in \mathbb{R}^2$ of positive plane Lebesgue measure,

    $$\dim_H(F \cap (\lambda F + x)) = 2\frac{\log 2}{\log 3} - 1.$$

We showed in a previous example that $\dim_H(F \times F) = 2\frac{\log 2}{\log 3}$.

So the stated dimensions follow from the two preceding theorems.