Introduction to Fractal Geometry

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Subsection 1

Iterated Function Systems

Contractions and Contracting Similarities

- Let D be a closed subset of $\mathbb{R}^n$, often $D = \mathbb{R}^n$.
- A mapping $S : D \to D$ is called a **contraction** on D if there is a number c with $0 < c < 1$, such that

$$|S(x) - S(y)| \leq c|x - y|, \text{ for all } x, y \in D.$$

- Clearly any contraction is continuous.
- A contraction $S : D \to D$ is called a **contracting similarity** if equality holds, i.e., if

$$|S(x) - S(y)| = c|x - y|.$$

- Contracting similarities transform sets into geometrically similar sets.

Iterated Function Systems

- An **iterated function system** or **IFS** is a finite family of contractions $\{S_1, S_2, \ldots, S_m\}$, with $m \geq 2$.
- We call a non-empty compact subset F of D an **attractor** (or **invariant set**) for the IFS if

$$F = \bigcup_{i=1}^{m} S_i(F).$$

IFS's and Attractors

- The fundamental property of an iterated function system is that it determines a unique attractor, which is usually a fractal.
  Example: Take F to be the middle third Cantor set.
  Let $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ be given by
  $$S_1(x) = \frac{1}{3}x; \quad S_2(x) = \frac{1}{3}x + \frac{2}{3}.$$
  Then $S_1(F)$ and $S_2(F)$ are just the left and right "halves" of F.
  So $F = S_1(F) \cup S_2(F)$.
  Thus, F is an attractor of the IFS $\{S_1, S_2\}$.
  S_1 and S_2 represent the basic self-similarities of the Cantor set.
- We shall prove the fundamental property that an IFS has a unique (non-empty compact, i.e., closed and bounded) attractor.
- E.g., the middle third Cantor set is completely specified as the attractor of the mappings $\{S_1, S_2\}$ given above.

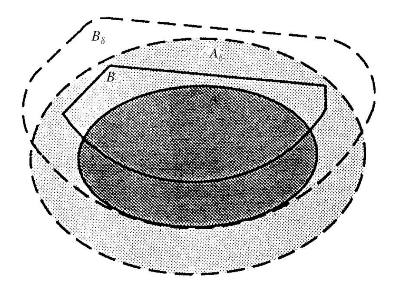
Distance Between Nonempty Compact Subsets

- Let $\mathcal{S}$ denote the class of all non-empty compact subsets of D.
- Recall that the δ-**neighborhood** of a set A is the set of points within distance δ of A, i.e.,

$$A_\delta = \{x \in D : |x - a| \leq \delta, \text{ for some } a \in A\}.$$

- We make $\mathcal{S}$ into a metric space by defining the distance between two sets A and B to be the least δ, such that the δ-neighborhood of A contains B and vice versa:

$$d(A, B) = \inf\{\delta : A \subseteq B_\delta \text{ and } B \subseteq A_\delta\}.$$



Properties of Distance

- A simple check shows that d is a metric or distance function, that is, satisfies the following three requirements:
  (i) $d(A, B) \geq 0$, with equality if and only if $A = B$;
  (ii) $d(A, B) = d(B, A)$;
  (iii) $d(A, B) \leq d(A, C) + d(C, B)$, for all $A, B, C \in \mathcal{S}$.
- The metric d is known as the **Hausdorff metric** on $\mathcal{S}$.
- In particular, if $d(A, B)$ is small, then A and B are close to each other as sets.

Existence of Unique Attractor

Theorem

Consider the iterated function system given by the contractions

$$\{S_1, \ldots, S_m\}$$

on $D \subseteq \mathbb{R}^n$, so that

$$|S_i(x) - S_i(y)| \leq c_i |x - y|, \quad (x, y) \in D,$$

with $c_i < 1$ for each i. Then there is a unique attractor F, i.e., a non-empty compact set such that

$$F = \bigcup_{i=1}^{m} S_i(F).$$

Existence of Unique Attractor (Cont'd)

Theorem (Cont'd)

Suppose we define a transformation S on the class $\mathcal{S}$ of non-empty compact sets by

$$S(E) = \bigcup_{i=1}^{m} S_i(E), \quad E \in \mathcal{S}.$$

Write S^k for the k-th iterate of S, i.e.,

$$\begin{aligned} S^0(E) &= E; \\ S^k(E) &= S(S^{k-1}(E)), \quad \text{for } k \geq 1. \end{aligned}$$

Then

$$F = \bigcap_{k=0}^{\infty} S^k(E),$$

for every set $E \in \mathcal{S}$, such that $S_i(E) \subseteq E$, for all i.

First Proof

- Note that sets in $\mathcal{S}$ are transformed by S into other sets of $\mathcal{S}$.
  If $A, B \in \mathcal{S}$, then, if the δ-neighborhood $(S_i(A))_\delta$ contains $S_i(B)$, for all i, then $(\bigcup_{i=1}^m S_i(A))_\delta$ contains $\bigcup_{i=1}^m S_i(B)$, and vice versa.
  So, applying the definition of the metric d, we get
  $$\begin{array}{rcl} d(S(A), S(B)) & = & d(\bigcup_{i=1}^m S_i(A), \bigcup_{i=1}^m S_i(B)) \\ & \leq & \max_{1 \leq i \leq m} d(S_i(A), S_i(B)). \end{array}$$
  By hypothesis,
  $$d(S(A), S(B)) \leq \left(\max_{1 \leq i \leq m} c_i \right) d(A, B).$$
  It may be shown that d is a complete metric on $\mathcal{S}$, that is every Cauchy sequence of sets in $\mathcal{S}$ is convergent to a set in $\mathcal{S}$.
  Since $0 < \max_{1 \leq i \leq m} c_i < 1$, the preceding inequality states that S is a contraction on the complete metric space $(\mathcal{S}, d)$.

First Proof (Cont'd)

- By Banach's Contraction Mapping Theorem, S has a unique fixed point, i.e., there is a unique set $F \in \mathcal{S}$, such that

$$S(F) = F.$$

This is the first statement in the conclusion.

Moreover $S^k(E) \to F$ as $k \to \infty$.

In particular, if $S_i(E) \subseteq E$, for all i, then $S(E) \subseteq E$.

So $S^k(E)$ is a decreasing sequence of non-empty compact sets containing F with intersection $\bigcap_{k=0}^{\infty} S^k(E)$ which must equal F.

Second Proof (Existence)

- Let E be any set in $\mathcal{S}$ such that $S_i(E) \subseteq E$, for all i.

  E.g., $E = D \cap B(0, r)$ will do, provided r is sufficiently large.

  Then
  $$S^k(E) \subseteq S^{k-1}(E).$$

  So $S^k(E)$ is a decreasing sequence of non-empty compact sets.

  They necessarily have non-empty compact intersection
  $$F = \bigcap_{k=1}^{\infty} S^k(E).$$

  But $S^k(E)$ is a decreasing sequence of sets.

  It follows that $S(F) = F$.

  So F satisfies the first conclusion and is an attractor of the IFS.

Second Proof (Uniqueness)

- For uniqueness, we derive, as in the first proof,

$$d(S(A), S(B)) \leq \left(\max_{1 \leq i \leq m} c_i\right) d(A, B).$$

  Suppose A and B are both attractors.

  Then $S(A) = A$ and $S(B) = B$.

  By the preceding inequality, $0 < \max_{1 \leq i \leq m} c_i < 1$.

  It follows that

$$d(A, B) = 0.$$

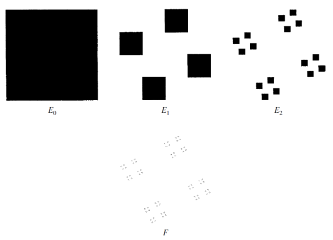
  This implies $A = B$.

Finding an IFS With a Given Attractor

- Finding an IFS that has a given F as its unique attractor can often be done by inspection, at least if F is self-similar or self-affine.

  Example: The Cantor dust is easily seen to be the attractor of the four similarities which give the basic self-similarities of the set:

$$\begin{aligned} S_1(x,y) &= (\tfrac{1}{4}x, \tfrac{1}{4}y + \tfrac{1}{2}); \\ S_2(x,y) &= (\tfrac{1}{4}x + \tfrac{1}{4}, \tfrac{1}{4}y); \\ S_3(x,y) &= (\tfrac{1}{4}x + \tfrac{1}{2}, \tfrac{1}{4}y + \tfrac{3}{4}); \\ S_4(x,y) &= (\tfrac{1}{4}x + \tfrac{3}{4}, \tfrac{1}{4}y + \tfrac{1}{4}). \end{aligned}$$



- In general it may not be possible to find an IFS with a given set as attractor.
- But we can normally find one with an attractor that is a close approximation to the required set.

Finding the Attractor of a Given IFS: Pre-fractals

- The transformation S introduced in the preceding theorem is the key to computing the attractor of an IFS.
- The sequence of iterates $S^k(E)$ converges to the attractor F for any initial set E in $\mathcal{S}$, in the sense that $d(S^k(E), F) \to 0$.

  We have
  $$d(S(A), S(B)) \leq \left(\max_{1\leq i\leq m} c_i\right) d(A, B).$$

  Let $c = \max_{1\leq i\leq m} c_i < 1$.

  Then $d(S(E), F) = d(S(E), S(F)) \leq cd(E, F)$.

  So $d(S^k(E), F) \leq c^k d(E, F)$.

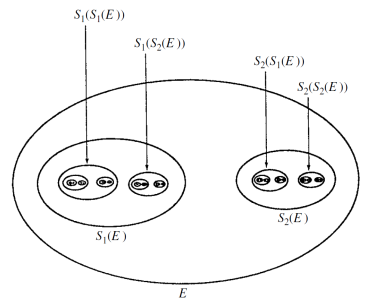
  Thus, the $S^k(E)$ provide increasingly good approximations to F.
- If F is a fractal, these approximations are sometimes called **pre-fractals** for F.

Finding the Attractor of a Given IFS (Cont'd)

- For each k,

$$S^k(E) = \bigcup_{\mathcal{I}_k} S_{i_1} \circ \cdots \circ S_{i_k}(E) = \bigcup_{\mathcal{I}_k} S_{i_1}(S_{i_2}(\cdots(S_{i_k}(E))\cdots)),$$

where the union is over the set $\mathcal{I}_k$ of all k-term sequences $(i_1, \ldots, i_k)$ with $1 \leq i_j \leq m$.



Finding the Attractor of a Given IFS (Cont'd)

- Suppose $S_i(E)$ is contained in E, for all i.

  Let x be a point of F.

  We know that $F = \bigcap_{k=0}^{\infty} S^k(E)$.

  Hence, there is a (not necessarily unique) sequence $(i_1, i_2, \ldots)$, such that, for all k,

$$x \in S_{i_1} \circ \cdots \circ S_{i_k}(E).$$

  This sequence provides a natural coding for x, with

$$x = x_{i_1,i_2,\ldots} = \bigcap_{k=1}^{\infty} S_{i_1} \circ \cdots \circ S_{i_k}(E).$$

  So $F = \bigcup \{x_{i_1,i_2,\ldots}\}$.

  This expression for $x_{i_1,i_2,\ldots}$ is independent of E provided that $S_i(E)$ is contained in E, for all i.

Finding the Attractor of a Given IFS (Cont'd)

- Suppose the union $F = \bigcup_{i=1}^{m} S_i(F)$ is disjoint.

  Then F must be totally disconnected (provided the S_i are injections).

  Indeed, suppose

  $$x_{i_1,i_2,\ldots} \neq x_{i'_1,i'_2,\ldots}.$$

  Then, we may find k such that

  $$(i_1,\ldots,i_k) \neq (i'_1,\ldots,i'_k).$$

  So the disjoint closed sets $S_{i_1} \circ \cdots \circ S_{i_k}(F)$ and $S_{i'_1} \circ \cdots \circ S_{i'_k}(F)$ disconnect the two points.

Example

- Consider again

$$S_1(x) = \frac{1}{3}x, \quad S_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Let F be the Cantor set.
Suppose $E = [0,1]$. In this case,

$$S^k(E) = E_k,$$

the set of 2^k basic intervals of length 3^{-k} obtained at the k-th stage of the usual Cantor set construction.
Moreover, $x_{i_1,i_2,\ldots}$ is the point with base-3 expansion $0.a_1a_2\ldots$, where

$$a_k = \begin{cases} 0, & \text{if } i_k = 1, \\ 2, & \text{if } i_k = 2. \end{cases}$$

The pre-fractals $S^k(E)$ provide the usual construction of many fractals for a suitably chosen initial set E.
The $S_{i_1} \circ \cdots \circ S_{i_k}(E)$ are called the **level-k sets** of the construction.

Drawing IFS Attractors: Method 1

- Take any initial set E (such as a square) and draw the k-th approximation $S^k(E)$ to F for a suitable value of k.
- The set $S^k(E)$ is made up of m^k small sets.
- Either these can be drawn in full, or a representative point of each can be plotted.
- In some cases, E can be chosen as a line segment in such a way that $S_1(E), \ldots, S_m(E)$ join up to form a polygonal curve with endpoints the same as those of E.

  Then the sequence of polygonal curves $S^k(E)$ provides increasingly good approximations to the fractal curve F.

Drawing IFS Attractors: Method 2

- Take x_0 as any initial point.
- Select a contraction S_{i_1} from $S_1, \ldots, S_m$ at random.
- Let $x_1 = S_{i_1}(x_0)$.
- Continue in this way:
  - Choose S_{i_k} from $S_1, \ldots, S_m$ at random (with equal probability, say);
  - Let $x_k = S_{i_k}(x_{k-1})$ for $k = 1, 2, \ldots$.
- For large enough k, the points x_k will be indistinguishably close to F, with x_k close to $S_{i_k} \circ \cdots \circ S_{i_1}(F)$.
- So the sequence $\{x_k\}$ will appear randomly distributed across F.
- A plot of the sequence $\{x_k\}$ from, say, the hundredth term onwards may give a good impression of F.

Subsection 2

Dimensions of Self-Similar Sets

Similarities and Self-Similar Sets

- One of the advantages of using an iterated function system is that the dimension of the attractor is often relatively easy to calculate or estimate in terms of the defining contractions.
- We discuss the case where $S_1, \ldots, S_m : \mathbb{R}^n \to \mathbb{R}^n$ are **similarities**.
- Suppose we have

$$|S_i(x) - S_i(y)| = c_i|x - y|, \quad x, y \in \mathbb{R}^n,$$

  where $0 < c_i < 1$ (c_i is called the **ratio** of S_i).
- Thus, each S_i transforms subsets of $\mathbb{R}^n$ into geometrically similar sets.

Self-Similar Sets

- Suppose $S_1, \ldots, S_m : \mathbb{R}^n \to \mathbb{R}^n$ are similarities.
- The attractor of such a collection of similarities is called a (**strictly**) **self-similar set**.
- It is a union of a number of smaller similar copies of itself.
- Standard examples include:
  - The middle third Cantor set;
  - The Sierpiński triangle;
  - The von Koch curve.

Condition Giving the Hausdorff Dimension

- We show that, under certain conditions, a self-similar set F:
  - Has Hausdorff and box dimensions equal to the value of s satisfying

$$\sum_{i=1}^{m} c_i^s = 1;$$

  - Has positive and finite $\mathcal{H}^s$-measure.

A "heuristic" calculation indicates the plausibility of this.

Suppose $F = \bigcup_{i=1}^{m} S_i(F)$, with the union "nearly disjoint".

Then

$$\mathcal{H}^s(F) = \sum_{i=1}^{m} \mathcal{H}^s(S_i(F)) = \sum_{i=1}^{m} c_i^s \mathcal{H}^s(F),$$

using the Scaling Property.

Assume that, at $s = \dim_H F$, we have $0 < \mathcal{H}^s(F) < \infty$.

Then s satisfies the claimed condition.

The Open Set Condition

- For the preceding argument to give the right answer, we require a condition that ensures that the components $S_i(F)$ of F do no overlap "too much".
- We say that the S_i satisfy the **open set condition** if, there exists a non-empty bounded open set V, such that

$$V \supseteq \bigcup_{i=1}^{m} S_i(V)$$

  with the union disjoint.

  Example: In the middle third Cantor set example, the open set condition holds for S_1 and S_2 with V as the open interval $(0, 1)$.
- We show that, if the similarities S_i satisfy the open set condition, the Hausdorff dimension of the attractor is given by $\sum_{i=1}^{m} c_i^s = 1$.

A Geometric Result

Lemma

Let $\{V_i\}$ be a collection of disjoint open subsets of $\mathbb{R}^n$ such that each V_i:

- Contains a ball of radius $a_1 r$;
- Is contained in a ball of radius $a_2 r$.

Then any ball B of radius r intersects at most $(1+2a_2)^n a_1^{-n}$ of the closures $\overline{V}_i$.

- Suppose $\overline{V}_i$ meets B.

  $\overline{V}_i$ is contained in the ball concentric with B of radius $(1+2a_2)r$.

  Suppose that q of the sets $\overline{V}_i$ intersect B.

  We sum the volumes of the corresponding interior balls of radii $a_1 r$.

  It follows that $q(a_1 r)^n \leq (1+2a_2)^n r^n$.

  This gives the stated bound for q.

Computing the Hausdorff Dimension

Theorem

Suppose that the open set condition holds for the similarities S_i on $\mathbb{R}^n$ with ratios $0 < c_i < 1$ for $1 \leq i \leq m$. Suppose F is the attractor of the IFS $\{S_1, \ldots, S_m\}$, that is

$$F = \bigcup_{i=1}^{m} S_i(F).$$

Then $\dim_H F = \dim_B F = s$, where s is given by

$$\sum_{i=1}^{m} c_i^s = 1.$$

Moreover, for this value of s, $0 < \mathcal{H}^s(F) < \infty$.

Computing the Hausdorff Dimension (Cont'd)

- Let s satisfy $\sum_{i=1}^{m} c_i^s = 1$.
  Let $\mathcal{I}_k$ be the set of all sequences $(i_1, \ldots, i_k)$ with $1 \leq i_j \leq m$.
  For any set A and $(i_1, \ldots, i_k) \in \mathcal{I}_k$, we write

  $$A_{i_1,\ldots,i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(A).$$

  By using $F = \bigcup_{i=1}^{m} S_i(F)$ repeatedly, we get $F = \bigcup_{\mathcal{I}_k} F_{i_1,\ldots,i_k}$.
  We get an upper estimate for the Hausdorff measure of F.
  $S_{i_1} \circ \cdots \circ S_{i_k}$ is a similarity of ratio $c_{i_1} \cdots c_{i_k}$.
  So

  $$\begin{array}{rcl} \sum_{\mathcal{I}_k} |F_{i_1,\ldots,i_k}|^s & = & \sum_{\mathcal{I}_k} (c_{i_1} \cdots c_{i_k})^s |F|^s \\ & = & (\sum_{i_1} c_{i_1}^s) \cdots (\sum_{i_k} c_{i_k}^s) |F|^s \\ & = & |F|^s. \end{array}$$

  For any $\delta > 0$, we choose k with $|F_{i_1,\ldots,i_k}| \leq (\max_i c_i)^k |F| \leq \delta$.
  So $\mathcal{H}_\delta^s(F) \leq |F|^s$. Hence, $\mathcal{H}^s(F) \leq |F|^s$.

Computing the Hausdorff Dimension (Cont'd)

- Let I be the set of all infinite sequences

$$\mathcal{I} = \{(i_1, i_2, \ldots) : 1 \leq i_j \leq m\}.$$

Let $I_{i_1,\ldots,i_k} = \{(i_1, \ldots, i_k, q_{k+1}, \ldots) : 1 \leq q_j \leq m\}$ be the "cylinder" consisting of those sequences in $\mathcal{I}$ with initial terms $(i_1, \ldots, i_k)$.

We may put a mass distribution μ on $\mathcal{I}$, such that

$$\mu(I_{i_1,\ldots,i_k}) = (c_{i_1} \cdots c_{i_k})^s.$$

We have $(c_{i_1} \cdots c_{i_k})^s = \sum_{i=1}^m (c_{i_1} \cdots c_{i_k} c_i)^s$.

That is,

$$\mu(I_{i_1,\ldots,i_k}) = \sum_{i=1}^{m} \mu(I_{i_1,\ldots,i_k,i}).$$

So μ is indeed a mass distribution on subsets of $\mathcal{I}$, with $\mu(\mathcal{I}) = 1$.

Computing the Hausdorff Dimension (Cont'd)

- Transfer μ to a mass distribution $\widetilde{\mu}$ on F by

$$\widetilde{\mu}(A) = \mu\{(i_1, i_2, \ldots) : x_{i_1,i_2,\ldots} \in A\},$$

for subsets A of F (recall that $x_{i_1,i_2,\ldots} = \bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k}$).
The $\widetilde{\mu}$-mass of a set is the μ-mass of the corresponding sequences.
It is easily checked that $\widetilde{\mu}(F) = 1$.
Claim: $\widetilde{\mu}$ satisfies the conditions of the Mass Distribution Principle.
Let V be the open set in the open set condition.
We have

$$\overline{V} \supseteq S(\overline{V}) = \bigcup_{i=1}^{m} S_i(\overline{V}).$$

So the decreasing sequence of iterates $S^k(\overline{V})$ converges to F.
In particular, for each finite sequence $(i_1, \ldots, i_k)$:
  - $\overline{V} \supseteq F$;
  - $\overline{V}_{i_1,\ldots,i_k} \supseteq F_{i_1,\ldots,i_k}$.

Computing the Hausdorff Dimension (Cont'd)

- Let B be any ball of radius $r < 1$.

  We estimate $\widetilde{\mu}(B)$ by considering the sets $V_{i_1,\ldots,i_k}$ with diameters comparable with that of B and with closures intersecting $F \cap B$.

  We curtail each infinite sequence $(i_1, i_2, \ldots) \in \mathcal{I}$ after the first term i_k for which $(\min_{1 \leq i \leq m} c_i) r \leq c_{i_1} c_{i_2} \cdots c_{i_k} \leq r$.

  Let $\mathcal{Q}$ denote the finite set of all sequences obtained in this way.

  Then, for every infinite sequence $(i_1, i_2, \ldots) \in \mathcal{I}$, there is exactly one value of k with $(i_1, \ldots, i_k) \in \mathcal{Q}$.

  But $V_1, \ldots, V_m$ are disjoint.

  So, for each $(i_1, \ldots, i_k)$, $V_{i_1,\ldots,i_k,1}, \ldots, V_{i_1,\ldots,i_k,m}$ are also disjoint.

  Using this in a nested way, it follows that the collection of open sets $\{V_{i_1,\ldots,i_k} : (i_1, \ldots, i_k) \in \mathcal{Q}\}$ is disjoint.

  Similarly, $F \subseteq \bigcup_{\mathcal{Q}} F_{i_1,\ldots,i_k} \subseteq \bigcup_{\mathcal{Q}} \overline{V}_{i_1,\ldots,i_k}$.

Computing the Hausdorff Dimension (Cont'd)

- We choose a_1 and a_2 so that V contains a ball of radius a_1 and is contained in a ball of radius a_2.

  Then, for all $(i_1, \ldots, i_k) \in Q$, $V_{i_1,\ldots,i_k}$ contains a ball of radius $c_{i_1} \cdots c_{i_k} a_1$.

  So it also contains one of radius $(\min_i c_i) a_1 r$.

  Moreover, it is contained in a ball of radius $c_{i_1} \cdots c_{i_k} a_2$.

  Hence it is also contained in a ball of radius $a_2 r$.

  Let $\mathcal{Q}_1$ denote the set of those sequences $(i_1, \ldots, i_k)$ in $\mathcal{Q}$ such that B intersects $\overline{V}_{i_1,\ldots,i_k}$.

  By the preceding lemma, there are at most

$$q = (1 + 2a_2)^n a_1^{-n} (\min_i c_i)^{-n}$$

  sequences in $\mathcal{Q}_1$.

Computing the Hausdorff Dimension (Cont'd)

- Then we have

$$\begin{array}{rcl} \widetilde{\mu}(B) & = & \widetilde{\mu}(F \cap B) \\ & = & \mu\{(i_1, i_2, \ldots) : x_{i_1, i_2, \ldots} \in F \cap B\} \\ & \leq & \mu\{\bigcup_{\mathcal{Q}_1} I_{i_1, \ldots, i_k}\}, \end{array}$$

since, if $x_{i_1, i_2, \ldots} \in F \cap B \subseteq_{\mathcal{Q}_1} \overline{V}_{i_1, \ldots, i_k}$, then, there is an integer k, such that $(i_1, \ldots, i_k) \in \mathcal{Q}_1$.

Thus,

$$\begin{array}{rcl} \widetilde{\mu}(B) & \leq & \sum_{\mathcal{Q}_1} \mu(I_{i_1, \ldots, i_k}) \\ & = & \sum_{\mathcal{Q}_1} (c_{i_1} \cdots c_{i_k})^s \\ & \leq & \sum_{\mathcal{Q}_1} r^s \\ & \leq & r^s q. \end{array}$$

But any set U is contained in a ball of radius $|U|$. So $\widetilde{\mu}(U) \leq |U|^s q$. By Mass Distribution, $\mathcal{H}^s(F) \geq q^{-1} > 0$ and $\dim_H F = s$.

Computing the Hausdorff Dimension (Conclusion)

- If $\mathcal{Q}$ is any set of finite sequences such that, for every $(i_1, i_2, \ldots) \in \mathcal{I}$, there is exactly one integer k with $(i_1, \ldots, i_k) \in \mathcal{Q}$, it follows inductively from $\sum_{i=1}^m c_i^s = 1$ that $\sum_{\mathcal{Q}} (c_{i_1} c_{i_2} \cdots c_{i_k})^s = 1$.

  If $\mathcal{Q}$ is chosen as above, $\mathcal{Q}$ contains at most $(\min_i c_i)^{-s} r^{-s}$ sequences.

  For each sequence $(i_1, \ldots, i_k) \in \mathcal{Q}$, we have

  $$|\overline{V}_{i_1,\ldots,i_k}| = c_{i_1} \cdots c_{i_k} |\overline{V}| \leq r|\overline{V}|.$$

  So F may be covered by $(\min_i c_i)^{-s} r^{-s}$ sets of diameter $r|\overline{V}|$, for each $r < 1$.

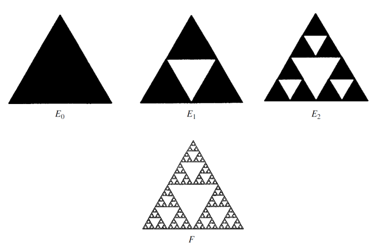
  By the equivalent definition of box dimension, $\overline{\dim}_B F \leq s$.

  Noting that $s = \dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F \leq s$, yields the result.
- If the open set condition is not assumed, it may be shown that we still have $\dim_H F = \dim_B F$ though this value may be less than s.

Example: Sierpiński Triangle

- The Sierpiński triangle or gasket F is constructed from an equilateral triangle by repeatedly removing inverted equilateral triangles. Then

$$\dim_H F = \dim_B F = \frac{\log 3}{\log 2}.$$



The set F is the attractor of the three obvious similarities of ratios $\frac{1}{2}$ which map the triangle E_0 onto the triangles of E_1.

The open set condition holds, taking V as the interior of E_0.

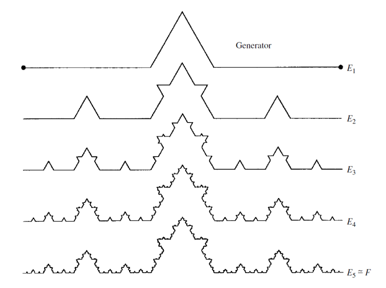
The solution of $3(\frac{1}{2})^s = \sum_1^3 (\frac{1}{2})^s = 1$ is $s = \frac{\log 3}{\log 2}$.

Thus, by the theorem, $\dim_H F = \dim_B F = \frac{\log 3}{\log 2}$.

Example: Modified von Koch Curve

- Fix $0 < a \leq \frac{1}{3}$ and construct a curve F by repeatedly replacing the middle proportion a of each interval by the other two sides of an equilateral triangle. Then $\dim_H F = \dim_B F$ is the solution of

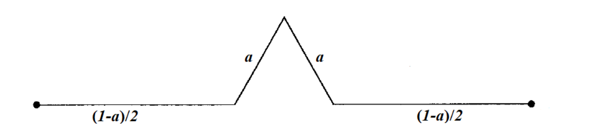
$$2a^s + 2\left(\frac{1}{2}(1-a)\right)^s = 1.$$



The curve F is the attractor of the similarities that map the unit interval onto each of the four intervals in E_1.

The open set condition holds, taking V as the interior of the isosceles triangle of base length 1 and height $\frac{1}{2}a\sqrt{3}$.

Example: Modified von Koch Curve (Cont'd)



- Note that:
  - The left and right segments have scaling factors $\frac{1-a}{2}$;
  - The segments forming the two sides of he equilateral triangle have scaling factors a.

So the equation we get for s is

$$2\left(\frac{1-a}{2}\right)^s + 2a^s = 1.$$

Specifying Self-Similar Sets Diagrammatically

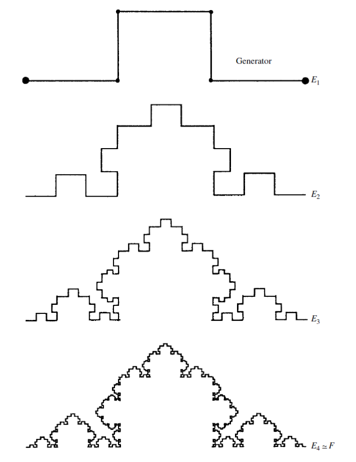
- A **generator** consists of a number of straight line segments and two points specially identified.
- We associate with each line segment the similarity that maps the two special points onto the endpoints of the segment.
- A sequence of sets approximating to the self-similar attractor may be built up by iterating the process of replacing each line segment by a similar copy of the generator.
- The similarities are defined by the generator only to within:
  - Reflection;
  - 180° rotation.
- But the orientation may be specified by displaying the first step of the construction.

Example 1

- Stages in the construction of a fractal curve from a generator. The lengths of the segments in the generator are $\frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}$. The Hausdorff and box dimensions of F are given by

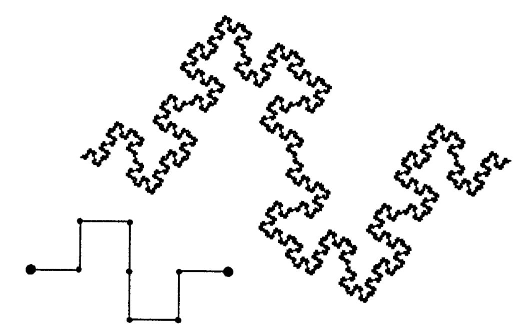
$$3\left(\frac{1}{3}\right)^s + 2\left(\frac{1}{4}\right)^s = 1.$$

Thus, $s = 1.34\ldots$.



Example 2

- A fractal curve and its generator.



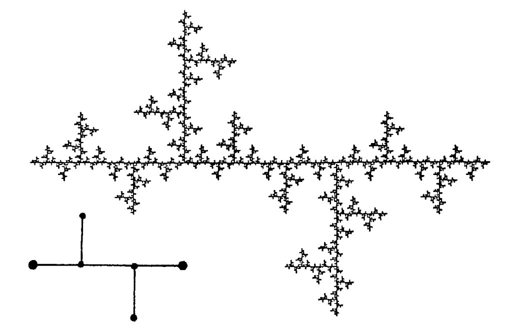
The Hausdorff and box dimensions of the curve satisfy

$$8\left(\frac{1}{4}\right)^s = 1.$$

Thus, they are equal to $\frac{\log 8}{\log 4} = \frac{3}{2}$.

Example 3

- A fractal curve and its generator.



The Hausdorff and box dimensions of the curve satisfy

$$5\left(\frac{1}{3}\right)^s = 1.$$

Thus, they are equal to $\frac{\log 5}{\log 3} = 1.465\ldots$.

Subsection 3

Non-Similarity Contractions

Dimension Upper Bound

Proposition

Let F be the attractor of an IFS consisting of contractions $\{S_1, \ldots, S_m\}$ on a closed subset D of $\mathbb{R}^n$, such that

$$|S_i(x) - S_i(y)| \leq c_i|x - y|, \quad x, y \in D,$$

with $0 < c_i < 1$ for each i. Then $\dim_H F \leq s$ and $\overline{\dim}_B F \leq s$, where $\sum_{i=1}^m c_i^s = 1$.

- These estimates are essentially those of the first and last paragraphs of the proof of the previous theorem.

  The difference is that we have, for each set A, instead of an equality, the inequality

$$|A_{i_1,\ldots,i_k}| \leq c_{i_1} \cdots c_{i_k}|A|.$$

Introducing Dimension Lower Bound

- We next obtain a lower bound for dimension in the case where the components $S_i(F)$ of F are disjoint.
- This will certainly be the case if, there is some non-empty compact set E, such that:
  - $S_i(E) \subseteq E$, for all i;
  - The $S_i(E)$ are disjoint.

Dimension Lower Bound

Proposition

Consider the IFS consisting of contractions $\{S_1, \ldots, S_m\}$ on a closed subset D of $\mathbb{R}^n$, such that

$$b_i|x-y| \leq |S_i(x) - S_i(y)|, \quad x, y \in D,$$

with $0 < b_i < 1$ for each i. Assume that the (non-empty compact) attractor F satisfies

$$F = \bigcup_{i=1}^{m} S_i(F),$$

with this union disjoint. Then F is totally disconnected and $\dim_H F \geq s$, where

$$\sum_{i=1}^{m} b_i^s = 0.$$

Dimension Lower Bound (Cont'd)

- Let $d > 0$ be the minimum distance between any pair of the disjoint compact sets $S_1(F), \ldots, S_m(F)$, i.e.,

$$d = \min_{i \neq j} \inf \{|x - y| : x \in S_i(F), y \in S_j(F)\}.$$

Let $F_{i_1,\ldots,i_k} = S_{i_1} \circ \cdots S_{i_k}(F)$.
Define μ by

$$\mu(F_{i_1 \ldots i_k}) = (b_{i_1} \cdots b_{i_k})^s.$$

We have

$$\begin{aligned} \textstyle\sum_{i=1}^m \mu(F_{i_1,\ldots,i_k,i}) &= \textstyle\sum_{i=1}^m (b_{i_1} \cdots b_{i_k} b_i)^s \\ &= (b_{i_1} \cdots b_{i_k})^s \\ &= \mu(F_{i_1,\ldots,i_k}) \\ &= \mu(\textstyle\bigcup_{i=1}^k F_{i_1,\ldots,i_k,i}). \end{aligned}$$

So μ defines a mass distribution on F with $\mu(F) = 1$.

Dimension Lower Bound (Cont'd)

- If $x \in F$, there is a unique infinite sequence $i_1, i_2, \ldots$ such that $x \in F_{i_1,\ldots,i_k}$ for each k.

  For $0 < r < d$ let k be the least integer such that

  $$b_{i_1} \cdots b_{i_k} d \leq r < b_{i_1} \cdots b_{i_{k-1}} d.$$

  If $i_1', \ldots, i_k'$ is distinct from $i_1, \ldots, i_k$, the sets $F_{i_1,\ldots,i_k}$ and $F_{i_1',\ldots,i_k'}$ are disjoint and separated by a gap of at least $b_{i_1} \cdots b_{i_{k-1}} d > r$.

  To see this, note that if j is the least integer such that $i_j \neq i_j'$, then $F_{i_j,\ldots,i_k} \subseteq F_{i_j}$ and $F_{i_j',\ldots,i_k'} \subseteq F_{i_j'}$ are separated by d.

  So $F_{i_1,\ldots,i_k}$ and $F_{i_1',\ldots,i_k'}$ are separated by at least $b_{i_1} \cdots b_{i_{j-1}} d$.

Dimension Lower Bound (Conclusion)

- It follows that $F \cap B(x, r) \subseteq F_{i_1,\ldots,i_k}$.

  So we get

$$\mu(F \cap B(x, r)) \leq \mu(F_{i_1,\ldots,i_k}) = (b_{i_1} \cdots b_{i_k})^s \leq d^{-s} r^s.$$

  If U intersects F, then $U \subseteq B(x, r)$, for some $x \in F$ with $r = |U|$.

  Thus,

$$\mu(U) \leq d^{-s} |U|^s.$$

  So, by the Mass Distribution Principle, $\mathcal{H}^s(F) > 0$ and $\dim_H F \geq s$.

  The separation indicated above implies that F is totally disconnected.

Example: "Non-Linear" Cantor Set

- Suppose $D = [\frac{1}{2}(1+\sqrt{3}), (1+\sqrt{3})]$.
  Let $S_1, S_2 : D \to D$ be given by
  $$S_1(x) = 1 + \frac{1}{x}, \quad S_2(x) = 2 + \frac{1}{x}.$$
  Then
  $$0.44 < \dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F < 0.66,$$
  where F is the attractor of $\{S_1, S_2\}$.
  We note that
  $$\begin{array}{rcl} S_1(D) & = & [\frac{1}{2}(1+\sqrt{3}), \sqrt{3}]; \\ S_2(D) & = & [\frac{1}{2}(3+\sqrt{3}), 1+\sqrt{3}]. \end{array}$$
  So we can use the preceding propositions to estimate $\dim_H F$.

Example: "Non-Linear" Cantor Set (Cont'd)

- Let $x, y \in D$ be distinct points.
  By the Mean Value Theorem, $\frac{S_i(x)-S_i(y)}{x-y} = S_i'(z_i)$, for some $z_i \in D$.
  Thus, for $i = 1, 2$,

$$\inf_{x \in D} |S_i'(x)| \leq \frac{|S_i(x) - S_i(y)|}{|x - y|} \leq \sup_{x \in D} |S_i'(x)|.$$

  But $S_1'(x) = S_2'(x) = -\frac{1}{x^2}$.
  So, for both $i = 1$ and $i = 2$,

$$\begin{aligned} \frac{1}{2}(2 - \sqrt{3}) &= \frac{1}{(1+\sqrt{3})^2} \\ &\leq \frac{|S_i(x)-S_i(y)|}{|x-y|} \\ &\leq \frac{1}{(\frac{1}{2}(1+\sqrt{3}))^2} \\ &= 2(2 - \sqrt{3}) \end{aligned}$$

Example: "Non-Linear" Cantor Set (Cont'd)

- According to the preceding propositions, lower and upper bounds for the dimensions are given by the solutions of

$$2\left(\frac{1}{2}(2-\sqrt{3})\right)^s = 1 \quad \text{and} \quad 2(2(2-\sqrt{3}))^s = 1.$$

  These are

$$\begin{aligned} s &= \frac{\log 2}{\log(2(2+\sqrt{3}))} = 0.34; \\ s &= \frac{\log 2}{\log\left(\frac{1}{2}(2+\sqrt{3})\right)} = 1.11. \end{aligned}$$

  For a subset of the real line, an upper bound greater than 1 is not of much interest.

Example: "Non-Linear" Cantor Set (Cont'd)

- One way of getting better estimates is to note that F is also the attractor of the four mappings on $[0,1]$

$$S_i \circ S_j = i + \frac{1}{j + \frac{1}{x}} = i + \frac{x}{jx+1}, \quad i,j = 1,2.$$

By calculating derivatives and using the mean-value theorem as before, we get that $(S_i \circ S_j)'(x) = \frac{1}{(jx+1)^2}$.

So

$$\frac{|x-y|}{(j(1+\sqrt{3})+1)^2} \leq |S_i \circ S_j(x) - S_i \circ S_j(y)| \leq \frac{|x-y|}{(\frac{1}{2}j(1+\sqrt{3})+1)^2}.$$

Example: "Non-Linear" Cantor Set (Cont'd)

- Lower and upper bounds for the dimensions are now given by the solutions of

$$\begin{aligned} 2(2+\sqrt{3})^{-2s} + 2(3+2\sqrt{3})^{-2s} &= 1; \\ 2(\tfrac{1}{2}(3+\sqrt{3}))^{-2s} + 2(2+\sqrt{3})^{-2s} &= 1. \end{aligned}$$

So we obtain

$$0.44 < \dim_H F < 0.66.$$

This is a considerable improvement on the previous estimates.

In fact, it turns out that $\dim_H F = 0.531$.

This value that may be obtained by looking at yet higher-order iterates of the S_i.

Subsection 4

Continued Fractions

Partial Fraction Expansions

- Any number x that is not an integer may be written as $x = a_0 + \frac{1}{x_1}$, where a_0 is an integer and $x_1 > 1$.
- Similarly, if x_1 is not an integer, then $x_1 = a_1 + \frac{1}{x_2}$ with $x_2 > 1$.
- So $x = a_0 + \frac{1}{a_1 + \frac{1}{x_2}}$.
- Proceeding in this way,

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{k-1} + \frac{1}{x_k}}}}},$$

  for each k, provided that at no stage is x_k an integer.
- The integers $a_0, a_1, a_2, \ldots$ form the **partial quotients** of x.
- We write

$$x = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3 + \cdots}$$

  for the **continued fraction expansion** of x.

Approximations and Examples

- The expansion of x into continued fractions terminates if and only if x is rational.
- Otherwise taking a finite number of terms,

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_k}}}}$$

provides a sequence of rational approximations to x.

- This sequence converge to x as $k \to \infty$.

Examples

- Examples of continued fractions include

$$\begin{aligned} \sqrt{2} &= 1+\frac{1}{2+}\frac{1}{2+}\frac{1}{2+\cdots} \\ &= 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\ddots}}}}, \\ \sqrt{3} &= 1+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+\cdots} \\ &= 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}. \end{aligned}$$

- A **quadratic surd** is a root of a quadratic equation with integer coefficients.
- Any quadratic surd has eventually periodic partial quotients.

Partial Quotients and Attractors of IFSs

- Sets of numbers defined by conditions on their partial quotients may be thought of as fractal attractors of certain iterated function systems.
- Let F be the set of positive real numbers x with:
  - Non-terminating continued fraction expressions;
  - All of whose partial quotients equal to 1 or 2.

  Then F is a fractal with

  $$0.44 < \dim_H F < 0.66.$$

  F satisfies the following properties.
  - The complement of F is open. So F is closed.
  - We have $F \subseteq [1, 3]$. So F s bounded.
  - $x \in F$ precisely when

  $$x = 1 + \frac{1}{y} \quad \text{or} \quad x = 2 + \frac{1}{y}, \quad \text{with } y \in F.$$

Partial Quotients and Attractors of IFSs (Cont'd)

- Define
$$\begin{aligned} S_1(x) &= 1+\frac{1}{x};\\ S_2(x) &= 2+\frac{1}{x}. \end{aligned}$$
Then
$$F = S_1(F) \cup S_2(F).$$
That is, F is the attractor of the iterated function system $\{S_1, S_2\}$.
In fact F is exactly the set analyzed in the example at the end of the previous section.
There, it was shown that
$$0.44 < \dim_H F < 0.66.$$

Subsection 5

Dimensions of Graphs

Graphs of Functions: Dimension 1

- We consider functions $f : [a, b] \to \mathbb{R}$.
- Under certain circumstances the graph

$$\text{graph} f = \{(t, f(t)) : a \leq t \leq b\}$$

regarded as a subset of the (t, x)-coordinate plane may be a fractal.
- If f has a continuous derivative, then it is not difficult to see that graph$f$ has dimension 1 and, indeed, is a regular 1-set.
- The same is true if f is of bounded variation, i.e., if

$$\sum_{i=0}^{m-1} |f(t_i) - f(t_{i+1})| \leq \text{constant},$$

for all dissections $0 = t_0 < t_1 < \cdots < t_m = 1$.

Graphs of Functions: Fractals

- It is possible for a continuous function to be sufficiently irregular to have a graph of dimension strictly greater than 1.

  Example: Consider

  $$f(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k t),$$

  where $1 < s < 2$ and $\lambda > 1$.

  The function f is essentially Weierstrass's example of a continuous function that is nowhere differentiable.

  Its has box dimension s.

  It is believed to have Hausdorff dimension s.

Estimate for Box Dimension

- Given a function f and an interval $[t_1, t_2]$, we write R_f for the **maximum range** of f over an interval,

$$R_f[t_1, t_2] = \sup_{t_1 \le t, u \le t_2} |f(t) - f(u)|.$$

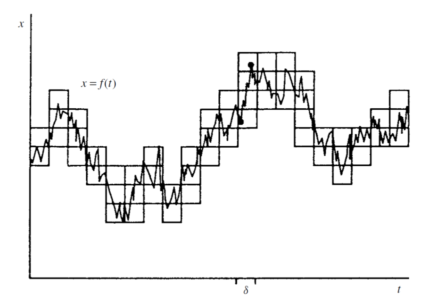
Proposition

Let $f : [0, 1] \to \mathbb{R}$ be continuous. Suppose that $0 < \delta < 1$, and m is the least integer greater than or equal to $\frac{1}{\delta}$. Then, if N_δ is the number of squares of the δ-mesh that intersect graph$f$,

$$\frac{1}{\delta}\sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta] \le N_\delta \le 2m + \frac{1}{\delta}\sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta].$$

Estimate for Box Dimension (Cont'd)

- We consider all mesh squares of side δ.



Let q be the number of those over $[i\delta, (i+1)\delta]$ intersecting graph$f$. Using the continuity of f, we have

$$R_f[i\delta, (i+1)\delta]/\delta \leq q \leq 2 + R_f[i\delta, (i+1)\delta]/\delta.$$

Summing over all such intervals gives the inequalities.

Hölder Condition (Upper Bound)

Corollary

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function.

(a) Suppose, for $c > 0$ and $1 \leq s \leq 2$,

$$|f(t) - f(u)| \leq c|t - u|^{2-s}, \quad 0 \leq t, u \leq 1.$$

Then $\mathcal{H}^s(\text{graph} f) < \infty$ and

$$\dim_H \text{graph} f \leq \underline{\dim}_B \text{graph} f \leq \overline{\dim}_B \text{graph} f \leq s.$$

This remains true if the condition on f holds when $|t - u| < \delta$, for some $\delta > 0$.

Hölder Condition (Upper Bound Cont'd)

(a) By hypothesis, for $0 \leq t_1, t_2 \leq 1$,

$$R_f[t_1, t_2] \leq c|t_1 - t_2|^{2-s}.$$

With notation as in the preceding proposition,

$$m < (1 + \delta^{-1}).$$

By the inequality in the proposition,

$$\begin{aligned} N_\delta &\leq 2m + \delta^{-1} mc\delta^{2-s} \\ &\leq (1 + \delta^{-1})(2 + c\delta^{-1}\delta^{2-s}) \\ &\leq c_1 \delta^{-s}, \end{aligned}$$

where c_1 is independent of δ.

The conclusion now follows from a previous result.

Hölder Condition (Lower Bound)

Corollary

Let $f : [0,1] \to \mathbb{R}$ be a continuous function.

(b) Suppose that there are numbers $c > 0$, $\delta_0 > 0$ and $1 \leq s < 2$, such that, for each $t \in [0,1]$ and $0 < \delta \leq \delta_0$, there exists u such that $|t - u| \leq \delta$ and

$$|f(t) - f(u)| \geq c\delta^{2-s}.$$

Then

$$s \leq \underline{\dim}_B \mathrm{graph} f.$$

Hölder Condition (Lower Bound Cont'd)

(b) By hypothesis, for $0 \leq t_1, t_2 \leq 1$,

$$R_f[t_1, t_2] \geq c|t_1 - t_2|^{2-s}.$$

Note that $\delta^{-1} \leq m$.

By the inequality of the preceding proposition,

$$\begin{aligned} N_\delta &\geq \delta^{-1} m c \delta^{2-s} \\ &\geq \delta^{-1}\delta^{-1} c \delta^{2-s} \\ &= c\delta^{-s}. \end{aligned}$$

Now one of the equivalent definitions of box-counting dimensions in a preceding theorem gives

$$s \leq \underline{\dim}_B \mathrm{graph} f.$$

Example: The Weierstrass Function

- Fix $\lambda > 1$ and $1 < s < 2$.

  Define $f : [0,1] \to \mathbb{R}$ by

  $$f(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k t).$$

  Then, provided λ is large enough,

  $$\dim_B \text{graph} f = s.$$

  Given $0 < h < \lambda^{-1}$, let N be the integer such that

  $$\lambda^{-(N+1)} \leq h < \lambda^{-N}.$$

  The following hold:
  - By the Mean-Value Theorem, $|\sin u - \sin v| \leq |u - v|$;
  - $|\sin u| \leq 1$.

Example: The Weierstrass Function (Cont'd)

- Applying the first on the first N terms of the sum and the second on the remaining terms, we obtain

$$\begin{aligned} |f(t+h)-f(t)| &\leq \sum_{k=1}^{N} \lambda^{(s-2)k} |\sin(\lambda^k(t+h)) - \sin(\lambda^k t)| \\ &\quad + \sum_{k=N+1}^{\infty} \lambda^{(s-2)k} |\sin(\lambda^k(t+h)) - \sin(\lambda^k t)| \\ &\leq \sum_{k=1}^{N} \lambda^{(s-2)k} \lambda^k h + \sum_{k=N+1}^{\infty} 2\lambda^{(s-2)k}. \end{aligned}$$

Summing these geometric series,

$$|f(t+h)-f(t)| \leq \frac{h\lambda^{(s-1)N}}{1-\lambda^{1-s}} + \frac{2\lambda^{(s-2)(N+1)}}{1-\lambda^{s-2}} \leq ch^{2-s},$$

where c is independent of h.

A previous corollary now gives that

$$\overline{\dim}_B \mathrm{graph} f \leq s.$$

Example: The Weierstrass Function (Cont'd)

- In the same way, but splitting the sum into three parts - the first $N-1$ terms, the N-th term, and the rest - we get that, for $\lambda^{-(N+1)} \leq h < \lambda^{-N}$,

$$|f(t+h)-f(t)-\lambda^{(s-2)N}(\sin\lambda^N(t+h)-\sin\lambda^N t)| \leq \frac{\lambda^{(s-2)N-s+1}}{1-\lambda^{1-s}}+\frac{2\lambda^{(s-2)(N+1)}}{1-\lambda^{s-2}}.$$

  Suppose $\lambda > 2$ is large enough for the right-hand side to be less than $\frac{1}{20}\lambda^{(s-2)N}$, for all N.

  For $\delta < \lambda^{-1}$, take N such that $\lambda^{-N} \leq \delta < \lambda^{-(N-1)}$.

  For each t, we may choose h, with $\lambda^{-(N+1)} \leq h < \lambda^{-N} < \delta$, such that

$$|\sin\lambda^N(t+h)-\sin\lambda^N t| > \frac{1}{10}.$$

Example: The Weierstrass Function (Cont'd)

- We chose:
  - λ, such that $\frac{\lambda^{(s-2)N-s+1}}{1-\lambda^{1-s}} + \frac{2\lambda^{(s-2)(N+1)}}{1-\lambda^{s-2}} \leq \frac{1}{20}\lambda^{(s-2)N}$;
  - $\lambda^{-(N+1)} \leq h < \lambda^{-N} < \delta$, such that $|\sin \lambda^N(t+h) - \sin \lambda^N t| > \frac{1}{10}$.

Therefore, by

$$|f(t+h) - f(t) - \lambda^{(s-2)N}(\sin \lambda^N(t+h) - \sin \lambda^N t)| \leq \frac{\lambda^{(s-2)N-s+1}}{1-\lambda^{1-s}} + \frac{2\lambda^{(s-2)(N+1)}}{1-\lambda^{s-2}},$$

we get

$$\begin{aligned} |f(t+h) - f(t)| &\geq \frac{1}{10}\lambda^{(s-2)N} - \frac{1}{20}\lambda^{(s-2)N} \\ &= \frac{1}{20}\lambda^{(s-2)N} \\ &\geq \frac{1}{20}\lambda^{s-2}\delta^{2-s}. \end{aligned}$$

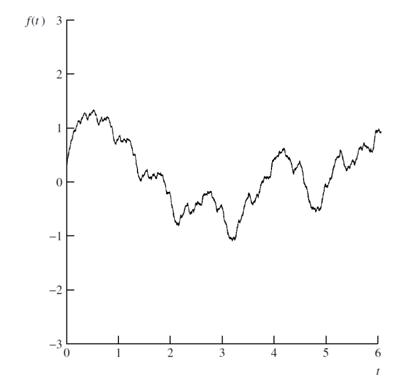
It follows from a preceding corollary that

$$\underline{\dim}_B \mathrm{graph} f \geq s.$$

Illustration: Weierstrass Function I

- The Weierstrass function

$$f(t) = \sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^{-0.9k} \sin\left(\left(\frac{3}{2}\right)^k t\right).$$

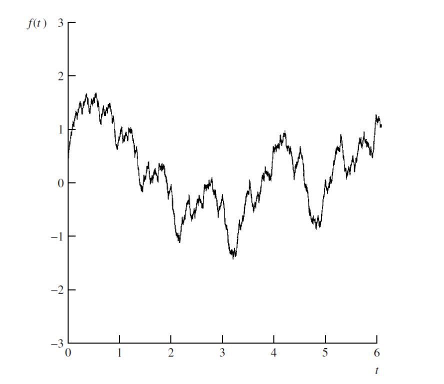


- Here $s = 1.1$ and $\dim_B \mathrm{graph} f = 1.1$.

Illustration: Weierstrass Function II

- The Weierstrass function

$$f(t) = \sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^{-0.7k} \sin\left(\left(\frac{3}{2}\right)^{k} t\right).$$

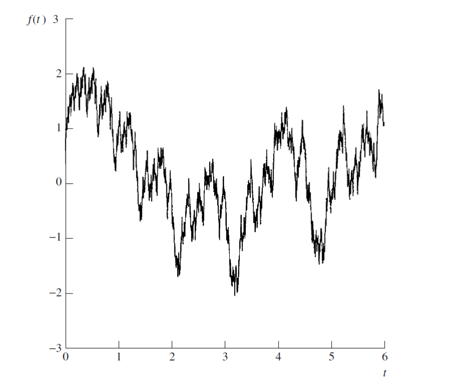


- Here $s = 1.3$ and $\dim_B \text{graph} f = 1.3$.

Illustration: Weierstrass Function III

- The Weierstrass function

$$f(t) = \sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^{-0.5k} \sin\left(\left(\frac{3}{2}\right)^{k} t\right).$$

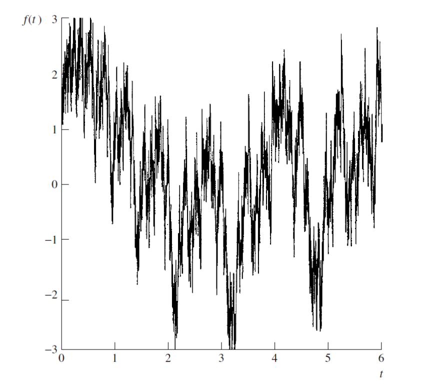


- Here $s = 1.5$ and $\dim_B \mathrm{graph} f = 1.5$.

Illustration: Weierstrass Function IV

- The Weierstrass function

$$f(t) = \sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^{-0.3k} \sin\left(\left(\frac{3}{2}\right)^{k} t\right).$$



- Here $s = 1.7$ and $\dim_B \text{graph} f = 1.7$.

Self-Affine Sets as Graphs of Functions

- We saw that self-affine sets defined by iterated function systems are often fractals.
- By a suitable choice of affine transformations, they can also be graphs of functions.
- Let $\{S_i, \ldots, S_m\}$ be affine transformations represented in matrix notation with respect to (t, x) coordinates by

$$S_i \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \frac{1}{m} & 0 \\ a_i & c_i \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} + \begin{bmatrix} \frac{i-1}{m} \\ b_i \end{bmatrix}.$$

- This can be written as

$$S_i(t, x) = \left(\frac{t}{m} + \frac{i-1}{m}, a_i t + c_i x + b_i \right).$$

Self-Affine Sets as Graphs of Functions (Cont'd)

- We defined
$$S_i(t,x) = \left(\frac{t}{m} + \frac{i-1}{m}, a_i t + c_i x + b_i\right).$$
- The S_i transform vertical lines to vertical lines.
Indeed, we have for $t = t_0$,
$$S_i(t_0,x) = \left(\frac{t_0 + i - 1}{m}, c_i x + (a_i t_0 + b_i)\right).$$
- The vertical strip $0 \leq t \leq 1$ is mapped onto the strip $\frac{i-1}{m} \leq t \leq \frac{i}{m}$.
- We obtain that the transformation involves:
  - A contraction by c_i in the t direction;
  - A contraction by $\frac{1}{m}$ in the x-direction.
- We suppose that
$$\frac{1}{m} < c_i < 1$$
so that contraction in the t is stronger than in the x direction.

Self-Affine Sets as Graphs of Functions (Cont'd)

- The fixed point of S_1 is $p_1 = (0, \frac{b_1}{1-c_1})$.

$$S_1(t,x) = (t,x) \ \Rightarrow \left\{ \begin{array}{rcl} \frac{t}{m} & = & t \\ a_1 t + c_1 x + b_1 & = & x \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} t & = & 0 \\ x & = & \frac{b_1}{1-c_1} \end{array} \right.$$

- The fixed point of S_m is $p_m = (1, \frac{a_m+b_m}{1-c_m})$.

$$\begin{array}{l} S_m(t,x) = (t,x) \ \Rightarrow \left\{ \begin{array}{rcl} \frac{t+m-1}{m} & = & t \\ a_m t + c_m x + b_m & = & x \end{array} \right\} \\ \Rightarrow \left\{ \begin{array}{rcl} \frac{m-1}{m} & = & \frac{m-1}{m} t \\ a_m t + b_m & = & (1-c_m)x \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} t & = & 1 \\ x & = & \frac{a_m+b_m}{1-c_m} \end{array} \right. \end{array}$$

- We assume that the matrix entries have been chosen so that

$$S_i(p_m) = S_{i+1}(p_1), \quad 1 \le i \le m-1.$$

- Then the segments $[S_i(p_1), S_i(p_m)]$ form a polygonal curve E_1.

Self-Affine Sets as Graphs of Functions (Cont'd)

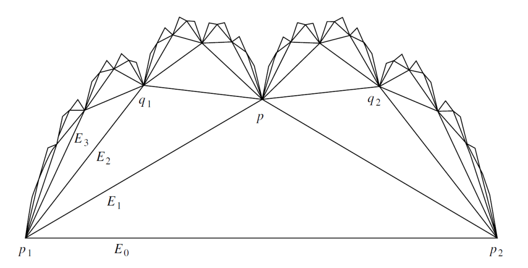
- To avoid trivial cases, we assume that the points

$$p_1 = S_1(p_1), \ldots, S_m(p_1), S_m(p_m) = p_m$$

  are not all collinear.
- The attractor F of the iterated function system $\{S_i, \ldots, S_m\}$ may be constructed by repeatedly replacing line segments by affine images of the "generator" E_1.
- The displayed condition ensures that the segments join up with the result that F is the graph of some continuous function $f : [0,1] \to \mathbb{R}$.
- The imposed conditions do not necessarily imply that the S_i are contractions with respect to Euclidean distance.
- It is possible to redefine distance in the (x, t) plane in such a way that the S_i become contractions.
- Then the IFS theory guarantees a unique attractor.

Illustration

- Stages in the construction of a self-affine curve F.



- The affine transformations S_1 and S_2 map the generating triangle p_1pp_2 onto the triangles p_1q_1p and pq_2p_2, respectively, and transform vertical lines to vertical lines.
- The rising sequence of polygonal curves $E_0, E_1, \ldots$ are given by

$$E_{k+1} = S_1(E_k) \cup S_2(E_k).$$

- They provide increasingly good approximations to F.

Example: Self-Affine Curves

- Let $F = \text{graph} f$ be the self-affine curve described above. Then

$$\dim_B F = 1 + \frac{\log(c_1 + \cdots + c_m)}{\log m}.$$

Let T_i be the "linear part" of S_i, given by the matrix $\begin{bmatrix} \frac{1}{m} & 0 \\ a_i & c_i \end{bmatrix}$.

Let $I_{i_1,\ldots,i_k}$ be the interval of the t-axis consisting of those t with base-m expansion beginning $0.i_1' \cdots i_k'$ where $i_j' = i_j - 1$.

Then the part of F above $I_{i_1,\ldots,i_k}$ is the affine image $S_{i_1} \circ \cdots \circ S_{i_k}(F)$, which is a translate of $T_{i_1} \circ \cdots \circ T_{i_k}(F)$.

The matrix representing $T_{i_1} \circ \cdots \circ T_{i_k}$ is seen by induction to be

$$\begin{bmatrix} m^{-k} & 0 \\ m^{1-k} a_{i_1} + m^{2-k} c_{i_1} a_{i_2} + \cdots + c_{i_1} c_{i_2} \cdots c_{i_{k-1}} a_{i_k} & c_{i_1} c_{i_2} \cdots c_{i_k} \end{bmatrix}.$$

This is a shear transformation, contracting vertical lines by a factor $c_{i_1} c_{i_2} \cdots c_{i_k}$.

Example: Self-Affine Curves (Cont'd)

- Observe that the bottom left-hand entry is bounded by

$$
\begin{aligned}
&|m^{1-k}a_{i_1} + m^{2-k}c_{i_1}a_{i_2} + \cdots + c_{i_1}c_{i_2}\cdots c_{i_{k-1}}a_{i_k}| \\
&\leq |m^{1-k}a + m^{2-k}c_{i_1}a + \cdots + c_{i_1}\cdots c_{i_{k-1}}a| \quad (a = \max|a_i|) \\
&\leq ((mc)^{1-k} + (mc)^{2-k} + \cdots + 1)c_{i_1}\cdots c_{i_{k-1}}a \quad (c = \min\{c_i\} > \tfrac{1}{m}) \\
&\leq rc_{i_1}\cdots c_{i_{k-1}}. \quad (r = \tfrac{a}{1-(mc)^{-1}})
\end{aligned}
$$

  Thus the image $T_{i_1} \circ \cdots \circ T_{i_k}(F)$ is contained in a rectangle of height $(r+h)c_{i_1}\cdots c_{i_k}$ where h is the height of F.

  On the other hand, if q_1, q_2, q_3 are three non-collinear points chosen from $S_1(p_1), \ldots, S_m(p_1), p_m$, then $T_{i_1} \circ \cdots \circ T_{i_k}(F)$ contains the points $T_{i_1} \circ \cdots \circ T_{i_k}(q_j)$, $j = 1, 2, 3$.

  The height of the triangle with these vertices is at least $c_{i_1}\cdots c_{i_k}d$, where d is the vertical distance from q_2 to the segment $[q_1, q_3]$.

Example: Self-Affine Curves (Cont'd)

- Thus the range of the function f over $I_{i_1,\ldots,i_k}$ satisfies

$$dc_{i_1}\cdots c_{i_k} \leq R_f[I_{i_1,\ldots,i_k}] \leq r_1 c_{i_1}\cdots c_{i_k}, \text{ with } r_1 = r + h.$$

For fixed k, sum this over the m^k intervals $I_{i_1,\ldots,i_k}$ of lengths m^{-k}.
We get, using a previous proposition,

$$m^k d \sum c_{i_1}\cdots c_{i_k} \leq N_{m^{-k}}(F) \leq 2m^k + m^k r_1 \sum c_{i_1}\cdots c_{i_k},$$

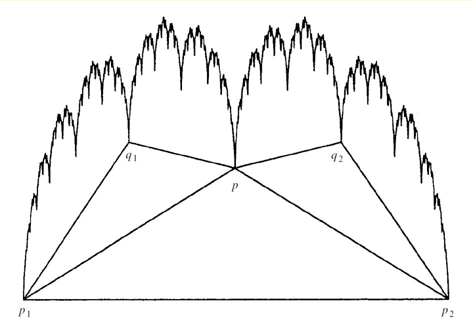
where $N_{m^{-k}}(F)$ is the number of mesh squares of side m^{-k} that intersect F.
For each j, the number c_{i_j} ranges through the values $c_1,\ldots,c_m$.
So $\sum c_{i_1}\cdots c_{i_k} = (c_1+\cdots+c_m)^k$.
Thus,

$$dm^k(c_1+\cdots+c_m)^k \leq N_{m^{-k}}(F) \leq 2m^k + r_1 m^k (c_1+\cdots+c_m)^k.$$

Taking logarithms and using one of the definitions of box dimension gives the value stated.

Example

- Self-affine curve defined by the two affine transformations that maps the triangle p_1pp_2 onto p_1q_1p and pq_2p_2 respectively.

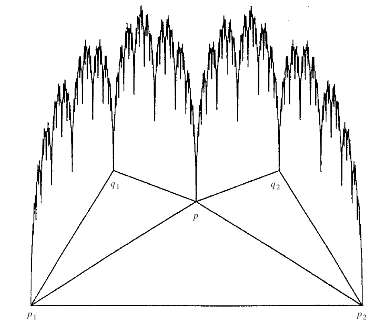


- The vertical contraction of both transformations is 0.7.
- This gives

$$\dim_B \mathrm{graph} f = 1 + \frac{\log(0.7 + 0.7)}{\log 2} = 1.49.$$

Example

- Self-affine curve defined by the two affine transformations that maps the triangle p_1pp_2 onto p_1q_1p and pq_2p_2 respectively.



- The vertical contraction of both transformations is 0.8.
- This gives

$$\dim_B \text{graph} f = 1 + \frac{\log(0.8 + 0.8)}{\log 2} = 1.68.$$

Subsection 6

Repellers and Iterated Function Systems

Iterates

- Let D be a subset of $\mathbb{R}^n$ (often $\mathbb{R}^n$ itself).
- Let $f : D \to D$ be a continuous mapping.
- f^k denotes the k-**th iterate** of f, so that

$$f^0(x) = x, \quad f^1(x) = f(x), \quad f^2(x) = f(f(x)), \ldots.$$

- Clearly $f^k(x)$ is in D, for all k, if x is a point of D.

Examples

- Typically, $x, f(x), f^2(x), \ldots$ are the values of some quantity at times $0, 1, 2, \ldots$.
- Thus the value at time $k+1$ is given in terms of the value at time k by the function f.

  For example, $f^k(x)$ might represent:
  - The size after k years of a biological population;
  - The value of an investment subject to certain interest and tax conditions.

Discrete Dynamical Systems and Orbits

- An iterative scheme $\{f^k\}$ is called a **discrete dynamical system**.
- We are interested in the behavior of the **sequence of iterates**, or **orbits**, $\{f^k(x)\}_{k=1}^{\infty}$ for various initial points $x \in D$.
- Of special interest is the asymptotic behavior (as $k \to \infty$).

  Example: Let $f(x) = \cos x$.

  Consider any x.

  The sequence $f^k(x)$ converges to $0.739\ldots$ as $k \to \infty$.

  We can discover this by repeatedly pressing the cosine button on a calculator.

Asymptotic Behavior

- Sometimes the distribution of iterates appears almost random.
- Alternatively, $f^k(x)$ may converge to a **fixed point** w, i.e., a point of D with $f(w) = w$.
- More generally, $f^k(x)$ may converge to an orbit of **period-p points** $\{w, f(w), \ldots, f^{p-1}(w)\}$, where p is the least positive integer with $f^p(w) = w$, in the sense that $|f^k(x) - f^k(w)| \to 0$ as $k \to \infty$.
- Sometimes, however, $f^k(x)$ may appear to move about at random, but always remaining close to a certain set, which may be a fractal.

Attractors

- We shall call a subset F of D an **attractor** for f if:
  - F is a closed set;
  - F is **invariant** under f, i.e., such that $f(F) = F$;
  - The distance from $f^k(x)$ to F converges to zero as k tends to infinity, for all x in an open set V containing F.
- The largest such open set V satisfying the last condition above is called the **basin of attraction** of F.
- It is usual to require that F is minimal in the sense that it has no proper subset satisfying these conditions.

Repellers

- Consider a function $f : D \to D$.
- Denote by f^{-1} the (perhaps multi-valued) inverse of f.
- We shall call a subset F of D a **repeller** for f if:
  - F is a closed set;
  - F is invariant under f;
  - The distance from $(f^{-1})^k(x)$ to F converges to zero as k tends to infinity, for all x in an open set V containing F.
- So a repeller is a closed invariant set F from which all nearby points not in F are iterated away from F.
- An attractor or repeller may just be a single point or a period-p orbit.
- However, even relatively simple maps f can have fractal attractors.

A Candidate Attractor Set

- Note that $f(D) \subseteq D$.
- So
$$f^k(D) \subseteq f^{k-1}(D) \subseteq \cdots \subseteq f(D) \subseteq D.$$
- It follows that
$$\bigcap_{i=1}^{k} f^i(D) = f^k(D).$$
- Thus, the set
$$F = \bigcap_{k=1}^{\infty} f^k(D)$$
is invariant under f.
- Now $f^k(x) \in \bigcap_{i=1}^{k} f^i(D)$, for all $x \in D$.
- So the iterates $f^k(x)$ approach F as $k \to \infty$.
- Thus, F is often an attractor of f.

Chaotic Behavior

- Very often, if f has a fractal attractor or repeller F, then f exhibits "chaotic" behavior on F.
- f would be regarded as chaotic on F if the following hold:
  (i) The orbit $\{f^k(x)\}$ is dense in F, for some $x \in F$.
  (ii) The periodic points of f in F (points for which $f^p(x) = x$, for some positive integer p) are dense in F.
  (iii) f has **sensitive dependence on initial conditions**.
  That is, there is a number $\delta > 0$, such that, for every x in F, there are points y in F arbitrarily close to x, such that
  $$|f^k(x) - f^k(y)| \geq \delta, \quad \text{for some } k.$$
  Thus, points that are initially close do not remain close under iterates of f.

Chaotic Behavior (Cont'd)

- Implications of the conditions:
  - Condition (i) implies that F cannot be decomposed into smaller closed invariant sets;
  - Condition (ii) suggests a skeleton of regularity in the structure of F;
  - Condition (iii) reflects the unpredictability of iterates of points on F.
- Condition (iii) implies that accurate long-term numerical approximation to orbits of f is impossible, since a tiny numerical error is magnified under iteration.

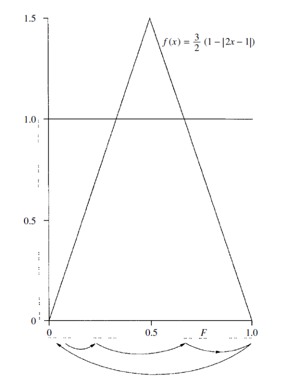
Example: Repellers as Attractors

- The mapping $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{3}{2}(1 - |2x - 1|)$$

is called the **tent map** because of the form of its graph.

f maps $\mathbb{R}$ in a two-to-one manner onto $(-\infty, \frac{3}{2})$.



Example: Repellers as Attractors (Cont'd)

- Define an iterated function system $S_1, S_2 : [0,1] \to [0,1]$ by the contractions
$$\begin{aligned} S_1(x) &= \tfrac{1}{3}x; \\ S_2(x) &= 1 - \tfrac{1}{3}x. \end{aligned}$$
Then, for $0 \leq x \leq 1$,
$$\begin{aligned} f(S_1(x)) &= \tfrac{3}{2}(1 - |2\tfrac{1}{3}x - 1|) \\ &= \tfrac{3}{2}(1 - (1 - \tfrac{2}{3}x)) \\ &= x; \\ f(S_2(x)) &= \tfrac{3}{2}(1|2(1 - \tfrac{1}{3}x) - 1|) \\ &= \tfrac{3}{2}(1 - |1 - \tfrac{3}{2}x|) \\ &= \tfrac{3}{2}(1 - 1 + \tfrac{2}{3}x) \\ &= x. \end{aligned}$$
Thus S_1 and S_2 are the two branches of f^{-1}.

Example: Repellers as Attractors (Cont'd)

- We started with
$$f(x) = \frac{3}{2}(1 - |2x - 1|).$$
We defined the two branches of f^{-1},
$$S_1(x) = \frac{1}{3}x; \quad S_2(x) = 1 - \frac{1}{3}x.$$
A previous theorem implies that there is a unique non-empty compact attractor $F \subseteq [0,1]$ satisfying $F = S_1(F) \cup S_2(F)$.
Write $S(E) = S_1(E) \cup S_2(E)$, for any set E.
Then F is given by
$$F = \bigcap_{k=0}^{\infty} S^k([0,1]).$$
Clearly the attractor F is the middle third Cantor set.
It has Hausdorff and box dimensions $\frac{\log 2}{\log 3}$.
It follows from $F = S_1(F) \cup S_2(F)$ that $f(F) = F$.

Example: Repellers as Attractors (Cont'd)

Claim: F is a repeller.

Suppose $x < 0$.

$$f(x) = \frac{3}{2}(1 - |2x - 1|) = \frac{3}{2}(1 - (-2x + 1)) = 3x.$$

So $f^k(x) = 3^k x \to -\infty$ as $k \to \infty$.

Suppose $x > 1$.

$$f(x) = \frac{3}{2}(1 - |2x - 1|) = \frac{3}{2}(1 - (2x - 1)) = 3(1 - x) < 0.$$

Again $f^k(x) \to -\infty$.

If $x \in [0, 1] \backslash F$, then for some k, we have

$$x \notin S^k[0, 1] = \bigcup \{S_{i_1} \circ \cdots \circ S_{i_k}[0, 1] : i_j = 1, 2\}.$$

So $f^k(x) \notin [0, 1]$. Again $f^k(x) \to -\infty$ as $k \to \infty$.

All points outside F are iterated to $-\infty$. So F is a repeller.

The Chaotic Nature of f

- Denote the points of F by $x_{i_1,i_2,\ldots}$ with $i_j = 1, 2$.
  If $i_1 = i_1', \ldots, i_k = i_k'$,
  $$|x_{i_1,i_2,\ldots} - x_{i_1',i_2',\ldots}| \leq 3^{-k}.$$
  Note that $x_{i_1,i_2,\ldots} = S_{i_1}(x_{i_2,i_3,\ldots})$.
  It follows that
  $$f(x_{i_1,i_2,\ldots}) = x_{i_2,i_3,\ldots}.$$
  Suppose that $(i_1, i_2, \ldots)$ is an infinite sequence with every finite sequence of 1s and 2s appearing as a consecutive block of terms.
  Example:
  $$(1, 2,\ 1, 1,\ 1, 2,\ 2, 1,\ 2, 2,\ 1, 1, 1,\ 1, 1, 2, \ldots)$$
  where the spacing is just to indicate the form of the sequence.

The Chaotic Nature of f (Cont'd)

- For each point $x_{i'_1,i'_2,\ldots}$ in F and each integer q, we may find k, such that $(i'_1, i'_2, \ldots, i'_q) = (i_{k+1}, \ldots, i_{k+q})$. Then

$$|x_{i_{k+1},i_{k+2},\ldots} - x_{i'_1,i'_2,\ldots}| < 3^{-q}.$$

So the iterates

$$f^k(x_{i_1,i_2,\ldots}) = x_{i_{k+1},i_{k+2},\ldots}$$

come arbitrarily close to each point of F for suitable large k.

So f has dense orbits in F.

Similarly, $x_{i_1,\ldots,i_k,i_1,\ldots,i_k,i_1,\ldots}$ is a periodic point of period k.

So the periodic points of f are dense in F.

The Chaotic Nature of f (Cont'd)

- The iterates have sensitive dependence on initial conditions.

  In fact, on the one hand,

$$f^k(x_{i_1,\ldots,i_k,1,\ldots}) \in \left[0, \frac{1}{3}\right].$$

  And, on the other,

$$f^k(x_{i_1,\ldots,i_k,2,\ldots}) \in \left[\frac{2}{3}, 1\right].$$

  So Conditions (i)-(iii) specifying chaotic behavior of f on F are satisfied.

  We conclude that F is a chaotic repeller for f.

- The study of f by its effect on points of F represented by sequences $(i_1, i_2, \ldots)$ is known as **symbolic dynamics**.

IFS and Dynamical Systems

- Suppose $S_1, \ldots, S_m$ is a set of bijective contractions on a domain D.
- Suppose they have an attractor F, with $S_1(F), \ldots, S_m(F)$ disjoint.
- Then F is a repeller for any mapping f, such that, for x is near $S_i(F)$,

$$f(x) = S_i^{-1}(x).$$

- By examining the effect of f on the point $x_{i_1,i_2,\ldots}$, it may be shown that f acts chaotically on F.
- For many dynamical systems f, it is possible to decompose the domain D into parts, such that the branches of f^{-1} on each part look rather like an iterated function system.
- Such a decomposition of the domain is called **Markov partition**.

Subsection 7

General Theory of Julia Sets

Complex Polynomials and Iterates

- Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $n \geq 2$ with complex coefficients,
$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0.$$
- With minor modifications, the theory remains true if f is a rational function $f(z) = \frac{p(z)}{q(z)}$, where p, q are polynomials, on the extended complex plane $\mathbb{C} \cup \{\infty\}$.
- Much of the theory holds if f is any meromorphic function, that is, a function that is analytic on $\mathbb{C}$ except at isolated poles.
- We write f^k for the k-fold composition $f \circ \cdots \circ f$ of the function f.
- So $f^k(w)$ is the k-th iterate $f(f(\cdots(f(w))\cdots))$ of w.

Julia Sets and Fatou Sets

- Julia sets are defined in terms of the behavior of $f^k(z)$ for large k.
- The **filled-in Julia set** of the polynomial f is defined by

$$K(f) = \{z \in \mathbb{C} : f^k(z) \not\to \infty\}.$$

- The **Julia set** of f is the boundary of the filled-in Julia set,

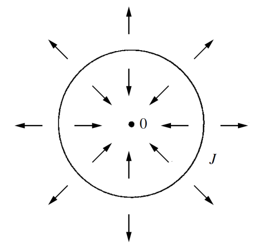
$$J(f) = \partial K(f).$$

- We write K for $K(f)$ and J for $J(f)$ when the function is clear.
- We have $z \in J(f)$ if, in every neighborhood of z, there are points w and v, such that $f^k(w) \to \infty$ and $f^k(v) \not\to \infty$.
- The **Fatou set** or **stable set** $F(f)$ is the complement of the Julia set.

Example

- Let $f(z) = z^2$. Then $f^k(z) = z^{2k}$.

  We have:
  - If $|z| < 1$, $f^k(z) \to 0$ as $k \to \infty$;
  - If $|z| > 1$, $f^k(z) \to \infty$;
  - If $|z| = 1$, $f^k(z)$ remains on the circle $|z| = 1$, for all k.



  Thus, the filled-in Julia set K is the unit disc $|z| \leq 1$.

  The Julia set J is its boundary, the unit circle, $|z| = 1$.

  The Julia set J is the boundary between the sets of points which iterate to 0 and ∞.

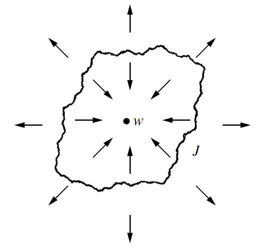
  Of course, in this special case, J is not a fractal.

Example

- Suppose that we modify the preceding example slightly, taking

$$f(z) = z^2 + c, \ c \text{ a small complex number.}$$

It can be shown that:
- If z is small, $f^k(z) \to w$, where w is the fixed point of f close to 0;
- If z is large, $f^k(z) \to \infty$.



Again, the Julia set is the boundary between these two types of behavior.

However, it turns out that now J is a fractal curve.

Fixed-Points and Periodic Points

- If $f(w) = w$, we call w a **fixed point** of f.
- If $f^p(w) = w$, for some $p \geq 1$, we call w a **periodic point** of f.
- The least such p is called the **period** of w.
- We call $w, f(w), \ldots, f^p(w)$ a **period** p **orbit**.

Attractive and Repelling Points

- Let w be a periodic point of period p, with

$$(f^p)'(w) = \lambda,$$

  where the prime denotes complex differentiation.
  - The point w is called **attractive** if $0 \leq |\lambda| < 1$, in which case nearby points are attracted to the orbit under iteration by f;
  - The point w is called **repelling** if $|\lambda| > 1$, in which case points close to the orbit move away.
- The study of sequences $f^k(z)$ for various initial z is known as **complex dynamics**.
- The position of z relative to the Julia set $J(f)$ is a key to this behavior.

Complex Polynomials and Unboundedness

Lemma

Given a polynomial

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0,$$

there exists a number r, such that if $|z| \geq r$, then $|f(z)| \geq 2|z|$. In particular, if $|f^m(z)| \geq r$, for some $m \geq 0$, then $f^k(z) \to \infty$ as $k \to \infty$. Thus, either $f^k(z) \to \infty$ or $\{f^k(z) : k = 0, 1, 2, \ldots\}$ is a bounded set.

- We may choose r sufficiently large to ensure that if $|z| \geq r$, then

$$\frac{1}{2}|a_n||z|^n \geq 2|z|$$

and

$$(|a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0|) \leq \frac{1}{2}|a_n||z|^n.$$

Complex Polynomials and Unboundedness (Cont'd)

- Then, if $|z| \geq r$,

$$\begin{array}{rcl} |f(z)| & \geq & |a_n||z|^n - (|a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0|) \\ & \geq & \frac{1}{2}|a_n||z|^n \\ & \geq & 2|z|. \end{array}$$

Furthermore, suppose $|f^m(z)| \geq r$, for some m.

Applying this inductively, we get

$$|f^{m+k}(z)| \geq 2^m|f^k(z)| \geq r.$$

So $f^k(z) \to \infty$.

Structure of Julia Sets

Proposition

Let $f(z)$ be a polynomial. Then the filled in Julia set $K(f)$ and the Julia set $J(f)$ are non-empty and compact, with $J(f) \subseteq K(f)$. Furthermore, $J(f)$ has an empty interior.

- Consider r given by the preceding lemma.

  By the lemma, K is contained in the disc $B(0, r)$.

  So K is bounded. Hence, its boundary J is bounded.

  If $z \notin K$, then $f^k(z) \to \infty$. So $|f^m(z)| > r$, for some integer m.

  By continuity of f^m, $|f^m(w)| > r$, for all w in a sufficiently small disc centered at z. By the preceding lemma, for such w, $f^k(w) \to \infty$.

  Thus, $w \notin K$. Hence, the complement of K is open. So K is closed.

  As the boundary of K, the Julia set J is closed and contained in K.

  Thus K and J are closed and bounded. So they are compact.

Structure of Julia Sets (Cont'd)

- The equation $f(z) = z$ has at least one solution z_0.

  So $f^k(z_0) = z_0$, for all k.

  This shows that $z_0 \in K$ and K is non-empty.

  Let $z_1 \in \mathbb{C} \backslash K$.

  Then, for some $0 \leq \lambda \leq 1$, the point $\lambda z_0 + (1-\lambda) z_1$, lying on the line joining z_0 and z_1, will be on the boundary of K.

  Taking λ as the infimum value for which $\lambda z_0 + (1-\lambda) z_1 \in K$ will do.

  Thus, $J = \partial K$ is non-empty.

  Finally, suppose U is a non-empty open subset of $J \subseteq K$.

  Then U lies in the interior of K.

  Therefore it has empty intersection with its boundary J.

  This contradicts $\emptyset \neq U \subseteq J$.

Invariance of J Under f and f^{-1}

Proposition

The Julia set $J = J(f)$ of f is forward and backward invariant under f, i.e., $J = f(J) = f^{-1}(J)$.

- Let $z \in J$. Then $f^k(z) \not\to \infty$.

  There exist $w_n \to z$ with $f^k(w_n) \to \infty$ as $k \to \infty$, for all n.

  Thus, we have:
  - $f^k(f(z)) \not\to \infty$;
  - $f^k(f(w_n)) \to \infty$.

  Moreover, by continuity of f, $f(w_n)$ can be chosen as close as we like to $f(z)$. Thus, $f(z) \in J$. So $f(J) \subseteq J$.

  This also implies
  $$J \subseteq f^{-1}(f(J)) \subseteq f^{-1}(J).$$

Invariance of J Under f and f^{-1} (Cont'd)

- Similarly, let z and w_n be as above and $f(z_0) = z$.

  Using the mapping properties of polynomials on $\mathbb{C}$, we may find $v_n \to z_0$ with $f(v_n) = w_n$.

  Hence, as $k \to \infty$:
  - $f^k(z_0) = f^{k-1}(z) \not\to \infty$;
  - $f^k(v_n) = f^{k-1}(w_n) \to \infty$.

  So $z_0 \in J$. Thus, $f^{-1}(J) \subseteq J$.

  This implies
  $$J = f(f^{-1}(J)) \subseteq f(J).$$

Julia Sets of Iterates

Proposition

$J(f^p) = J(f)$ for every positive integer p.

- By a previous lemma, either $f^k(z) \to \infty$ or $\{f^k(z) : k = 0, 1, 2, \ldots\}$ is a bounded set.

  This implies that

  $$f^k(z) \to \infty \quad \text{if and only if} \quad (f^p)^k(z) = f^{kp}(z) \to \infty.$$

  Thus f and f^p have identical filled-in Julia sets.

  Consequently, they also have identical Julia sets.

Normal Families of Functions

- Let U be an open subset of $\mathbb{C}$.
- Recall that a complex function is *analytic* on U if it is differentiable on U in the complex sense.
- Let $g_k : U \to \mathbb{C}$, $k = 1, 2, \ldots$ be a family of complex analytic functions.
- The family $\{g_k\}$ is said to be **normal** on U if every sequence of functions selected from $\{g_k\}$ has a subsequence which converges uniformly on every compact subset of U, either to a bounded analytic function or to ∞.
- This means that the subsequence converges either to a finite analytic function or to ∞ on each connected component of U.
- Note that, in the former case, the derivatives of the subsequence must converge to the derivative of the limit function.

Families of Functions Normal at a Point

- Let $g_k : U \to \mathbb{C}$, $k = 1, 2, \ldots$ be a family of complex analytic functions.
- The family $\{g_k\}$ is **normal at the point** w of U if, there is some open subset V of U containing w, such that $\{g_k\}$ is a normal family on V.
- This is equivalent to there being a neighborhood V of w on which every sequence $\{g_k\}$ has a subsequence convergent to a bounded analytic function or to ∞.

Montel's Theorem

- The key result which we will use in our development of Julia sets is the remarkable theorem of Montel, which asserts that non-normal families of functions take virtually all complex values.

Montel's Theorem

Let $\{g_k\}$ be a family of complex analytic functions on an open domain U. If $\{g_k\}$ is not a normal family, then for all $w \in \mathbb{C}$, with at most one exception, there exists $z \in U$ and k, such that

$$g_k(z) = w.$$

Characterization of Julia Sets

Proposition

$J(f) = \{z \in \mathbb{C} : \text{the family } \{f^k\} \text{ is not normal at } z\}$.

- Suppose $z \in J$. Then, in every neighborhood V of z, there are points w, such that $f^k(w) \to \infty$, whilst $f^k(z)$ remains bounded.
  Thus, no subsequence of $\{f^k\}$ is uniformly convergent on V.
  So $\{f^k\}$ is not normal at z.
  Suppose that $z \notin J$.
  - Assume, first, $z \in \text{int}K$. Let V be open, with $z \in V \subseteq \text{int}K$.
    Then $f^k(w) \in K$, for all $w \in V$ and all k.
    By Montel's Theorem $\{f^k\}$ is normal at w.
  - Suppose, next, $z \in \mathbb{C} \setminus K$.
    Then $|f^k(z)| > r$ for some k, where r is given by a previous lemma.
    So $|f^k(w)| > r$, for all w in some neighborhood V of z.
    By the same lemma, $f^k(w) \to \infty$ uniformly on V.
    So, again, $\{f^k\}$ is normal at w.

Mixing of f Near $J(f)$

Lemma

Let f be a polynomial, let $w \in J(f)$ and let U be any neighborhood of w. Then, for each $j = 1, 2, \ldots$, the set $W \equiv \bigcup_{k=j}^{\infty} f^k(U)$ is the whole of $\mathbb{C}$, except possibly for a single point. Any such exceptional point is not in $J(f)$, and is independent of w and U.

- By the preceding proposition, the family $\{f^k\}_{k=j}^{\infty}$ is not normal at w.
  So the first part follows immediately by Montel's Theorem.
  Suppose $v \notin W$. Assume $f(z) = v$.
  Since $f(W) \subseteq W$, it follows that $z \notin W$.
  Now $\mathbb{C} \setminus W$ consists of at most one point. So $z = v$.
  But f is a polynomial of degree n.
  Moreover, the only solution of $f(z) - v = 0$ is v.

Mixing of f Near $J(f)$ (Cont'd)

- It follows that
$$f(z) - v = c(z - v)^n,$$
for some constant c.
If z is sufficiently close to v, then
$$f^k(z) - v \to 0 \text{ as } k \to \infty.$$
Moreover, convergence is uniform on, say,
$$\{z : |z - v| < (2c)^{-1/(n-1)}\}.$$
Thus $\{f^k\}$ is normal at v.
So the exceptional point $v \notin J(f)$.
Clearly v only depends on the polynomial f.
In fact, if W omits a point v of $\mathbb{C}$, then $J(f)$ is the circle with center v and radius $c^{-1/(n-1)}$.

Towards Generating Pictures of Julia Sets

Corollary

(a) The following holds for all $z \in \mathbb{C}$ with at most one exception.

If U is an open set intersecting $J(f)$ then $f^{-k}(z)$ intersects U for infinitely many values of k.

(b) If $z \in J(f)$, then $J(f)$ is the closure of $\bigcup_{k=1}^{\infty} f^{-k}(z)$.

(a) Unless z is the exceptional point of the lemma, $z \in f^k(U)$.

Thus, $f^{-k}(z)$ intersects U, for infinitely many k.

Towards Generating Pictures of Julia Sets (Cont'd)

(b) If $z \in J(f)$, then $f^{-k}(z) \subseteq J(f)$, by a previous proposition.

It follows that

$$\bigcup_{k=1}^{\infty} f^{-k}(z) \subseteq J(f).$$

Hence, the closure of the union is contained in the closed set $J(f)$.

Conversely, let U be an open set containing $z \in J(f)$.

Then $f^{-k}(z)$ intersects U for some k, by Part (a).

(By the preceding lemma, z cannot be the exceptional point.)

So z is in the closure of $\bigcup_{k=1}^{\infty} f^{-k}(z)$.

$J(f)$ is Perfect

Proposition

$J(f)$ is a perfect set (i.e., closed and with no isolated points) and is therefore uncountable.

- Let $v \in J(f)$ and let U be a neighborhood of v.

  We must show that U contains other points of $J(f)$.

  We consider three cases.

  (i) Suppose, first, v is not a fixed or periodic point of f.
  By the preceding corollary and a previous proposition, U contains a point of $f^{-k}(v) \subseteq J(f)$, for some $k \geq 1$.
  This point must be different from v.

$J(f)$ is Perfect (Cont'd)

(ii) Suppose, next, $f(v) = v$.
Suppose $f(z) = v$ has no solution other than v.
Just as in the proof of the preceding lemma, $v \notin J(f)$.
Thus, there exists $w \neq v$, with $f(w) = v$.
By the preceding corollary, U contains a point u of $f^{-k}(w) = f^{-k-1}(v)$, for some $k \geq 1$.
Any such u is in $J(f)$, by backward invariance.
Moreover, it is distinct from v, since $f^k(v) = v \neq w = f^k(u)$.

(iii) Assume, finally, $f^p(v) = v$, for some $p > 1$.
By a previous proposition, $J(f) = J(f^p)$.
By applying Part (ii) to f^p, we see that U contains points of $J(f^p) = J(f)$ other than v.

Thus $J(f)$ has no isolated points.

Since it is closed, it is perfect.

Finally, every perfect set is uncountable.

$J(f)$ as the Closure of Repelling Periodic Points

Theorem

If f is a polynomial, $J(f)$ is the closure of the repelling periodic points of f.

- Let w be a repelling periodic point of f of period p.

  So w is a repelling fixed point of $g = f^p$.

  Suppose that $\{g^k\}$ is normal at w.

  Then w has an open neighborhood V on which a subsequence $\{g^{k_i}\}$ converges to a finite analytic function g_0 (it cannot converge to ∞ since $g^k(w) = w$ for all k).

  By a standard result from complex analysis, the derivatives also converge,

$$(g^{k_i})'(z) \to g_0'(z), \quad z \in V.$$

$J(f)$ as the Closure of Repelling Periodic Points (Cont'd)

- We have, for all $z \in V$,

$$(g^{k_i})'(z) \to g_0'(z), \quad z \in V.$$

By the chain rule, $|(g^{k_i})'(w)| = |(g'(w))^{k_i}|$.

But w is a repelling fixed point and $|g'(w)| > 1$.

So we get $|(g^{k_i})'(w)| = |(g'(w))^{k_i}| \to \infty$.

This contradicts the finiteness of $g_0'(w)$.

So $\{g_k\}$ cannot be normal at w.

By a previous proposition, $w \in J(g) = J(f^p) = J(f)$.

Since $J(f)$ is closed, it follows that the closure of the repelling periodic points is in $J(f)$.

$J(f)$ as the Closure of Repelling Periodic Points (Cont'd)

- Define

$$E = \{w \in J(f) : \text{exists } v \neq w \text{ with } f(v) = w \text{ and } f'(v) \neq 0\}.$$

Suppose that $w \in E$.

Then there is an open neighborhood V of w on which we may find a local analytic inverse $f^{-1} : V \to \mathbb{C} \backslash V$ so that $f^{-1}(w) = v \neq w$ (just choose values of $f^{-1}(z)$ in a continuous manner).

Define a family of analytic functions $\{h_k\}$ on V by

$$h_k(z) = \frac{f^k(z) - z}{f^{-1}(z) - z}.$$

Let U be any open neighborhood of w, with $U \subseteq V$.

Since $w \in J(f)$, the family $\{f^k\}$ is not normal on U.

Thus, by the definition, the family $\{h_k\}$ is not normal on U.

$J(f)$ as the Closure of Repelling Periodic Points (Cont'd)

- By Montel's theorem, $h_k(z)$ must take either the value 0 or 1 for some k and $z \in U$.
  - In the first case $f^k(z) = z$, for some $z \in U$.
  - In the second $f^k(z) = f^{-1}(z)$.
    So $f^{k+1}(z) = z$, for some $z \in U$.

  Thus, U contains a periodic point of f.

  So w is in the closure of the repelling periodic points, for all $w \in E$.

  But f is a polynomial.

  So E contains all of $J(f)$ except for a finite number of points.

  By the preceding proposition, $J(f)$ contains no isolated points.

  So $J(f) \subseteq \overline{E}$ is a subset of the closure of the repelling periodic points.

Basin of Attraction

- If w is an attractive fixed point of f, we write

$$A(w) = \{z \in \mathbb{C} : f^k(z) \to w \text{ as } k \to \infty\}$$

  for the **basin of attraction of** w.
- The **basin of attraction of infinity**, $A(\infty)$, is defined in the same way.
- Since w is attractive, there is an open set V containing w in $A(w)$.

  If $w = \infty$, we may take $\{z : |z| > r\}$, for sufficiently large r.
- This implies that $A(w)$ is open.

  Suppose $z \in A(w)$.

  Then $f^k(z) \in V$, for some k, where $V \subseteq A(w)$ is open.

  So $z \in f^{-k}(V)$, which is open.

$J(f)$ as the Boundary of a Basin of Attraction

Lemma

Let w be an attractive fixed point of f. Then $\partial A(w) = J(f)$. The same is true if $w = \infty$.

- If $z \in J(f)$, then $f^k(z) \in J(f)$ for all k.

  So it cannot converge to an attractive fixed point.

  Thus, $z \notin A(w)$.

  Suppose U is any neighborhood of z.

  The set $f^k(U)$ contains points of $A(w)$, for some k, by a previous lemma.

  So there are points arbitrarily close to z that iterate to w.

  Thus, $z \in \overline{A(w)}$.

  So $z \in \partial A(w)$.

$J(f)$ as the Boundary of a Basin of Attraction (Converse)

- Suppose $z \in \partial A(w)$ but $z \notin J(f)$.

  Then z has a connected open neighborhood V on which $\{f^k\}$ has a subsequence convergent either to an analytic function or to ∞.

  The subsequence converges to w on $V \cap A(w)$, which is open and nonempty.

  But an analytic function is constant on a connected set if it is constant on any open subset.

  Therefore this subsequence converges on V.

  All points of V are mapped into $A(w)$ by iterates of f.

  So $V \subseteq A(w)$. This contradicts $z \in \partial A(w)$.

  Example: Recall the case $f(z) = z^2$.

  The Julia set is the unit circle.

  It is the boundary of both $A(0)$ and $A(\infty)$.

Summary and Remarks on Chaotic Behavior

Summary

The Julia set $J(f)$ of the polynomial f is the boundary of the set of points $z \in \mathbb{C}$, such that $f^k(z) \to \infty$. It is an uncountable non-empty compact set containing no isolated points, and is invariant under f and f^{-1}, and $J(f) = J(f^p)$, for each positive integer p. If $z \in J(f)$, then $J(f)$ is the closure of $\bigcup_{k=1}^{\infty} f^{-k}(z)$. The Julia set is the boundary of the basin of attraction of each attractive fixed point of f, including ∞, and is the closure of the repelling periodic points of f.

- This collects together the results of this section.
- It may be shown that "f acts chaotically on J".
  - Periodic points of f are dense in J;
  - J contains points z with iterates $f^k(z)$ that are dense in J.
  - f has "sensitive dependence on initial conditions" on J.
    Thus $|f^k(z) - f^k(w)|$ will be large for certain k, regardless of how close $z, w \in J$ are, making accurate computation of iterates impossible.

Subsection 8

Quadratic Functions: The Mandelbrot Set

Quadratic Polynomials

- We study Julia sets of polynomials of the form $f_c(z) = z^2 + c$.
- This is not as restrictive as it first appears.
- Let $h(z) = \alpha z + \beta$, $\alpha \neq 0$.
- Then $f^{-1}(z) = \frac{z-\beta}{\alpha}$
- So we get

$$\begin{aligned} h^{-1}(f_c(h(z))) &= h^{-1}(f_c(\alpha z + \beta)) \\ &= h^{-1}(\alpha^2 z^2 + 2\alpha\beta z + \beta^2 + c) \\ &= \frac{\alpha^2 z^2 + 2\alpha\beta z + \beta^2 + c - \beta}{\alpha}. \end{aligned}$$

- By choosing appropriate values of α, β and c we can make this expression into any quadratic function f that we please.
- Then $h^{-1} \circ f_c \circ h = f$.
- So $h^{-1} \circ f_c^k \circ h = f^k$, for all k.

Quadratic Polynomials (Cont'd)

- We found that, for any quadratic function f,

$$h^{-1} \circ f_c^k \circ h = f^k, \quad \text{for all } k.$$

- This means that the sequence of iterates $\{f^k(z)\}$ of a point z under f is just the image under h^{-1} of the sequence of iterates $\{f_c^k(h(z))\}$ of the point $h(z)$ under f_c.
- The mapping h transforms the dynamical picture of f to that of f_c.
- In particular, $f^k(z) \to \infty$ if and only if $f_c^k(z) \to \infty$.
- Thus, the Julia set of f is the image under h^{-1} of the Julia set of f_c.

Conjugacy and Branches of f_c^{-1}

- The transformation h is called a **conjugacy** between f and f_c.
- Any quadratic function is conjugate to f_c for some c.
- So, by studying the Julia sets of f_c for $c \in \mathbb{C}$, we effectively study the Julia sets of all quadratic polynomials.
- Since h is a similarity transformation, the Julia set of any quadratic polynomial is geometrically similar to that of f_c, for some $c \in \mathbb{C}$.
- When $z \neq c$, $f_c^{-1}(z)$ takes two distinct values

$$\pm(z-c)^{1/2}.$$

- These are called the two **branches** of $f_c^{-1}(z)$.
- Thus, if U is a small open set with $c \notin U$, then:
    - The pre-image $f_c^{-1}(U)$ has two parts,;
    - Both parts are mapped bijectively and smoothly by f_c onto U.

The Mandelbrot Set

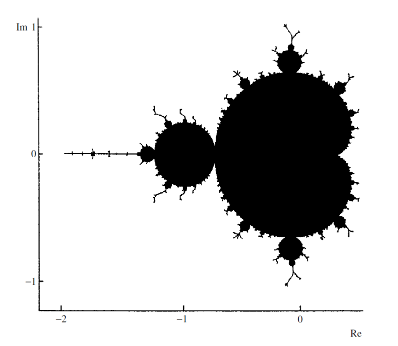
- We define the **Mandelbrot set** M to be the set of parameters c for which the Julia set of f_c is connected

$$M = \{c \in \mathbb{C} : J(f_c) \text{ is connected}\}.$$

- At first, M appears to relate to one rather specific property of $J(f_c)$.
- As we will see, M contains an enormous amount of information about the structure of Julia sets.

An Equivalent Definition

- The definition of M is awkward for computational purposes.
- We show that $c \in M$ if and only if $f_c^k(0) \not\to \infty$.
- This equivalent definition is much more useful for:
  - Determining whether a parameter c lies in M;
  - Investigating the extraordinarily intricate form of M.



Loops, Interior and Exterior

- A curve in the complex plane is:
  - **Smooth** if it is differentiable;
  - **Simple** if it is non-self-intersecting.
- A **loop** is a smooth, closed, simple curve in the complex plane.
- We refer to the parts of $\mathbb{C}$ inside and outside such a curve as the **interior** and **exterior** of the loop.
- A **figure of eight** is a smooth closed curve with a single point of self-intersection.

Inverse Action as related to Loops

Lemma

Let C be a loop in the complex plane.

(a) If c is inside C then $f_c^{-1}(C)$ is a loop, with the inverse image of the interior of C as the interior of $f_c^{-1}(C)$.

(b) If c lies on C then $f_c^{-1}(C)$ is a figure of eight with self-intersection at 0, such that the inverse image of the interior of C is the interior of the two loops.

(c) If c is outside C, then $f_c^{-1}(C)$ comprises two disjoint loops, with the inverse image of the interior of C the interior of the two loops.

- Note that $f_c^{-1}(z) = \pm(z-c)^{1/2}$ and $(f_c^{-1})'(z) = \pm\frac{1}{2}(z-c)^{-1/2}$.

  The latter is finite and non-zero, if $z \neq c$.

  Hence, if we select one of the two branches of f_c^{-1}, the set $f_c^{-1}(C)$ is locally a smooth curve, provided $c \notin C$.

Inverse Action as related to Loops (Part (a))

(a) Suppose c is inside C.

Take an initial point w on C.

Choose one of the two values for $f_c^{-1}(w)$.

Allow $f_c^{-1}(z)$ to vary continuously as z moves around C.

The point $f_c^{-1}(z)$ traces out a smooth curve.

When z returns to w, however, $f_c^{-1}(w)$ takes its second value.

As z traverses C again, $f_c^{-1}(z)$ continues on its smooth path.

The path closes as z returns to w the second time.

Now $c \notin C$.

So $0 \notin f_c^{-1}(C)$.

It follows that $f_c'(z) \neq 0$ on $f_c^{-1}(C)$.

Thus, f_c is locally smooth and bijective near points on $f_c^{-1}(C)$.

Inverse Action as related to Loops (Part (a) Cont'd)

- f_c is locally smooth and bijective near points on $f_c^{-1}(C)$.

  $f_c(z)$ cannot be a self-intersection point of C.

  So $z \in f_c^{-1}(C)$ cannot be a point of self-intersection of $f_c^{-1}(C)$.

  Thus, $f^{-1}(C)$ is a loop.

  But f_c is a continuous function that maps the loop $f_c^{-1}(C)$ and no other points onto the loop C.

  So the polynomial f_c must map the interior and exterior of $f_c^{-1}(C)$ into the interior and exterior of C, respectively.

  Hence, f_c^{-1} maps the interior of C to the interior of $f_c^{-1}(C)$.

Inverse Action as related to Loops (Parts (b) and (c))

(b) This is proved in a similar way to Part (a).
Suppose C_0 is a smooth piece of curve through c.
Then $f_c^{-1}(C_0)$ consists of two smooth pieces of curve through 0.
These pieces cross at right angles.
So they provide the self-intersection of the figure of eight.

(c) This is similar to Part (a).
$f_c^{-1}(z)$ can only pick up one of the two values, as z moves around C.
So we get two loops.

Fundamental Theorem of the Mandelbrot Set

Theorem

$$\begin{aligned} M &= \{c \in \mathbb{C} : \{f_c^k(0)\}_{k\geq 1} \text{ bounded}\} \\ &= \{c \in \mathbb{C} : f_c^k(0) \not\to \infty \text{ as } k \to \infty\}. \end{aligned}$$

We provide a sketch of the proof based on the lemma.

(a) We show that if $\{f_c^k(0)\}$ is bounded then $J(f_c)$ is connected.

Let C be a large circle in $\mathbb{C}$ such that:

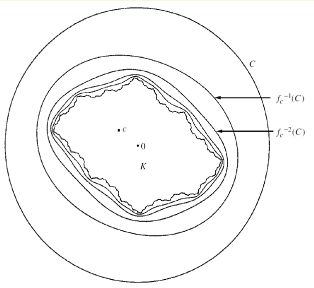
- All the points $\{f_c^k(0)\}$ lie inside C;
- $f_c^{-1}(C)$ is interior to C;
- Points outside C iterate to ∞ under f_c^k.

Now $c = f_c(0)$ is inside C.

Thus, Part (a) of the lemma gives that $f_c^{-1}(C)$ is a loop contained in the interior of C.

Fundamental Theorem of the Mandelbrot Set (Cont'd)

- Also, $f_c(c) = f_c^2(0)$ is inside C.
  Moreover, f_c^{-1} maps the exterior of C onto the exterior of $f_c^{-1}(C)$.
  So c is inside $f_c^{-1}(C)$.
  By Part (a) of the lemma, $f_c^{-2}(C)$ is a loop contained in the interior of $f_c^{-1}(C)$.
  Proceeding in this way, $\{f_c^{-k}(C)\}$ consists of a sequence of loops, each containing the next in its interior.



Fundamental Theorem of the Mandelbrot Set (Cont'd)

- Let K denote the closed set of points that are on or inside the loops $f_c^{-k}(C)$, for all k.

  If $z \in \mathbb{C} \backslash K$, some iterate $f_c^k(z)$ lies outside C.

  So $f_c^k(z) \to \infty$.

  Thus,
  $$A(\infty) = \{z : f_c^k(z) \to \infty \text{ as } k \to \infty\} = \mathbb{C} \backslash K.$$

  So K is the filled in Julia set of f_c.

  By a previous lemma, $J(f_c)$ is the boundary of $\mathbb{C} \backslash K$.

  This is, of course, the same as the boundary of K.

  But K is the intersection of a decreasing sequence of closed simply connected sets (i.e., connected with a connected complement).

  So, by a simple topological argument, K is simply connected.

  Therefore, K has a connected boundary.

  Thus, $J(f_c)$ is connected.

Fundamental Theorem of the Mandelbrot Set (Part (b))

(b) We now show that $J(f_c)$ is not connected if $\{f_c^k(0)\}$ is unbounded.

Let C be a large circle such that:

- $f_c^{-1}(C)$ is inside C;
- All points outside C iterate to ∞;
- For some p, the point $f_c^{p-1}(c) = f_c^p(0) \in C$ with $f_c^k(0)$ inside or outside C according as to whether k is less than or greater than p.

Just as in the first part of the proof, we construct a series of loops $\{f_c^{-k}(C)\}$, each containing the next in its interior.

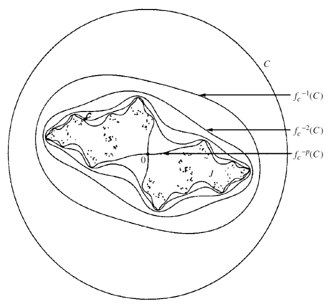
But the argument breaks down when we get to the loop $f_c^{1-p}(C)$.

We have $c \in f_c^{1-p}(C)$ and Part (a) of the lemma does not apply.

By Part (b), we get that:

- $E \equiv f^{-p}(C)$ is a figure of eight inside the loop $f_c^{1-p}(C)$;
- f_c maps the interior of each half of E onto the interior of $f_c^{1-p}(C)$.

Fundamental Theorem of the Mandelbrot Set (Cont'd)



The Julia set $J(f_c)$ must lie in the interior of the loops of E, since other points iterate to infinity. But $J(f_c)$ is invariant under f_c^{-1}.

So parts of it must be contained in each of the loops of E.

Thus, this figure of eight E disconnects $J(f_c)$.

In fact, by applying Part (c) of the previous lemma in the same way, we can see that $J(f_c)$ is totally disconnected.

Comments

- The reason for considering iterates of the origin in the theorem is that the origin is the critical point of f_c for each c, i.e., the point for which
$$f_c'(z) = 0.$$
- The critical points are the points where f_c fails to be a local bijection.
- This is the property that was crucial in distinguishing the two cases in the proof of the theorem.

Pictures of the Mandelbrot Set

- The equivalent definition of M provided by the theorem is the basis of computer pictures of the Mandelbrot set.
- Choose numbers $r > 2$ and k_0 of the order of 100, say.
- For each c compute successive terms of the sequence $\{f_c^k(0)\}$ until one of the following two cases occurs:
  - $|f_c^k(0)| \geq r$.
    In this case c is deemed to be outside M;
  - $k = k_0$.
    In this case we take $c \in M$.
- Repeating this process for values of c across a region enables a picture of M to be drawn.
- Often colors are assigned to the complement of M according to the first integer k such that $|f_c^k(0)| \geq r$.

Properties of the Mandelbrot Set

- The Mandelbrot set has a highly complicated form.
  - It has a main cardioid to which a series of prominent circular "buds" are attached.
  - Each of these buds is surrounded by further buds, and so on.
  - In addition, fine, branched "hairs" grow outwards from the buds.
  - These hairs carry miniature copies of the entire Mandelbrot set along their length.
- The Mandelbrot set is connected.
- Its boundary has Hausdorff dimension 2, a reflection on its intricacy.

Subsection 9

Julia Sets of Quadratic Functions

Dimension of the Julia Set

Theorem

Suppose $|c| > \frac{1}{4}(5+2\sqrt{6}) = 2.475\ldots$. Then $J(f_c)$ is totally disconnected, and is the attractor of the contractions given by the two branches of

$$f_c^{-1}(z) = \pm(z-c)^{1/2}, \quad \text{for } z \text{ near } J.$$

When $|c|$ is large,

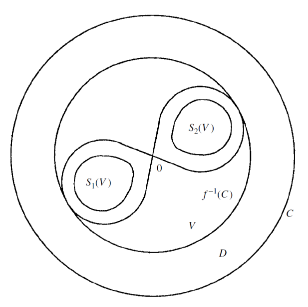
$$\dim_B J(f_c) = \dim_H J(f_c) \simeq \frac{2\log 2}{\log 4|c|}.$$

- Let C be the circle $|z| = |c|$ and D its interior $|z| < |c|$. Then

  $$f_c^{-1}(C) = \{(ce^{i\theta} - c)^{1/2} : 0 \leq \theta \leq 4\pi\}.$$

  This is a figure of eight with self-intersection point at 0.
  Its loops are on either side of a straight line through the origin.

Dimension of the Julia Set (Cont'd)



- By hypothesis, $|c| > 2$. Assume $|z| > |c|$. Then we have

$$|f_c(z)| \geq |z^2| - |c| \geq |c|^2 - |c| > |c|.$$

Therefore, $f_c^{-1}(C) \subseteq D$.
The interior of each of the loops of $f_c^{-1}(C)$ is mapped by f_c in a bijective manner onto D.

Dimension of the Julia Set (Cont'd)

- Define $S_1, S_2 : D \to D$ as the branches of $f_c^{-1}(z)$ inside each loop.
  Then $S_1(D)$ and $S_2(D)$ are the interiors of the two loops.
  Let V be the disc
  $$V = \{z : |z| < |2c|^{1/2}\}.$$
  We have chosen the radius of V so that V just contains $f_c^{-1}(C)$.
  So $S_1(D), S_2(D) \subseteq V \subseteq D$.
  Hence $S_1(V), S_2(V) \subseteq V$, with $S_1(\overline{V})$ and $S_2(\overline{V})$ disjoint.

Dimension of the Julia Set (Cont'd)

- Now we have, for $i = 1, 2$,

$$\begin{array}{rcl} |S_i(z_1) - S_i(z_2)| & = & |(z_1 - c)^{1/2} - (z_2 - c)^{1/2}| \\ & = & \frac{|z_1 - z_2|}{|(z_1 - c)^{1/2} + (z_2 - c)^{1/2}|}. \end{array}$$

Hence, if $z_1, z_2 \in \overline{V}$, taking least and greatest values,

$$\frac{1}{2}(|c| + |2c|^{1/2})^{-1/2} \leq \frac{|S_i(z_1) - S_i(z_2)|}{|z_1 - z_2|} \leq \frac{1}{2}(|c| - |2c|^{1/2})^{-1/2}.$$

The upper bound is less than 1, if $|c| > \frac{1}{4}(5 + 2\sqrt{6})$.
In this case S_1 and S_2 are contractions on the disc $\overline{V}$.
By a previous theorem, there is a unique non-empty compact attractor $F \subseteq \overline{V}$ satisfying $S_1(F) \cup S_2(F) = F$.
$S_1(\overline{V})$ and $S_2(\overline{V})$ are disjoint. So $S_1(F)$ and $S_2(F)$ are disjoint.
Thus, F is totally disconnected.

Dimension of the Julia Set (Cont'd)

- F is none other than the Julia set $J = J(f_c)$.

  To see this, note that $\overline{V}$ contains at least one point z of J (for example, a repelling fixed point of f_c).

  Taking into account $f_c^{-k}(\overline{V}) \subseteq \overline{V}$, we have

$$J = \text{closure}\left(\bigcup_{k=1}^{\infty} f_c^{-k}(z)\right) \subseteq \overline{V}.$$

  Using previous results, J is a non-empty compact subset of $\overline{V}$ satisfying $J = f_c^{-1}(J)$ or, equivalently, $J = S_1(J) \cup S_2(J)$.

  Thus $J = F$, the unique non-empty compact set satisfying

$$S_1(F) \cup S_2(F) = F.$$

Dimension of the Julia Set (Cont'd)

- Finally, we estimate the dimension of $J(f_c) = F$.

  By previous propositions, lower and upper bounds for $\dim_H J(f_c)$ are provided by the solutions of

$$2\left(\frac{1}{2}(|c| \pm |2c|^{1/2})^{-1/2}\right)^s = 1.$$

  That is, by

$$s = \frac{2\log 2}{\log 4}(|c| \pm |2c|^{1/2}).$$

  This gives the stated asymptotic estimate.

The Case of Small c

- We next turn to the case where c is small.

  We know that, if $c = 0$, then $J(f_c)$ is the unit circle.

  Suppose c is small.
  - If z is small enough, then $f_c^k(z) \to w$ as $k \to \infty$, where w is the attractive fixed point $\frac{1}{2}(1 - \sqrt{1-4c})$ close to 0;
  - If z is large, $f_c^k(z) \to \infty$.
- The circle "distorts" into a simple closed curve (i.e., having no points of self-intersection) separating these two types of behavior as c moves away from 0, provided that f_c retains an attractive fixed point, i.e., if $|f_c'(z)| < 1$ at one of the roots of $f_c(z) = z$.
- This happens if c lies inside the cardioid $z = \frac{1}{2}e^{i\theta}(1 - \frac{1}{2}e^{i\theta})$, $0 \leq \theta \leq 2\pi$, the main cardioid of the Mandelbrot set.
- For convenience, we treat the case of $|c| < \frac{1}{4}$, but the proof is easily modified if f_c has any attractive fixed point.

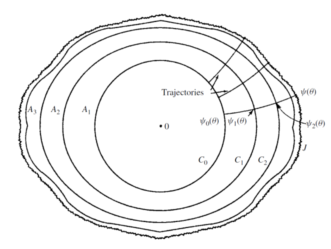
Julia Sets for Small c

Theorem

If $|c| < \frac{1}{4}$, then $J(f_c)$ is a simple closed curve.

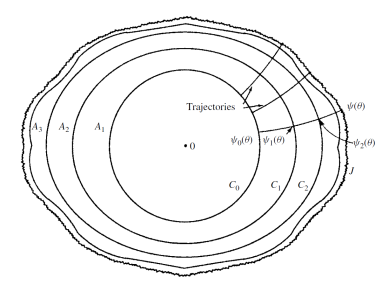
- Let C_0 be the curve $|z| = \frac{1}{2}$, which encloses both c and the attractive fixed point w of f_c.

  By direct calculation, the inverse image $f_c^{-1}(C_0)$ is a loop C_1 surrounding C_0. We may fill the annular region A_1 between C_0 and C_1 by a continuum of curves, which we call "trajectories", which leave C_0 and reach C_1 perpendicularly.



  For each θ, let $\psi_1(\theta)$ be the point on C_1 at the end of the trajectory leaving C_0 at $\psi_0(\theta) = \frac{1}{2}e^{i\theta}$.

Julia Sets for Small c (Cont'd)



- The inverse image $f_c^{-1}(A_1)$ is an annular region A_2, with:
  - Outer boundary the loop $C_2 = f_c^{-1}(C_1)$;
  - Inner boundary C_1.

  f_c maps A_2 onto A_1 in a two-to-one manner.

  The inverse image of the trajectories joining C_0 to C_1 provides a family of trajectories joining C_1 to C_2.

  $\psi_2(\theta) :=$ point on C_2 at the end of the trajectory leaving C_1 at $\psi_1(\theta)$.

Julia Sets for Small c (Cont'd)

- We continue in this way to get:
  - A sequence of loops C_k, each surrounding its predecessor;
  - Families of trajectories joining the points $\psi_k(\theta)$ on C_k to $\psi_{k+1}(\theta)$ on C_{k+1}, for each k.

  As $k \to \infty$, the curves C_k approach the boundary of the basin of attraction of w.

  By a previous lemma, this boundary is just the Julia set $J(f_c)$.

  Since $|f_c'(z)| > \gamma$, for some $\gamma > 1$ outside C_1, it follows that f_c^{-1} is contracting near J.

  Thus, the length of the trajectory joining $\psi_k(\theta)$ to $\psi_{k+1}(\theta)$ converges to 0 at a geometric rate as $k \to \infty$.

  Consequently, $\psi_k(\theta)$ converges uniformly to a continuous function $\psi(\theta)$ as $k \to \infty$.

  It follows that J is the closed curve given by $\psi(\theta)$, $0 \leq \theta \leq 2\pi$.

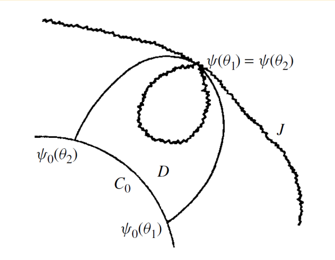
Julia Sets for Small c (Cont'd)

- It remains to show that ψ represents a simple curve. Suppose that $\psi(\theta_1) = \psi(\theta_2)$. Let D be the region bounded by C_0 and the two trajectories joining $\psi(\theta_1)$ and $\psi(\theta_2)$ to this common point.

  The boundary of D remains bounded under iterates of f_c.

  So by the maximum modulus theorem (that the modulus of an analytic function takes its maximum on the boundary point of a region) D remains bounded under iteration of f.

  Thus D is a subset of the filled-in Julia set. So the interior of D cannot contain any points of J. Thus the situation of the figure on the right cannot occur. So $\psi(\theta) = \psi(\theta_1) = \psi(\theta_2)$, for all θ between θ_1 and θ_2. It follows that $\psi(\theta)$ has no point of self-intersection.



Dimension of the Julia Set for Small c

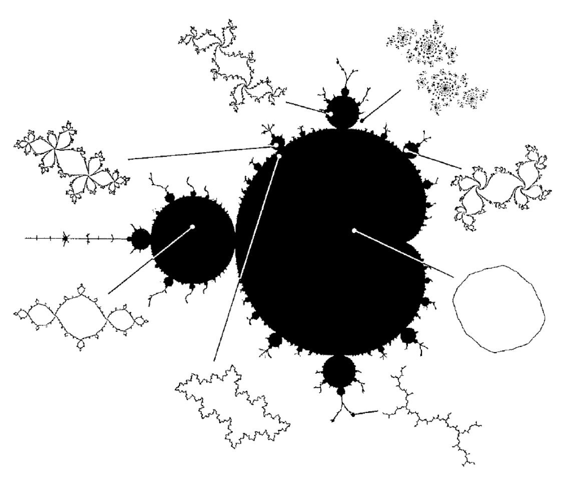
- By an extension of this argument, if c is in the main cardioid of M, then $J(f_c)$ is a simple closed curve.
- Such curves are sometimes referred to as **quasi-circles**.
- Of course, $J(f_c)$ will be a fractal curve if $c > 0$.
- It may be shown that, for small c, its dimension is given by

$$\begin{aligned} s &= \dim_B J(f_c) = \dim_H J(f_c) \\ &= 1 + \frac{|c|^2}{4 \log 2} + \text{terms in } |c|^3 \text{ and higher powers.} \end{aligned}$$

- Moreover, $0 < \mathcal{H}^s(J) < \infty$, with $\dim_B J(f_c) = \dim_H J(f_c)$ given by a real analytic function of c.

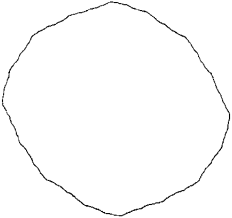
Examples

- Julia sets $J(f_c)$ for c at various points in the Mandelbrot set.

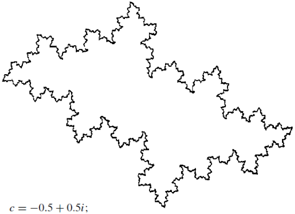


Examples

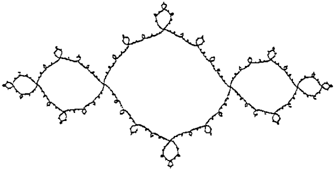
- Julia sets of the quadratic function $f_c(z) = z^2 + c$.



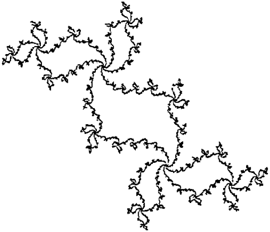
$c = -0.1 + 0.1i$; f_c has an attractive fixed point, and J is a quasi-circle.



$c = -0.5 + 0.5i$;
f_c has an attractive fixed point, and J is a quasi-circle.



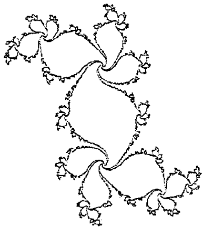
$c = -1 + 0.05i$; f_c has an attractive period-2 orbit.



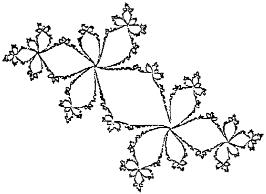
$c = -0.2 + 0.75i$, f_c has an attractive period-3 orbit.

Examples

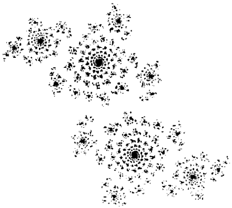
- Julia sets of the quadratic function $f_c(z) = z^2 + c$.



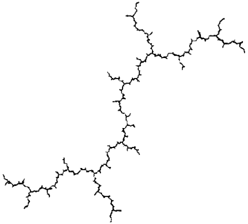
$c = 0.25 + 0.52i$; f_c has an attractive period-4 orbit.



$c = -0.5 + 0.55i$; f_c has an attractive period-5 orbit.



$c = 0.66i$; f_c has no attractive orbits and J is totally disconnected.



$c = -i$, $f_c^2(0)$ is periodic and J is a dendrite