Introduction to Functional Analysis

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Subsection 1

Metric Spaces

Metric Spaces and Metrics

- The symbol $\times$ denotes the **Cartesian product** of sets: $A \times B$ is the set of all ordered pairs (a,b), where $a \in A$ and $b \in B$.

  Hence $X \times X$ is the set of all ordered pairs of elements of X.

Definition (Metric Space, Metric)

A **metric space** is a pair (X,d), where X is a set and d is a **metric on** X (or **distance function on** X), that is, a function defined on $X \times X$, such that for all $x,y,z \in X$, we have:

(M1) d is real-valued, finite and nonnegative.

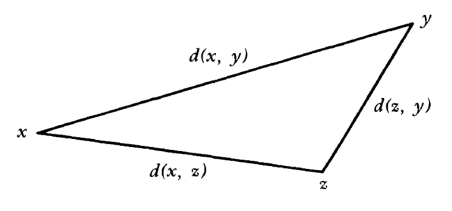
(M2) $d(x,y) = 0$ if and only if $x = y$.

(M3) $d(x,y) = d(y,x)$. (**Symmetry**)

(M4) $d(x,y) \leq d(x,z) + d(z,y)$. (**Triangle Inequality**)

Comments on the Definition of Metric Spaces

- X is usually called the **underlying set** of (X, d). Its elements are called **points**.
- For fixed x, y we call the nonnegative number $d(x, y)$ the **distance** from x to y.
- Properties (M1) to (M4) are the **axioms of a metric**.
- The name "triangle inequality" is motivated by elementary geometry:



- From (M4) we obtain by induction the **generalized triangle inequality** $d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$.
- Instead of (X, d) we may simply write X if no confusion is likely.

Subspaces and Induced Metrics

- A **subspace** $(Y, \widetilde{d})$ of (X, d) is obtained if we take a subset $Y \subseteq X$ and restrict d to $Y \times Y$.
- Thus, the metric on Y is the restriction $\widetilde{d} = d|_{Y \times Y}$.
- $\widetilde{d}$ is called the **metric induced on Y by d**.

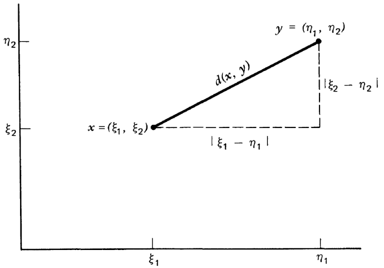
  Example: (**Real line** $\mathbb{R}$) This is the set of all real numbers $\mathbb{R}$, taken with the usual metric defined by

$$d(x, y) = |x - y|.$$

The Euclidean Plane

- (**Euclidean plane** $\mathbb{R}^2$) The metric space $\mathbb{R}^2$, called the **Euclidean plane**, is obtained if we take the set $\mathbb{R}^2$ of ordered pairs of real numbers, written $x=(\xi_1,\xi_2)$, $y=(\eta_1,\eta_2)$, etc., and the **Euclidean metric** defined by

$$d(x,y)=\sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2}.$$



- Another metric space is obtained if we choose the same set as before, but another metric d_1 defined by

$$d_1(x,y)=|\xi_1-\eta_1|+|\xi_2-\eta_2|.$$

  d_1 is sometimes called the **taxicab metric**.
- Hence, from a given set (having more than one element) we can obtain various metric spaces by choosing different metrics.
- $\mathbb{R}^2$ is sometimes denoted by E^2.

Higher Dimensional Euclidean Spaces

- (**Three-dimensional Euclidean space** $\mathbb{R}^3$) This metric space consists of the set $\mathbb{R}^3$ of ordered triples of real numbers $x=(\xi_1,\xi_2,\xi_3)$, $y=(\eta_1,\eta_2,\eta_3)$ etc., and the Euclidean metric defined by

$$d(x,y)=\sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2+(\xi_3-\eta_3)^2}.$$

- (n-**Dimensional Euclidean space** $\mathbb{R}^n$) The previous examples are special cases of n-dimensional Euclidean space $\mathbb{R}^n$. This space is obtained if we take the set $\mathbb{R}^n$ of all ordered n-tuples of real numbers, written $x=(\xi_1,\ldots,\xi_n)$, $y=(\eta_1,\ldots,\eta_n)$, etc., and the Euclidean metric defined by

$$d(x,y)=\sqrt{(\xi_1-\eta_1)^2+\cdots+(\xi_n-\eta_n)^2}.$$

Unitary Spaces and Complex Plane

- (**unitary space** $\mathbb{C}^n$) n-**dimensional unitary space** $\mathbb{C}^n$ is the space $\mathbb{C}^n$ of all ordered n-tuples of complex numbers with metric defined by
$$d(x,y) = \sqrt{|\xi_1 - \eta_1|^2 + \cdots + |\xi_n - \eta_n|^2}.$$
- (**complex plane** $\mathbb{C}$) When $n = 1$ this is the **complex plane** $\mathbb{C}$ with the usual metric defined by
$$d(x,y) = |x - y|.$$
- $\mathbb{C}^n$ is also called **complex Euclidean** n-**space**.

The Sequence Space ℓ^∞

- (**Sequence space ℓ^∞**) As a set X, we take the set of all bounded sequences of complex numbers, i.e., every element of X is a complex sequence $x = (\xi_1, \xi_2, \ldots)$, briefly $x = (\xi_j)$, such that for all $j = 1, 2, \ldots$, we have $|\xi_j| \le c_x$, where c_x is a real number which may depend on x, but does not depend on j.

  We choose the metric defined by

$$d(x,y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j|,$$

  where $y = (\eta_j) \in X$ and $\mathbb{N} = \{1, 2, \ldots\}$, and sup denotes the supremum (least upper bound).

  The metric space thus obtained is generally denoted by ℓ^∞.
- ℓ^∞ is a **sequence space** because each element of X (each point of X) is a sequence.

The Function Space $C[a,b]$

- (**Function space** $C[a,b]$) As a set X we take the set of all real-valued functions $x, y, \ldots$ which are functions of an independent real variable t and are defined and continuous on a given closed interval $J=[a,b]$.

  Choosing the metric defined by

  $$d(x,y)=\max_{t\in J}|x(t)-y(t)|,$$

  where max denotes the maximum, we obtain a metric space which is denoted by $C[a,b]$ (C suggesting "continuous").
- This is a **function space** because every point of $C[a,b]$ is a function.
- A significant difference between calculus and functional analysis is that:
  - in calculus, we consider a single function or a few functions at a time;
  - in functional analysis, a function becomes a single point in a large space.

Discrete Metric Spaces

- (**Discrete metric space**) We take any set X and on it the so-called **discrete metric** for X, defined by

$$d(x,x)=0, \quad d(x,y)=1, \text{ if } x\neq y.$$

  This space (X,d) is called a **discrete metric space**.

- Discrete metric spaces rarely occur in applications.

  They are used, however, in examples for illustrating certain concepts.

Subsection 2

Further Examples of Metric Spaces

Sequence Space s

- This space consists of the set of all (bounded or unbounded) sequences of complex numbers and the metric d defined by
$$d(x,y)=\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{|\xi_j-\eta_j|}{1+|\xi_j-\eta_j|},$$
where $x=(\xi_j)$ and $y=(\eta_j)$.
- It is easy to see that Axioms (M1) to (M3) are satisfied.
- To verify (M4), consider the function f defined on $\mathbb{R}$ by $f(t)=\frac{t}{1+t}$.
Since $f'(t)=\frac{1}{(1+t)^2}>0$, f is increasing.
Consequently, $|a+b|\leq|a|+|b|$ implies $f(|a+b|)\leq f(|a|+|b|)$.
Now we get
$$\frac{|a+b|}{1+|a+b|}\leq\frac{|a|+|b|}{1+|a|+|b|}=\frac{|a|}{1+|a|+|b|}+\frac{|b|}{1+|a|+|b|}\leq\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}.$$

Sequence Space s (Cont'd)

- We got
$$\frac{|a+b|}{1+|a+b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$
In this inequality take $a=\xi_j-\zeta_j$ and $b=\zeta_j-\eta_j$, where $z=(\zeta_j)$. Then, since $a+b=\xi_j-\eta_j$,
$$\frac{|\xi_j-\eta_j|}{1+|\xi_j-\eta_j|} \le \frac{|\xi_j-\zeta_j|}{1+|\xi_j-\zeta_j|} + \frac{|\zeta_j-\eta_j|}{1+|\zeta_j-\eta_j|}.$$
Multiply both sides by $\frac{1}{2^j}$ and sum over j from 1 to ∞, to get
$$d(x,y) \le d(x,z) + d(z,y).$$
So (M4) holds and s is a metric space.

Space $B(A)$ of Bounded Functions

- By definition, each element $x \in B(A)$ is a function defined and bounded on a given set A, and the metric is defined by

$$d(x,y) = \sup_{t \in A} |x(t) - y(t)|,$$

  where sup denotes the supremum.
- We write $B[a,b]$ for $B(A)$ in the case of an interval $A = [a,b] \subseteq \mathbb{R}$.
- We show that $B(A)$ is a metric space.
  - Clearly, (M1) and (M3) hold.
  - For (M2), note, first, that $d(x,x) = 0$.
    Conversely, suppose $d(x,y) = 0$. This implies $x(t) - y(t) = 0$, for all $t \in A$. Hence, $x = y$.

Space $B(A)$ of Bounded Functions (Cont'd)

- For (M4), note that for every $t \in A$,

$$\begin{array}{rcl} |x(t) - y(t)| & \leq & |x(t) - z(t)| + |z(t) - y(t)| \\ & \leq & \max_{t \in A} |x(t) - z(t)| + \max_{t \in A} |z(t) - y(t)|. \end{array}$$

  This shows that $x - y$ is bounded on A.

  The bound given by the expression on the right does not depend on t.

  Taking the supremum on the left, we obtain (M4).

Space ℓ^p

- Let $p \geq 1$ be a fixed real number. By definition, each element in the space ℓ^p is a sequence $x = (\xi_j) = (\xi_1, \xi_2, \ldots)$ of numbers such that $|\xi_1|^p + |\xi_2|^p + \cdots$ converges; thus, $\sum_{j=1}^{\infty} |\xi_j|^p < \infty$.
  The metric is defined by
  $$d(x,y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{1/p},$$
  where $y = (\eta_j)$ and $\sum |\eta_j|^p < \infty$.
- If we take only real sequences, we get the **real space** ℓ^p.
- If we take complex sequence, we get the **complex space** ℓ^p.
- In the case $p = 2$ we have the famous **Hilbert sequence space** ℓ^2 with metric defined by
  $$d(x,y) = \sqrt{\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2}.$$

Strategy for Proving ℓ^p is a Metric Space

- Clearly,

$$d(x,y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{1/p}$$

satisfies (M1) to (M3) provided the series on the right converges.

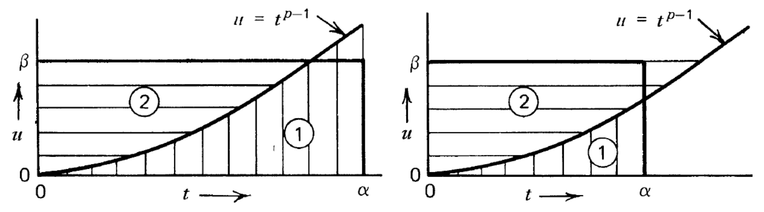
We shall prove that it does converge and that (M4) is satisfied.

Proceeding stepwise, we shall derive

(a) an auxiliary inequality;
(b) the Hölder inequality from (a);
(c) the Minkowski inequality from (b);
(d) the triangle inequality (M4) from (c).

The Auxiliary Inequality

- Two numbers $p, q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$ are called **conjugate exponents**.
- Note that $1 = \frac{p+q}{pq} \Rightarrow pq = p + q \Rightarrow (p-1)(q-1) = 1 \Rightarrow \frac{1}{p-1} = q - 1$.
- As a consequence, $u = t^{p-1}$ implies $t = u^{q-1}$.
- Let α and β be any positive numbers. Since $\alpha\beta$ is the area of the rectangle, we get, by integration:



$$\alpha\beta \leq \int_0^\alpha t^{p-1} dt + \int_0^\beta u^{q-1} du = \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

- This inequality is trivially true if $\alpha = 0$ or $\beta = 0$.

The Hölder Inequality

- Let $(\widetilde{\xi}_j)$ and $(\widetilde{\eta}_j)$ be such that $\sum|\widetilde{\xi}_j|^p = 1$, $\sum|\widetilde{\eta}_j|^q = 1$.
  By the preceding inequality, with $\alpha = |\widetilde{\xi}_j|$ and $\beta = |\widetilde{\eta}_j|$,
  $$|\widetilde{\xi}_j\widetilde{\eta}_j| \leq \frac{1}{p}|\widetilde{\xi}_j|^p + \frac{1}{q}|\widetilde{\eta}_j|^q \overset{\text{sum over } j}{\Rightarrow} \sum|\widetilde{\xi}_j\widetilde{\eta}_j| \leq \frac{1}{p} + \frac{1}{q} = 1.$$
- Take any nonzero $x = (\xi_j) \in \ell^p$ and $y = (\eta_j) \in \ell^q$ and set $\widetilde{\xi}_j = \frac{\xi_j}{(\sum|\xi_k|^p)^{1/p}}$, $\widetilde{\eta}_j = \frac{\eta_j}{(\sum|\eta_m|^q)^{1/q}}$. Then $\sum|\widetilde{\xi}_j|^p = 1$ and $\sum|\widetilde{\eta}_j|^q = 1$.
  So, by what was shown above,
  $$\sum\left|\frac{\xi_j}{(\sum|\xi_k|^p)^{1/p}}\frac{\eta_j}{(\sum|\eta_m|^q)^{1/q}}\right| \leq 1.$$
  Equivalently, we have the **Hölder inequality** for sums
  $$\sum_{j=1}^{\infty}|\xi_j\eta_j| \leq (\sum_{k=1}^{\infty}|\xi_k|^p)^{1/p}(\sum_{m=1}^{\infty}|\eta_m|^q)^{1/q}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1.$$

The Hölder Inequality

- We proved the Hölder inequality for sums:

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \le \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} \left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{1/q}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1.$$

- If $p = 2$, then $q = 2$, we get the **Cauchy-Schwarz inequality** for sums:

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \le \sqrt{\sum_{k=1}^{\infty} |\xi_k|^2} \sqrt{\sum_{m=1}^{\infty} |\eta_m|^2}.$$

The Minkowski Inequality

- We prove the **Minkowski inequality** for sums

$$(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p)^{1/p} \leq (\sum_{k=1}^{\infty} |\xi_k|^p)^{1/p} + (\sum_{m=1}^{\infty} |\eta_m|^p)^{1/p},$$

where $x = (\xi_j) \in \ell^p$ and $y = (\eta_j) \in \ell^p$, and $p \geq 1$.

For $p = 1$, use the triangle inequality for numbers.

Let $p > 1$. Write $\xi_j + \eta_j = \omega_j$.

By the triangle inequality,

$$|\omega_j|^p = |\xi_j + \eta_j||\omega_j|^{p-1} \leq (|\xi_j| + |\eta_j|)|\omega_j|^{p-1}.$$

Summing from $j = 1$ to a fixed n,

$$\sum |\omega_j|^p \leq \sum |\xi_j||\omega_j|^{p-1} + \sum |\eta_j||\omega_j|^{p-1}.$$

The Minkowski Inequality (Cont'd)

- We obtained $\sum |\omega_j|^p \leq \sum |\xi_j||\omega_j|^{p-1} + \sum |\eta_j||\omega_j|^{p-1}$.
  Apply the Hölder inequality to the first sum on the right

  $$\sum |\xi_j||\omega_j|^{p-1} \leq [\sum |\xi_k|^p]^{1/p} [\sum (|\omega_m|^{p-1})^q]^{1/q}.$$

  Since $pq = p+q$, $(p-1)q = p$.
  Similarly, we obtain

  $$\sum |\eta_j||\omega_j|^{p-1} \leq [\sum |\eta_k|^p]^{1/p} [\sum |\omega_m|^p]^{1/q}.$$

  Together,

  $$\sum |\omega_j|^p \leq ([\sum |\xi_k|^p]^{1/p} + [\sum |\eta_k|^p]^{1/p})(\sum |\omega_m|^p)^{1/q}.$$

  Dividing by the last factor on the right and noting that $1 - \frac{1}{q} = \frac{1}{p}$, we obtain the inequality with n instead of ∞.
  Now let $n \to \infty$. On the right, the two series converge since $x, y \in \ell^p$.
  So the series on the left also converges, and yields the result.

The Triangle Inequality in ℓ^p

- **The Triangle Inequality**: Taking any $x, y, z \in \ell^p$, with $x = (\xi_j)$, $y = (\eta_j)$ and $z = (\zeta_j)$, we now obtain

$$\begin{array}{rcl} d(x,y) & = & (\sum |\xi_j - \eta_j|^p)^{1/p} \\ & \leq & (\sum [|\xi_j - \zeta_j| + |\zeta_j - \eta_j|]^p)^{1/p} \\ & \leq & (\sum |\xi_j - \zeta_j|^p)^{1/p} + (\sum |\zeta_j - \eta_j|^p)^{1/p} \\ & = & d(x,z) + d(z,y). \end{array}$$

  This completes the proof that ℓ^p is a metric space.

Subsection 3

Open Set, Closed Set, Neighborhood

Open and Closed Balls and Spheres

- We first consider important types of subsets of a given metric space $X = (X, d)$.

Definition (Ball and Sphere)

Given a point $x_0 \in X$ and a real number $r > 0$, we define three types of sets:

(a) $B(x_0; r) = \{x \in X : d(x, x_0) < r\}$ (**Open Ball**)

(b) $\widetilde{B}(x_0; r) = \{x \in X : d(x, x_0) \le r\}$ (**Closed Ball**)

(c) $S(x_0; r) = \{x \in X : d(x, x_0) = r\}$ (**Sphere**)

In all three cases, x_0 is called the **center** and r the **radius**.

- We see that an open ball of radius r is the set of all points in X whose distance from the center of the ball is less than r.
- The definition immediately implies that $S(x_0; r) = \widetilde{B}(x_0; r) - B(x_0; r)$.

Open and Closed Sets, Neighborhoods and Interior

Definition (Open and Closed Set)

A subset M of a metric space X is said to be **open** if it contains a ball about each of its points. A subset K of X is said to be **closed** if its complement (in X) is open, that is, $K^C = X - K$ is open.

- An open ball is an open set.

  Consider $B(x_0;\varepsilon)$ and $x \in B(x_0;\varepsilon)$.

  Choose positive $\delta < \varepsilon - d(x_0, x)$.

  Then $B(x;\delta) \subseteq B(x_0;\varepsilon)$.
- A closed ball is a closed set.

  Consider $\widetilde{B}(x_0;\varepsilon)$ and $x \in \widetilde{B}(x_0;\varepsilon)^C$.

  Choose positive $\delta < d(x_0, x) - \varepsilon$.

  Then $B(x;\delta) \subseteq B(x_0;\varepsilon)^C$.

Neighborhoods

- An open ball $B(x_0;\varepsilon)$ is often called an ε-**neighborhood** of x_0.
- By a **neighborhood** of x_0 we mean any subset of X which contains an ε-neighborhood of x_0.
- Every neighborhood of x_0 contains x_0.
- If N is a neighborhood of x_0 and $N \subseteq M$, then M is also a neighborhood of x_0.

Interior

- We call x_0 an **interior point** of a set $M \subseteq X$ if M is a neighborhood of x_0.
- The **interior** of M is the set of all interior points of M and is denoted by M^0 or $\text{Int}(M)$.
- $\text{Int}(M)$ is open.
- $\text{Int}(M)$ is, in fact, the largest open set contained in M.

Properties of Open Sets

- The collection of all open subsets of X, call it $\mathscr{T}$, has the following properties:
  (T1) $\varnothing \in \mathscr{T}$, $X \in \mathscr{T}$;
  (T2) The union of any members of $\mathscr{T}$ is a member of $\mathscr{T}$;
  (T3) The intersection of finitely many members of $\mathscr{T}$ is a member of $\mathscr{T}$.

(T1) Follows by noting that $\varnothing$ is open since $\varnothing$ has no elements.

Obviously, X is open.

(T2) Any point x of the union U of open sets belongs to (at least) one of these sets, call it M. M contains a ball B about x since M is open. Then $B \subseteq U$, by the definition of a union.

(T3) Finally, if y is any point of the intersection of open sets $M_1, \ldots, M_n$, then each M_i contains a ball about y. A smallest of these balls is contained in that intersection.

Topological Spaces and Continuous Functions

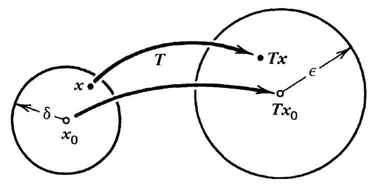
- We define a **topological space** $(X, \mathscr{T})$ to be a set X and a collection $\mathscr{T}$ of subsets of X, such that $\mathscr{T}$ satisfies the axioms (T1)-(T3).

  The set $\mathscr{T}$ is called a **topology** for X.
- A metric space is a topological space, according to the preceding slide.

Definition (Continuous Mapping)

Let $X = (X, d)$ and $Y = (Y, \widetilde{d})$ be metric spaces.

A mapping $T : X \to Y$ is said to be **continuous at a point** $x_0 \in X$ if, for every $\varepsilon > 0$, there is a $\delta > 0$, such that $d(x, x_0) < \delta$ implies $\widetilde{d}(Tx, Tx_0) < \varepsilon$.



T is said to be **continuous** if it is continuous at every point of X.

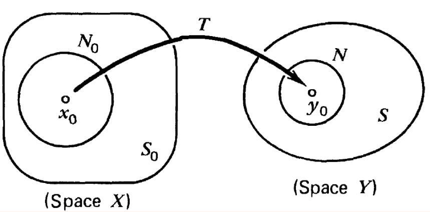
The Continuous Mapping Theorem

Theorem (Continuous Mapping)

A map T of a metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X.

- Suppose that T is continuous. Let $S \subseteq Y$ be open and S_0 the inverse image of S. If $S_0 = \emptyset$, it is open.

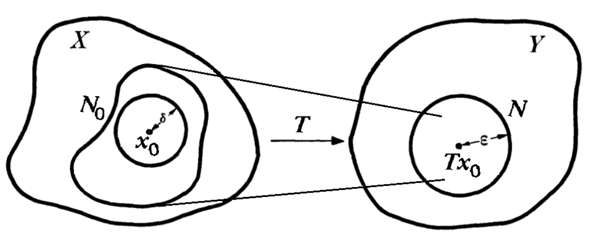
  Let $S_0 \neq \emptyset$. For any $x_0 \in S_0$, let $y_0 = Tx_0$. Since S is open, it contains an ε-neighborhood N of y_0. Since T is continuous, x_0 has a δ-neighborhood N_0 which is mapped into N.

  Since $N \subseteq S$, we have $N_0 \subseteq S_0$, Since $x_0 \in S_0$ was arbitrar, S_0 is open.

The Continuous Mapping Theorem (Converse)

- Conversely, assume that the inverse image of every open set in Y is an open set in X. Then for every $x_0 \in X$ and any ε-neighborhood N of Tx_0, the inverse image N_0 of N is open, since N is open, and N_0 contains x_0.



Hence N_0 also contains a δ-neighborhood of x_0, which is mapped into N because N_0 is mapped into N. Consequently, by the definition, T is continuous at x_0. Since $x_0 \in X$ was arbitrary, T is continuous.

Accumulation Points and Closures

- Let M be a subset of a metric space X.

  A point x_0 of X (which may or may not be a point of M) is called an **accumulation point** of M (or **limit point** of M) if every neighborhood of x_0 contains at least one point $y \in M$ distinct from x_0.
- The set consisting of the points of M and the accumulation points of M is called the **closure** of M and is denoted by $\overline{M}$.
- The closure of $\overline{M}$ of M is the smallest closed set containing M.

Dense and Separable Sets

Definition (Dense Set, Separable Space)

A subset M of a metric space X is said to be **dense in** X if $\overline{M} = X$.
X is said to be **separable** if it has a countable subset which is dense in X.

- By the definition, if M is dense in X, then every ball in X, no matter how small, will contain points of M.

  In other words, if M is dense in X, there is no point $x \in X$ which has a neighborhood that does not contain points of M.

Examples

- (**Real Line** $\mathbb{R}$) The real line $\mathbb{R}$ is separable.

  The set $\mathbb{Q}$ of all rational numbers is countable and is dense in $\mathbb{R}$.
- (**Complex Plane** $\mathbb{C}$) The complex plane $\mathbb{C}$ is separable.

  A countable dense subset of $\mathbb{C}$ is the set of all complex numbers whose real and imaginary parts are both rational.
- (**Discrete metric space**) A discrete metric space X is separable if and only if X is countable.

  The kind of metric implies that no proper subset of X can be dense in X. Hence the only dense set in X is X itself, and the statement follows.

Example: The Space ℓ^∞

- The space ℓ^∞ is not separable.

  Let $y = (\eta_1, \eta_2, \eta_3, \ldots)$ be a sequence of zeros and ones. Then $y \in \ell^\infty$. With y we associate the real number $\widehat{y}$, whose binary representation is $\frac{\eta_1}{2^1} + \frac{\eta_2}{2^2} + \frac{\eta_3}{2^3} + \cdots$. We now use the facts that:
  - the set of points in the interval $[0,1]$ is uncountable;
  - each $\widehat{y} \in [0,1]$ has a binary representation;
  - different $\widehat{y}$'s have different binary representations.

  Hence there are uncountably many sequences of zeros and ones.

  In ℓ^∞, any two different ones must be at distance 1 apart.

  Let each of these sequences be the center of a small ball of radius $\frac{1}{3}$.

  These balls do not intersect and we have uncountably many of them.

  If M is any dense set in ℓ^∞, each of these nonintersecting balls must contain an element of M. Hence M cannot be countable.

  Since M was an arbitrary dense set, ℓ^∞ cannot have dense subsets which are countable. Consequently, ℓ^∞ is not separable.

Example: The Space ℓ^p

- The space ℓ^p with $1 \le p < +\infty$ is separable.

  Let M be the set of all sequences y of the form

  $$y = (\eta_1, \eta_2, \ldots, \eta_n, 0, 0, \ldots),$$

  where n is any positive integer and the η_j's are rational.

  - M is countable.
  - We show that M is dense in ℓ^p: Let $x = (\xi_j) \in \ell^p$ be arbitrary.

    For every $\varepsilon > 0$, there is an $n = n(\varepsilon)$, such that $\sum_{j=n+1}^{\infty} |\xi_j|^p < \frac{\varepsilon^p}{2}$, because on the left we have the remainder of a converging series.

    Since the rationals are dense in $\mathbb{R}$, for each ξ_j, there is a rational η_j close to it. Hence, there is a $y \in M$ satisfying $\sum_{j=1}^{n} |\xi_j - \eta_j|^p < \frac{\varepsilon^p}{2}$.

    It follows that

    $$[d(x,y)]^p = \sum_{j=1}^{n} |\xi_j - \eta_j|^p + \sum_{j=n+1}^{\infty} |\xi_j|^p < \varepsilon^p.$$

    We thus have $d(x,y) < \varepsilon$. So M is dense in ℓ^p.

Subsection 4

Convergence, Cauchy Sequence, Completeness

Convergence of Sequences and Limits

Definition (Convergence and Limit)

A sequence (x_n) in a metric space $X = (X, d)$ is said to **converge** or to **be convergent** if there is an $x \in X$, such that $\lim_{n \to \infty} d(x_n, x) = 0$.
x is called the **limit** of (x_n) and we write $\lim_{n \to \infty} x_n = x$ or, simply, $x_n \to x$.
We say that (x_n) **converges to** x or **has the limit** x.
If (x_n) is not convergent, it is said to be **divergent**.

- The metric d yields the sequence of real numbers $a_n = d(x_n, x)$ whose convergence defines that of (x_n).

  Hence, if $x_n \to x$, an $\varepsilon > 0$ being given, there is an $N = N(\varepsilon)$, such that all x_n, with $n > N$, lie in the ε-neighborhood $B(x; \varepsilon)$ of x.
- The limit of a convergent sequence must be a point in X:

  Let X be the open interval $(0, 1)$ on $\mathbb{R}$ with metric $d(x, y) = |x - y|$.

  The sequence $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ is not convergent, since $0 \notin X$.

Boundedness

- The **diameter** $\delta(M)$ of a nonempty subset M of a metric space $X = (X,d)$ is defined by

$$\delta(M) = \max_{x,y \in M} d(x,y).$$

- A nonempty subset $M \subseteq X$ is a **bounded set** if its diameter is finite.
- A sequence (x_n) in X is a **bounded sequence** if the corresponding point set is a bounded subset of X.
- Obviously, if M is bounded, then $M \subseteq B(x_0;r)$, where $x_0 \in X$ is any point and r is a (sufficiently large) real number, and conversely.

Boundedness and Limits

Lemma (Boundedness, Limit)

Let $X=(X,d)$ be a metric space. Then:

(a) A convergent sequence in X is bounded and its limit is unique.

(b) If $x_n \to x$ and $y_n \to y$ in X, then $d(x_n,y_n) \to d(x,y)$.

(a) Suppose that $x_n \to x$. Then, taking $\varepsilon = 1$, we can find an N, such that $d(x_n,x)<1$, for all $n>N$. Hence, by the triangle inequality, for all n, we have $d(x_n,x)<1+a$, where $a=\max\{d(x_1,x),\ldots,d(x_N,x)\}$. This shows that (x_n) is bounded.

If $x_n \to x$ and $x_n \to z$, we get $0 \le d(x,z) \le d(x,x_n)+d(x_n,z) \to 0+0$. So, the uniqueness $x=z$ of the limit follows from (M2).

(b) We have $d(x_n,y_n) \le d(x_n,x)+d(x,y)+d(y,y_n)$. So, $d(x_n,y_n)-d(x,y)$ $\le d(x_n,x)+d(y,y_n)$. By interchanging x_n and x as well as y_n and y and multiplying by -1, we get $-d(x_n,x)-d(y_n,y) \le d(x_n,y_n)-d(x,y)$. Together, $|d(x_n,y_n)-d(x,y)| \le d(x_n,x)+d(y_n,y) \to 0$ as $n \to \infty$.

Convergence and Cauchy Sequences in $\mathbb{R}$ and in $\mathbb{C}$

- Recall that a sequence (x_n) of real or complex numbers converges on the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$, respectively, if and only if it satisfies the **Cauchy convergence criterion**:

  For every given $\varepsilon > 0$, there is an $N = N(\varepsilon)$, such that $|x_m - x_n| < \varepsilon$, for all $m, n > N$.

- In a measure space $X = (X, d)$, a sequence (x_n) is called a **Cauchy sequence** if, for all $\varepsilon > 0$, there exists $N > 0$, such that $d(x_m, x_n) < \varepsilon$, for all $m, n > N$.
- The Cauchy criterion simply says that a sequence of real or complex numbers converges on $\mathbb{R}$ or in $\mathbb{C}$ if and only if it is a Cauchy sequence.
- Unfortunately, in more general spaces, there may be Cauchy sequences which do not converge.

Cauchy Sequences and Completeness

Definition (Cauchy Sequence, Completeness)

A sequence (x_n) in a metric space $X = (X, d)$ is said to be **Cauchy** (or **fundamental**) if, for every $\varepsilon > 0$, there is an $N = N(\varepsilon)$, such that $d(x_m, x_n) < \varepsilon$, for every $m, n > N$.
The space X is said to be **complete** if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

- Expressed in terms of completeness, the Cauchy convergence criterion implies the following:

Theorem (Real Line, Complex Plane)

The real line and the complex plane are complete metric spaces.

- The definition shows that complete metric spaces are precisely those in which the Cauchy Condition continues to be necessary and sufficient for convergence.

Some Examples

- Omission of a point a from the real line yields the incomplete space $\mathbb{R} - \{a\}$.
- By the omission of all irrational numbers we have the **rational line** $\mathbb{Q}$, which is incomplete.
- An open interval (a, b) with the metric induced from $\mathbb{R}$ is another incomplete metric space.
- Take $X = (0, 1]$, with the usual metric defined by $d(x, y) = |x - y|$, and the sequence (x_n), where $x_n = \frac{1}{n}$ and $n = 1, 2, \ldots$.
  - This is a Cauchy sequence, but it does not converge, because the point 0 (to which it "wants to converge") is not a point of X.
  - This also illustrates that the concept of convergence is not an intrinsic property of the sequence itself but also depends on the space in which the sequence lies.

Necessity of the Cauchy Condition

Theorem (Convergent Sequence)

Every convergent sequence in a metric space is a Cauchy sequence.

- If $x_n \to x$, then for every $\varepsilon > 0$, there is an $N = N(\varepsilon)$, such that $d(x_n, x) < \frac{\varepsilon}{2}$, for all $n > N$. Hence, by the triangle inequality, we obtain for $m, n > N$,

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

  This shows that (x_n) is Cauchy.
- The completeness of the real line $\mathbb{R}$ is the main reason why in calculus we use $\mathbb{R}$ rather than the **rational line** $\mathbb{Q}$ (the set of all rational numbers with the metric induced from $\mathbb{R}$).

Closures and Closed Sets

Theorem (Closure, Closed Set)

Let M be a nonempty subset of a metric space (X,d) and $\overline{M}$ its closure as defined in the previous section. Then:

(a) $x \in \overline{M}$ if and only if there is a sequence (x_n) in M, such that $x_n \to x$.

(b) M is closed if and only if $x_n \in M$ and $x_n \to x$ imply that $x \in M$.

(a) Let $x \in \overline{M}$. If $x \in M$, a sequence of that type is $(x, x, \ldots)$. If $x \notin M$, it is a point of accumulation of M. Hence, for each $n = 1, 2, \ldots$, the ball $B(x; \frac{1}{n})$ contains an $x_n \in M$, and $x_n \to x$ because $\lim_{n\to\infty} \frac{1}{n} = 0$.
Conversely, if (x_n) is in M and $x_n \to x$, then $x \in M$ or every neighborhood of x contains points $x_n \neq x$, so that x is a point of accumulation of M. Hence $x \in \overline{M}$, by the definition of the closure.

(b) M is closed if and only if $M = \overline{M}$, so that (b) follows from (a).

Complete Subspaces

Theorem (Complete Subspace)

A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.

- Let M be complete. For every $x \in \overline{M}$, there is a sequence (x_n) in M which converges to x. Since (x_n) is Cauchy and M is complete, (x_n) converges in M and the limit being unique. Hence $x \in M$. This proves that M is closed because $x \in M$ was arbitrary.

  Conversely, let M be closed and (x_n) Cauchy in M. Then $x_n \to x \in X$. So $x \in \overline{M}$. Then $x \in M$, since $M = \overline{M}$, by assumption. Hence the arbitrary Cauchy sequence (x_n) converges in M. This proves completeness of M.

Convergence of Sequences and Continuity

Theorem (Continuous Mapping)

A mapping $T : X \to Y$ of a metric space (X,d) into a metric space $(Y,\widetilde{d})$ is continuous at a point $x_0 \in X$ if and only if $x_n \to x_0$ implies $Tx_n \to Tx_0$.

- Assume T to be continuous at x_0. Then for $\varepsilon > 0$, there is a $\delta > 0$, such that $d(x,x_0) < \delta$ implies $\widetilde{d}(Tx, Tx_0) < \varepsilon$. Let $x_n \to x_0$. Then, there is an N, such that, for all $n > N$, we have $d(x_n,x_0) < \delta$. Hence, for all $n > N$, $\widetilde{d}(Tx_n, Tx_0) < \varepsilon$. By definition, this means that $Tx_n \to Tx_0$.

  Conversely, assume that $x_n \to x_0$ implies $Tx_n \to Tx_0$. We prove that T is continuous at x_0. Suppose this is false. Then there is an $\varepsilon > 0$, such that, for every $\delta > 0$, there is an $x \neq x_0$ satisfying $d(x,x_0) < \delta$ but $\widetilde{d}(Tx, Tx_0) \geq \varepsilon$. In particular, for $\delta = \frac{1}{n}$, there is an x_n satisfying $d(x_n,x_0) < \frac{1}{n}$ but $\widetilde{d}(Tx_n, Tx_0) \geq \varepsilon$. Clearly $x_n \to x$ but (Tx_n) does not converge to Tx_0. This contradicts $Tx_n \to Tx_0$ and proves the theorem.

Subsection 5

Examples, Completeness Proofs

Completeness Proofs

- In various applications a set X is given (for instance, a set of sequences or a set of functions), and X is made into a metric space.

  This is done by choosing a metric d on X.

  The remaining task is then to find out whether (X, d) has the desirable property of being complete.
- To prove completeness, we take an arbitrary Cauchy sequence (x_n) in X and show that it converges in X.

  For different spaces, such proofs may vary in complexity, but they have the same general pattern:
  (i) Construct an element x (to be used as a limit).
  (ii) Prove that x is in the space considered.
  (iii) Prove convergence $x_n \to x$ (in the sense of the metric).
- Often, we get help from the completeness of the real line or the complex plane.

Completeness of $\mathbb{R}^n$ and $\mathbb{C}^n$

- Euclidean space $\mathbb{R}^n$ and unitary space $\mathbb{C}^n$ are complete.
  We first consider $\mathbb{R}^n$.
  The metric on $\mathbb{R}^n$ is defined, for all $x = (\xi_j)$ and all $y = (\eta_j)$, by
  $$d(x,y) = (\sum_{j=1}^{n} (\xi_j - \eta_j)^2)^{1/2}.$$
  Consider any Cauchy sequence (x_m) in $\mathbb{R}^n$, with $x_m = (\xi_1^{(m)}, \ldots, \xi_n^{(m)})$.
  Since (x_m) is Cauchy, for every $\varepsilon > 0$, there is an N, such that
  $$d(x_m, x_r) = (\sum_{j=1}^{n} (\xi_j^{(m)} - \xi_j^{(r)})^2)^{1/2} < \varepsilon, \quad \text{for all } m, r > N.$$
  So, for $m, r > N$, $j = 1, \ldots, n$, $(\xi_j^{(m)} - \xi_j^{(r)})^2 < \varepsilon^2$ and $|\xi_j^{(m)} - \xi_j^{(r)}| < \varepsilon$.
  This shows that for each fixed j, $1 \le j \le n$, the sequence $(\xi_j^{(1)}, \xi_j^{(2)}, \ldots)$ is a Cauchy sequence of real numbers.

Completeness of $\mathbb{R}^n$ and $\mathbb{C}^n$ (Cont'd)

- We showed that, for each fixed j, $1 \le j \le n$, the sequence $(\xi_j^{(1)}, \xi_j^{(2)}, \ldots)$ is a Cauchy sequence of real numbers.

  So, it converges say, $\xi_j^{(m)} \overset{m\to\infty}{\to} \xi_j$.

  Define $x = (\xi_1, \ldots, \xi_n)$.

  Clearly, $x \in \mathbb{R}^n$.

  Let $r \to \infty$ in $(\sum_{j=1}^n (\xi_j^{(m)} - \xi_j^{(r)})^2)^{1/2} < \varepsilon$.

  Then, we get
  $$d(x_m, x) \le \varepsilon, \quad \text{for all } m > N.$$
  This shows that x is the limit of (x_m) and proves completeness of $\mathbb{R}^n$ because (x_m) was an arbitrary Cauchy sequence.
- Completeness of $\mathbb{C}^n$ follows by the same method of proof.

Completeness of ℓ^∞

- The space ℓ^∞ is complete.

  Let (x_m) be any Cauchy sequence in ℓ^∞, with $x_m = (\xi_1^{(m)}, \xi_2^{(m)}, \ldots)$.

  The metric on ℓ^∞ is given, for all $x = (\xi_j)$ and $y = (\eta_j)$, by

  $$d(x,y) = \sup_j |\xi_j - \eta_j|.$$

  Since (x_m) is Cauchy, for any $\varepsilon > 0$, there is an N, such that

  $$d(x_m, x_n) = \sup_j |\xi_j^{(m)} - \xi_j^{(n)}| < \varepsilon, \quad \text{for all } m, n > N.$$

  A fortiori, for every fixed j, $|\xi_j^{(m)} - \xi_j^{(n)}| < \varepsilon$, for all $m, n > N$.

  Hence, for every fixed j, the sequence $(\xi_j^{(1)}, \xi_j^{(2)}, \ldots)$ is a Cauchy sequence of numbers. So, it converges $\xi_j^{(m)} \stackrel{m \to \infty}{\longrightarrow} \xi_j$.

  We now define $x = (\xi_1, \xi_2, \ldots)$ and show that $x \in \ell^\infty$ and $x_m \to x$.

Completeness of ℓ^∞ (Cont'd)

- From $|\xi_j^{(m)} - \xi_j^{(n)}| < \varepsilon$ with $n \to \infty$, we have
$$|\xi_j^{(m)} - \xi_j| \leq \varepsilon, \quad \text{for all } m > N.$$
Since $x_m = (\xi_j^{(m)}) \in \ell^\infty$, there is k_m, such that $|\xi_j^{(m)}| \leq k_m$ for all j.
Hence, by the triangle inequality,
$$|\xi_j| \leq |\xi_j - \xi_j^{(m)}| + |\xi_j^{(m)}| \leq \varepsilon + k_m, \quad \text{for all } m > N.$$
This holds for all j, and the right-hand side does not involve j.
Hence (ξ_j) is a bounded sequence of numbers.
This implies that $x = (\xi_j) \in \ell^\infty$.
Since $|\xi_j^{(m)} - \xi_j| \leq \varepsilon$, we get $d(x_m, x) = \sup_j |\xi_j^{(m)} - \xi_j| \leq \varepsilon$, for all $m > N$.
This shows that $x_m \to x$.
Since (x_m) was an arbitrary Cauchy sequence, ℓ^∞ is complete.

Completeness of c

- The space c consisting of all convergent sequences $x=(\xi_j)$ of complex numbers, with the metric induced from the space ℓ^∞ is complete.

  c is a subspace of the complete space ℓ^∞.

  By the Complete Subspace Theorem, it suffices to show that c is closed in ℓ^∞.

  Consider any $x=(\xi_j)\in\overline{c}$, the closure of c.

  Then there are $x_n=(\xi_j^{(n)})\in c$, such that $x_n\to x$.

  Hence, given any $\varepsilon>0$, there is an N, such that

  $$|\xi_j^{(n)}-\xi_j|\leq d(x_n,x)<\frac{\varepsilon}{3},\quad \text{for all } n\geq N \text{ and all } j.$$

  In particular, this holds for $n=N$ and all j.

  Since $x_N\in c$, its terms $\xi_j^{(N)}$ form a convergent sequence.

  Such a sequence is Cauchy.

Completeness of c (Cont'd)

- We showed that $\xi_j^{(N)}$, $j = 1, 2, \ldots$, is Cauchy.

  Hence there is an N_1 such that

  $$|\xi_j^{(N)} - \xi_k^{(N)}| < \frac{\varepsilon}{3}, \quad \text{for all } j, k \geq N_1.$$

  The triangle inequality now yields for all $j, k \geq N_1$,

  $$|\xi_j - \xi_k| \leq |\xi_j - \xi_j^{(N)}| + |\xi_j^{(N)} - \xi_k^{(N)}| + |\xi_k^{(N)} - \xi_k| < \varepsilon.$$

  This shows that the sequence $x = (\xi_j)$ is convergent.

  Hence $x \in c$.

  Since $x \in \overline{c}$ was arbitrary, this proves closedness of c in ℓ^∞.

Completeness of ℓ^p

- The space ℓ^p is complete; here p is fixed and $1 \leq p < +\infty$.
  Let (x_n) be any Cauchy sequence in ℓ^p, where $x_m = (\xi_1^{(m)}, \xi_2^{(m)}, \ldots)$.
  For every $\varepsilon > 0$, there is an N, such that
  $$d(x_m, x_n) = (\sum_{j=1}^{\infty} |\xi_j^{(m)} - \xi_j^{(n)}|^p)^{1/p} < \varepsilon, \quad \text{for all } m, n > N.$$
  So, for every $j = 1, 2, \ldots$ we have
  $$|\xi_j^{(m)} - \xi_j^{(n)}| < \varepsilon.$$
  Choose a fixed j.
  Then $(\xi_j^{(1)}, \xi_j^{(2)}, \ldots)$ is a Cauchy sequence of numbers.
  It converges since $\mathbb{R}$ and $\mathbb{C}$ are complete, $\xi_j^{(m)} \overset{m \to \infty}{\to} \xi_j$.
  We define $x = (\xi_1, \xi_2, \ldots)$ and show that $x \in \ell^p$ and $x_m \to x$.

Completeness of ℓ^p (Cont'd)

- From $d(x_m,x_n)=(\sum_{j=1}^{\infty}|\xi_j^{(m)}-\xi_j^{(n)}|^p)^{1/p}<\varepsilon$ we have for $k=1,2,\ldots,$

$$\sum_{j=1}^{k}|\xi_j^{(m)}-\xi_j^{(n)}|^p<\varepsilon^p, \quad \text{for all } m,n>N.$$

Letting $n\to\infty$, we obtain, for $k=1,2,\ldots,$

$$\sum_{j=1}^{k}|\xi_j^{(m)}-\xi_j|^p\le\varepsilon^p, \quad \text{for all } m>N.$$

We may now let $k\to\infty$:

$$\sum_{j=1}^{\infty}|\xi_j^{(m)}-\xi_j|^p\le\varepsilon^p, \quad \text{for all } m>N.$$

This shows that $x_m-x=(\xi_j^{(m)}-\xi_j)\in\ell^p$.
Since $x_m\in\ell^p$, by the Minkowski inequality, $x=x_m+(x-x_m)\in\ell^p$.
Furthermore, the series in $\sum_{j=1}^{\infty}|\xi_j^{(m)}-\xi_j|^p\le\varepsilon^p$ represents $[d(x_m,x)]^p$.
So the inequality implies that $x_m\to x$.

Completeness of $C[a,b]$

- The function space $C[a,b]$ is complete ($[a,b]$ any closed interval in $\mathbb{R}$).

  Let (x_m) be any Cauchy sequence in $C[a,b]$. Denote $J := [a,b]$.

  Given $\varepsilon > 0$, there is an N, such that

  $$d(x_m, x_n) = \max_{t \in J} |x_m(t) - x_n(t)| < \varepsilon, \quad \text{for all } m, n > N.$$

  Hence, for any fixed $t = t_0 \in J$,

  $$|x_m(t_0) - x_n(t_0)| < \varepsilon, \quad \text{for all } m, n > N.$$

  So $(x_1(t_0), x_2(t_0), \ldots)$ is a Cauchy sequence of real numbers.

  Since $\mathbb{R}$ is complete, it converges, say, $x_m(t_0) \overset{m \to \infty}{\longrightarrow} x(t_0)$.

  In this way we can associate with each $t \in J$ a unique real number $x(t)$.

  This defines a function x on J.

  We show that $x \in C[a,b]$ and $x_m \to x$.

Completeness of $C[a,b]$ (Cont'd)

- We have
$$d(x_m, x_n) = \max_{t \in J} |x_m(t) - x_n(t)| < \varepsilon, \quad \text{for all } m, n > N.$$
Letting $n \to \infty$, we get
$$\max_{t \in J} |x_m(t) - x(t)| \le \varepsilon, \quad \text{for } m > N..$$
Hence, for every $t \in J$,
$$|x_m(t) - x(t)| \le \varepsilon, \quad \text{for all } m > N.$$
This shows that $(x_m(t))$ converges to $x(t)$ uniformly on J.
Since the x_m's are continuous on J and the convergence is uniform, the limit function x is continuous on J.
Hence, $x \in C[a,b]$.
Also $x_m \to x$.
This proves completeness of $C[a,b]$.

Uniform Convergence and Uniform Metric

- In the preceding example, we assumed the functions x to be real-valued, for simplicity.

  We may call this space the **real** $C[a,b]$.
- Similarly, we obtain the **complex** $C[a,b]$ if we take complex-valued continuous functions defined on $[a,b] \subseteq \mathbb{R}$.

  This space is also complete and the proof is almost the same as before.
- Furthermore, the proof shows the following:

Theorem (Uniform Convergence)

Convergence $x_m \to x$ in the space $C[a,b]$ is uniform convergence, i.e., (x_m) converges uniformly on $[a,b]$ to x.

- Hence the metric on $C[a,b]$ describes uniform convergence on $[a,b]$ and it is sometimes called the **uniform metric**.

Examples of Incomplete Metric Spaces

- (**Space** $\mathbb{Q}$) This is the set of all rational numbers with the usual metric given by $d(x,y) = |x - y|$, where $x, y \in \mathbb{Q}$, and is called the **rational line**. $\mathbb{Q}$ is not complete.
- (**Polynomials**) Let X be the set of all polynomials considered as functions of t on some finite closed interval $J = [a,b]$ and define a metric d on X by

$$d(x,y) = \max_{t \in J} |x(t) - y(t)|.$$

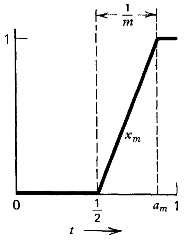
This metric space (X,d) is not complete:

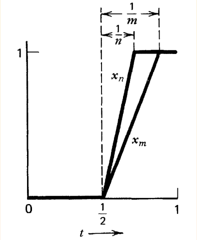
An example of a Cauchy sequence without limit in X is given by any sequence of polynomials which converges uniformly on J to a continuous function, not a polynomial.

Continuous Functions

- Let X be the set of all continuous real-valued functions on $J=[0,1]$, $d(x,y)=\int_0^1|x(t)-y(t)|dt$. The metric space (X,d) is not complete.

  Consider the functions x_m whose graphs are shown of the left.
  They form a Cauchy sequence.
  Note that $d(x_m,x_n)$ is the area of the triangle on right.
  Given $\varepsilon>0$, choose $m,n>\frac{1}{\varepsilon}$.





  Then
  $$d(x_m,x_n)=\frac{1}{2}\left(\frac{1}{m}-\frac{1}{n}\right)<\varepsilon.$$

  We show that this Cauchy sequence does not converge.

Continuous Functions (Cont'd)

- Note that
$$\begin{array}{l} x_m(t) = 0, \quad \text{if } t \in [0, \frac{1}{2}], \\ x_m(t) = 1, \quad \text{if } t \in [a_m, 1], \text{ where } a_m = \frac{1}{2} + \frac{1}{m}. \end{array}$$
Hence, for all $x \in X$,
$$\begin{array}{rcl} d(x_m, x) & = & \int_0^1 |x_m(t) - x(t)| dt \\ & = & \int_0^{1/2} |x(t)| dt + \int_{1/2}^{a_m} |x_m(t) - x(t)| dt \\ & & \qquad + \int_{a_m}^1 |1 - x(t)| dt. \end{array}$$
The integrands are nonnegative. So, the same holds for the integrals. Now, $d(x_m, x) \to 0$ implies that each integral approaches 0. By the continuity of x we should have
$$\begin{array}{l} x(t) = 0, \quad \text{if } t \in [0, \frac{1}{2}), \\ x(t) = 1, \quad \text{if } t \in (\frac{1}{2}, 1]. \end{array}$$
This contradicts the continuity of x.

Subsection 6

Completion of Metric Spaces

Isometric Mappings and Isometric Spaces

Definition (Isometric Mapping, Isometric Spaces)

Let $X = (X, d)$ and $\widetilde{X} = (\widetilde{X}, \widetilde{d})$ be metric spaces. Then:

(a) A mapping T of X into $\widetilde{X}$ is said to be **isometric** or an **isometry** if T preserves distances, that is, if for all $x, y \in X$, $\widetilde{d}(Tx, Ty) = d(x, y)$, where Tx and Ty are the images of x and y, respectively.

(b) The space X is said to be **isometric** with the space $\widetilde{X}$ if there exists a bijective isometry of X onto $\widetilde{X}$. The spaces X and $\widetilde{X}$ are then called **isometric spaces**.

- Isometric spaces may differ at most by the nature of their points but are indistinguishable from the viewpoint of metric.
- In any study in which the nature of the points does not matter, we may regard two isometric spaces as identical - as two copies of the same "abstract" space.

The Completion Theorem

- It turns out that every metric space can be completed. The space $\widehat{X}$ occurring below is called the **completion** of the given space X.

Theorem (Completion)

For a metric space $X = (X, d)$, there exists a complete metric space $\widehat{X} = (\widehat{X}, \widehat{d})$ which has a subspace W isometric with X and is dense in $\widehat{X}$. The space $\widehat{X}$ is unique up to isometries, i.e., if $\widetilde{X}$ is any complete metric space having a dense subspace $\widetilde{W}$ isometric with X, then $\widetilde{X}$ and $\widehat{X}$ are isometric.

- The proof is subdivided it into four steps (a) to (d):
  - (a) We construct $\widehat{X} = (\widehat{X}, \widehat{d})$;
  - (b) We construct an isometry T of X onto W, where $\overline{W} = \widehat{X}$;
  - (c) We prove completeness of $\widehat{X}$;
  - (d) We prove uniqueness of $\widehat{X}$, up to isometries.

The Completion Theorem: Outline of the Proof

- The task will be the assignment of suitable limits to Cauchy sequences in X that do not converge.
- However, we should not introduce "too many" limits, but take into account that certain sequences "may want to converge with the same limit" since the terms of those sequences "ultimately come arbitrarily close to each other".
- This intuitive idea can be expressed mathematically in terms of a suitable equivalence relation.

  This is not artificial but is suggested by the process of completion of the rational line.

Construction of $\widehat{X} = (\widehat{X}, \widehat{d})$

- Let (x_n) and (x'_n) be Cauchy sequences in X. Define (x_n) to be **equivalent** to (x'_n), written $(x_n) \sim (x'_n)$, if $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

  Let $\widehat{X}$ be the set of all equivalence classes $\widehat{x}, \widehat{y}, \ldots$.

  Write $(x_n) \in \widehat{x}$ to mean that (x_n) is a member of $\widehat{x}$.

  Set, for all $\widehat{x}, \widehat{y}$,
  $$\widehat{d}(\widehat{x}, \widehat{y}) = \lim_{n\to\infty} d(x_n, y_n),$$
  where $(x_n) \in \widehat{x}$ and $(y_n) \in \widehat{y}$.

  We show that this limit exists.

  We have $d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$, whence $d(x_n, y_n) - d(x_m, y_m) \le d(x_n, x_m) + d(y_m, y_n)$ and a similar inequality with m and n interchanged. Together, $|d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_m, y_n)$. Since (x_n) and (y_n) are Cauchy, we can make the right side as small as we please. This implies that the limit in $\lim_{n\to\infty} d(x_n, y_n)$ exists because $\mathbb{R}$ is complete.

Independence of Representative and the Metric Property

- We now show that the limit $\widehat{d}(\widehat{x},\widehat{y}) = \lim_{n\to\infty} d(x_n,y_n)$ is independent of the particular choice of representatives $(x_n) \in \widehat{x}$ and $(y_n) \in \widehat{y}$.

  Suppose $(x_n) \sim (x_n')$ and $(y_n) \sim (y_n')$.

  Then $|d(x_n,y_n) - d(x_n',y_n')| \le d(x_n,x_n') + d(y_n,y_n') \to 0$ as $n \to \infty$.

  This implies $\lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(x_n',y_n')$.
- We prove, next, that $\widehat{d}$ is a metric on $\widehat{X}$.
  - Obviously, $\widehat{d}$ satisfies (M1);
  - It also satisfies $\widehat{d}(\widehat{x},\widehat{x}) = 0$;
  - It satisfies (M3);
  - Further, $\widehat{d}(\widehat{x},\widehat{y}) = 0$ implies $(x_n) \sim (y_n)$ implies $\widehat{x} = \widehat{y}$, giving (M2);
  - Finally, $d(x_n,y_n) \le d(x_n,z_n) + d(z_n,y_n)$. Letting $n \to \infty$, we get $\widehat{d}(\widehat{x},\widehat{y}) \le \widehat{d}(\widehat{x},\widehat{z}) + \widehat{d}(\widehat{z},\widehat{y})$, i.e., (M4) for $\widehat{d}$.

Construction of an isometry $T : X \to W \subseteq \widehat{X}$

- With each $b \in X$ we associate the class $\widehat{b} \in \widehat{X}$ which contains the constant Cauchy sequence $(b, b, \ldots)$. This defines a mapping $T : X \to W$ onto the subspace $W = T(X) \subseteq \widehat{X}$. The mapping T is given by $b \mapsto \widehat{b} = Tb$, where $(b, b, \ldots) \in \widehat{b}$.
- We see that T is an isometry since $\widehat{d}(\widehat{b}, \widehat{c}) = d(b, c)$, where $\widehat{c}$ is the class of (y_n) where $y_n = c$, for all n.

  Any isometry is injective. $T : X \to W$ is also surjective since $T(X) = W$. Hence, W and X are isometric.
- W is dense in X: Consider any $\widehat{x} \in \widehat{X}$. Let $(x_n) \in \widehat{x}$. For every $\varepsilon > 0$, there is N, such that $d(x_n, x_N) < \frac{\varepsilon}{2}$, for all $n > N$. Let $(x_N, x_N, \ldots) \in \widehat{x}_N$. Then $\widehat{x}_N \in W$. Also, $\widehat{d}(\widehat{x}, \widehat{x}_N) = \lim_{n \to \infty} d(x_n, x_N) \le \frac{\varepsilon}{2} < \varepsilon$. This shows that every ε-neighborhood of the arbitrary $\widehat{x} \in \widehat{X}$ contains an element of W. Hence W is dense in X.

Completeness of $\widehat{X}$

- Let $(\widehat{x}_n)$ be any Cauchy sequence in $\widehat{X}$. Since W is dense in $\widehat{X}$, for every $\widehat{x}_n$, there is a $\widehat{z}_n \in W$, such that $\widehat{d}(\widehat{x}_n, \widehat{z}_n) < \frac{1}{n}$. Hence, by (M4),

$$\widehat{d}(\widehat{z}_m, \widehat{z}_n) \le \widehat{d}(\widehat{z}_m, \widehat{x}_m) + \widehat{d}(\widehat{x}_m, \widehat{x}_n) + \widehat{d}(\widehat{x}_n, \widehat{z}_n) < \frac{1}{m} + \widehat{d}(\widehat{x}_m, \widehat{x}_n) + \frac{1}{n}.$$

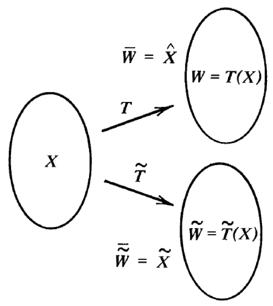
This is less than any given $\varepsilon > 0$ for sufficiently large m and n because (x_m) is Cauchy. Hence $(\widehat{z}_m)$ is Cauchy. Since $T : X \to W$ is isometric and $\widehat{z}_m \in W$, the sequence (z_m), where $z_m = T^{-1}\widehat{z}_m$, is Cauchy in X.

Let $\widehat{x} \in \widehat{X}$ be the class to which (z_m) belongs. We show $\widehat{x} = \lim_{n\to\infty} \widehat{x}_n$.

Since $\widehat{d}(\widehat{x}_n, \widehat{z}_n) < \frac{1}{n}$, $\widehat{d}(\widehat{x}_n, \widehat{x}) \le \widehat{d}(\widehat{x}_n, \widehat{z}_n) + \widehat{d}(\widehat{z}_n, \widehat{x}) < \frac{1}{n} + \widehat{d}(\widehat{z}, \widehat{x})$. But $(z_m) \in \widehat{x}$ and $\widehat{z}_n \in W$, so that $(z_n, z_n, \ldots) \in \widehat{z}_n$. It follows that $\widehat{d}(\widehat{x}_n, \widehat{x}) < \frac{1}{n} + \lim_{m\to\infty} d(z_n, z_m)$. Hence, the right side is smaller than any given $\varepsilon > 0$ for sufficiently large n. This shows that $\widehat{x} = \lim_{n\to\infty} \widehat{x}_n$.

Uniqueness of $\widehat{X}$ Except for Isometries

- Suppose $(\widetilde{X}, \widetilde{d})$ is another complete metric space with a subspace $\widetilde{W}$
  - dense in $\widetilde{X}$;
  - isometric with X.



For any $\widetilde{x}, \widetilde{y} \in \widetilde{X}$, we have sequences $(\widetilde{x}_n), (\widetilde{y}_n)$ in $\widetilde{W}$, such that $\widetilde{x}_n \to \widetilde{x}$ and $\widetilde{y}_n \to \widetilde{y}$.

Now $$|\widetilde{d}(\widetilde{x}, \widetilde{y}) - \widetilde{d}(\widetilde{x}_n, \widetilde{y}_n)| \le \widetilde{d}(\widetilde{x}, \widetilde{x}_n) + \widetilde{d}(\widetilde{y}, \widetilde{y}_n) \to 0.$$

Hence, $\widetilde{d}(\widetilde{x}, \widetilde{y}) = \lim_{n\to\infty} \widetilde{d}(\widetilde{x}_n, \widetilde{y}_n)$.

But $\widetilde{W}$ is isometric with $W \subseteq \widehat{X}$, and $\overline{W} = \widehat{X}$.

So the distances on $\widetilde{X}$ and $\widehat{X}$ must be the same.

Thus, $\widetilde{X}$ and $\widehat{X}$ are isometric.