Introduction to Functional Analysis

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Subsection 1

Banach Fixed Point Theorem

Fixed Points

- A **fixed point** of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself (is "kept fixed" by T), that is,

$$Tx = x,$$

  the image Tx coincides with x.

  Examples:

  - A translation has no fixed points.
  - A rotation of the plane has a single fixed point (the center of rotation).
  - The mapping $x \mapsto x^2$ of $\mathbb{R}$ into itself has two fixed points (0 and 1).
  - The projection $(\xi_1, \xi_2) \mapsto \xi_1$ of $\mathbb{R}^2$ onto the ξ_1-axis has infinitely many fixed points (all points of the ξ_1-axis).

Banach Fixed Point and Iteration

- The **Banach fixed point theorem**:
  - is an existence and uniqueness theorem for fixed points of certain mappings;
  - gives a constructive procedure, called an **iteration**, for obtaining better and better approximations to the fixed point.
- By definition, **iteration** is a method such that we choose an arbitrary x_0 in a given set and calculate recursively a sequence $x_0, x_1, x_2, \ldots$ from a relation of the form

$$x_{n+1} = Tx_n, \qquad n = 0, 1, 2, \ldots;$$

  that is, we choose an arbitrary x_0 and determine successively $x_1 = Tx_0, x_2 = Tx_1, \ldots$.
- Convergence proofs and error estimates for iteration procedures are very often obtained by an application of Banach's Fixed Point Theorem (or more difficult fixed point theorems).

Contractions

Definition (Contraction)

Let $X=(X,d)$ be a metric space. A mapping $T:X\to X$ is called a **contraction on** X if there is a positive real number $\alpha<1$, such that, for all $x,y\in X$,

$$d(Tx,Ty)\le\alpha d(x,y),\qquad \alpha<1.$$

- Geometrically this means that any points x and y have images that are closer together than those points x and y.

  More precisely, the ratio $\frac{d(Tx,Ty)}{d(x,y)}$ does not exceed a constant α which is strictly less than 1.

Banach Fixed Point Theorem

Banach Fixed Point Theorem (Contraction Theorem)

Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point.

- We construct a sequence (x_n). We show that it is Cauchy, so that it converges in the complete space X. Then we prove that its limit x is a fixed point of T and T has no further fixed points.
  We choose any $x_0 \in X$. Define the "iterative sequence" (x_n) by

$$x_0,\ x_1 = Tx_0,\ x_2 = Tx_1 = T^2 x_0, \ldots, x_n = T^n x_0, \ldots.$$

We show that (x_n) is Cauchy:

$$\begin{array}{rcl} d(x_{m+1}, x_m) & = & d(Tx_m, Tx_{m-1}) \le \alpha d(x_m, x_{m-1}) \\ & = & \alpha d(Tx_{m-1}, Tx_{m-2}) \le \alpha^2 d(x_{m-1}, x_{m-2}) \\ & \le & \cdots \le \quad \alpha^m d(x_1, x_0). \end{array}$$

Banach Fixed Point Theorem: Convergence

- Hence by the triangle inequality and the formula for the sum of a geometric progression we obtain for $n > m$,

$$\begin{array}{rcl} d(x_m, x_n) & \leq & d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ & \leq & (\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}) d(x_0, x_1) \\ & = & \alpha^m \frac{1-\alpha^{n-m}}{1-\alpha} d(x_0, x_1). \end{array}$$

Since $0 < \alpha < 1$, in the numerator we have $1 - \alpha^{n-m} < 1$. Consequently,

$$d(x_m, x_n) \leq \frac{\alpha^m}{1-\alpha} d(x_0, x_1), \qquad n > m.$$

On the right, $0 < \alpha < 1$ and $d(x_0, x_1)$ is fixed, so that we can make the right-hand side as small as we please by taking m sufficiently large and $n > m$. This proves that (x_m) is Cauchy. Since X is complete, (x_m) converges, say, $x_m \to x$.

Banach Fixed Point Theorem: Fixed Point

- We show that this limit x is a fixed point of the mapping T. We have

  $$d(x,Tx) \leq d(x,x_m) + d(x_m,Tx) \leq d(x,x_m) + \alpha d(x_{m-1},x).$$

  We can make the sum on the right smaller than any preassigned $\varepsilon > 0$ because $x_m \to x$. We conclude that $d(x,Tx) = 0$, so that $x = Tx$. This shows that x is a fixed point of T.

  x is the only fixed point of T because from $Tx = x$ and $T\widetilde{x} = \widetilde{x}$ we obtain $d(x,\widetilde{x}) = d(Tx,T\widetilde{x}) \leq \alpha d(x,\widetilde{x})$, which implies $d(x,\widetilde{x}) = 0$, since $\alpha < 1$. Hence, $x = \widetilde{x}$.

Prior and Posterior Estimates

Corollary (Iteration, Error Bounds)

Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. The iterative sequence

$$x_0,\ x_1 = Tx_0,\ x_2 = Tx_1 = T^2 x_0, \ldots, x_n = T^n x_0, \ldots,$$

with arbitrary $x_0 \in X$, converges to the unique fixed point x of T. Error estimates are the **prior estimate**

$$d(x_m, x) \leq \frac{\alpha^m}{1 - \alpha} d(x_0, x_1)$$

and the **posterior estimate**

$$d(x_m, x) \leq \frac{\alpha}{1 - \alpha} d(x_{m-1}, x_m).$$

Proof of Prior and Posterior Estimates

- The first statement is obvious from the previous proof.

  The prior estimate follows from $d(x_m, x_n) \leq \frac{\alpha^m}{1-\alpha} d(x_0, x_1)$ by letting $n \to \infty$.

  We derive the posterior estimate: Taking $m = 1$ and writing y_0 for x_0 and y_1 for x_1, we get from $d(x_m, x) \leq \frac{\alpha^m}{1-\alpha} d(x_0, x_1)$ the inequality $d(y_1, x) \leq \frac{\alpha}{1-\alpha} d(y_0, y_1)$. Setting $y_0 = x_{m-1}$, we have $y_1 = Ty_0 = x_m$ and we obtain $d(x_m, x) \leq \frac{\alpha}{1-\alpha} d(x_{m-1}, x_m)$.
- The prior error bound can be used at the beginning of a calculation for estimating the number of steps necessary to obtain a given accuracy.
- The posterior can be used at intermediate stages or at the end of a calculation. It is at least as accurate as the prior and may be better.

Contraction on a Closed Subspace

- Suppose T is a contraction only on a subset Y of X. If Y is closed, it is complete, so that T has a fixed point x in Y, and $x_m \to x$ as before, provided we impose a suitable restriction on the choice of x_0, so that the x_m's remain in Y.

Theorem (Contraction on a Ball)

Let T be a mapping of a complete metric space $X = (X, d)$ into itself. Suppose T is a contraction on a closed ball $Y = \{x : d(x, x_0) \le r\}$, that is, T satisfies $d(Tx, Ty) \le \alpha d(x, y)$, for all $x, y \in Y$. Moreover, assume that $d(x_0, Tx_0) < (1-\alpha)r$. Then the iterative sequence $x_n = T^n x_0$ converges to an $x \in Y$. This x is a fixed point of T and is the only fixed point of T in Y.

- We show that all x_m's as well as x lie in Y. We put $m = 0$ in $d(x_m, x_n) \le \frac{\alpha^m}{1-\alpha} d(x_0, x_1)$, $n > m$, to get $d(x_0, x_n) \le \frac{1}{1-\alpha} d(x_0, x_1)$. Change n to m to get $d(x_0, x_m) \le \frac{1}{1-\alpha} d(x_0, x_1) < r$. Hence all x_m's are in Y. Also $x \in Y$ since $x_m \to x$ and Y is closed. The assertion of the theorem now follows from the proof of Banach's theorem.

Contractions are Continuous

Lemma (Continuity)

A contraction T on a metric space X is a continuous mapping.

- Let $x_n \to x$. Consider $\varepsilon > 0$.

  Since $x_n \to x$, there exists an N, such that, for all $n > N$,

$$d(x, x_n) < \frac{\varepsilon}{\alpha}.$$

  Therefore, for all $n > N$,

$$d(Tx, Tx_n) \leq \alpha d(x, x_n) < \alpha \frac{\varepsilon}{\alpha} = \varepsilon.$$

  Therefore, $Tx_n \to Tx$ and T is continuous.

Subsection 2

Application of Banach's Theorem to Linear Equations

The Space and the Operator

- Consider the set X of all ordered n-tuples of real numbers, written
$$x=(\xi_1,\ldots,\xi_n), y=(\eta_1,\ldots,\eta_n), z=(\zeta_1,\ldots,\zeta_n), \text{ etc.}$$
On X we define a metric d by
$$d(x,z)=\max_j|\xi_j-\zeta_j|.$$
- $X=(X,d)$ is complete.
- On X we define $T:X\to X$ by
$$y=Tx=Cx+b,$$
where $C=(c_{jk})$ is a fixed real $n\times n$ matrix and $b\in X$ a fixed vector.
- Writing this in components, we have
$$\eta_j=\sum_{k=1}^n c_{jk}\xi_k+\beta_j,\ j=1,\ldots,n,$$
where $b=(\beta_j)$.

The Space and the Operator

- Setting $w = (\omega_j) = Tz$, we obtain

$$\begin{aligned} d(y,w) &= d(Tx,Tz) = \max_j |\eta_j - \omega_j| = \max_j |\textstyle\sum_{k=1}^n c_{jk}(\xi_k - \zeta_k)| \\ &\leq \max_i |\xi_i - \zeta_i| \max_j \textstyle\sum_{k=1}^n |c_{jk}| = d(x,z) \max_j \sum_{k=1}^n |c_{jk}|. \end{aligned}$$

This can be written $d(y,w) \leq \alpha d(x,z)$, where $\alpha = \max_j \sum_{k=1}^n |c_{jk}|$.

Theorem (Linear Equations)

If a system $x = Cx + b$, with $C = (c_{jk}), b$ given, of n linear equations in n unknowns $\xi_1, \ldots, \xi_n$ (components of x) satisfies $\sum_{k=1}^n |c_{jk}| < 1$, $j = 1, \ldots, n$, it has precisely one solution x. This solution can be obtained as the limit of the iterative sequence $(x^{(0)}, x^{(1)}, x^{(2)}, \ldots)$, where $x^{(0)}$ is arbitrary and $x^{(m+1)} = Cx^{(m)} + b$, $m = 0, 1, \ldots$. Error bounds are

$$d(x^{(m)}, x) \leq \frac{\alpha}{1-\alpha} d(x^{(m-1)}, x^{(m)}) \leq \frac{\alpha^m}{1-\alpha} d(x^{(0)}, x^{(1)}).$$

Application of the Method

- A system of n linear equations in n unknowns is usually written $Ax = c$, where A is an n-rowed square matrix.
- Many iterative methods with $\det A \neq 0$ are such that one writes $A = B - G$, with a suitable nonsingular matrix B.
- Then $Ax = c$ becomes $Bx = Gx + c$, or $x = B^{-1}(Gx + c)$.
- This suggests the iteration

$$x^{(m+1)} = Cx^{(m)} + b, \text{ where } C = B^{-1}G \text{ and } b = B^{-1}c.$$

- Two standard methods are:
  - The Jacobi iteration, which is largely of theoretical interest;
  - The Gauss-Seidel iteration, which is widely used in applied mathematics.

Jacobi Iteration

- **Jacobi iteration** is defined by

$$\xi_j^{(m+1)} = \frac{1}{a_{jj}}\left(\gamma_j - \sum_{\substack{k=1 \\ k\neq j}}^{n} a_{jk}\xi_k^{(m)}\right), \quad j=1,\ldots,n,$$

  where $c=(\gamma_j)$ and we assume $a_{jj}\neq 0$, for $j=1,\ldots,n$.
- This iteration is suggested by solving the j-th equation in $Ax=c$ for ξ_j.
- It is not difficult to verify that it can be written in the form

$$x^{(m+1)} = Cx^{(m)} + b, \quad C=-D^{-1}(A-D),\ b=D^{-1}c,$$

  where $D=\text{diag}(a_{jj})$ is the diagonal matrix whose nonzero elements are those of the principal diagonal of A.

Convergence of the Jacobi Iteration

- Condition $\sum_{k=1}^{n} |c_{jk}| < 1$ applied to this C is sufficient for the convergence of the Jacobi iteration.

  Expressing this directly in terms of the elements of A, we get the row sum criterion for the Jacobi iteration

$$\sum_{\substack{k=1\\k\neq j}}^{n} \left|\frac{a_{jk}}{a_{jj}}\right| < 1, \quad j = 1, \ldots, n,$$

  or $\sum_{\substack{k=1\\k\neq j}}^{n} |a_{jk}| < |a_{jj}|$, $j = 1, \ldots, n$.

- This shows that, roughly speaking, convergence is guaranteed if the elements in the principal diagonal of A are sufficiently large.
- In the Jacobi iteration some components of $x^{(m+1)}$ may already be available at a certain instant but are not used while the computation of the remaining components is still in progress.

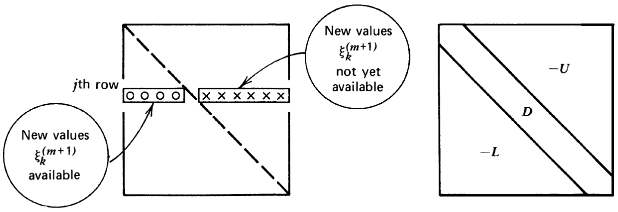
  We say the Jacobi iteration is a method of **simultaneous corrections**.

Gauss-Seidel Iteration

- This is a method of **successive corrections**: At every instant all of the latest known components are used.
- The method is defined by

$$\xi_j^{(m+1)} = \frac{1}{a_{jj}}\left(\gamma_j - \sum_{k=1}^{j-1} a_{jk}\xi_k^{(m+1)} - \sum_{k=j+1}^{n} a_{jk}\xi_k^{(m)}\right),$$

  where $j = 1, \ldots, n$ and we again assume $a_{jj} \neq 0$, for all j.
- We obtain a matrix form of by writing $A = -L + D - U$, where D is as in the Jacobi iteration and L and U are lower and upper triangular, respectively, with principal diagonal elements all zero:



Convergence of the Gauss-Seidel Iteration

- We now imagine that each equation in

$$\xi_j^{(m+1)} = \frac{1}{a_{jj}} \left(\gamma_j - \sum_{k=1}^{j-1} a_{jk} \xi_k^{(m+1)} - \sum_{k=j+1}^{n} a_{jk} \xi_k^{(m)} \right),$$

is multiplied by a_{jj}.
Then we can write the resulting system in the form

$$Dx^{(m+1)} = c + Lx^{(m+1)} + Ux^{(m)}$$

or $(D - L)x^{(m+1)} = c + Ux^{(m)}$.
Multiplication by $(D - L)^{-1}$ gives

$$x^{(m+1)} = Cx^{(m)} + b, \quad C = (D - L)^{-1} U, \; b = (D - L)^{-1} c.$$

- Condition $\sum_{k=1}^{n} |c_{jk}| < 1$ applied to $C = (D - L)^{-1} U$ is sufficient for the convergence of the Gauss-Seidel iteration.
- Since C is complicated, the remaining practical problem is to get simpler conditions sufficient for the validity of $\sum_{k=1}^{n} |c_{jk}| < 1$.

Subsection 3

Applications of Banach's Theorem to Differential Equations

From Banach's Theorem to Picard's Theorem

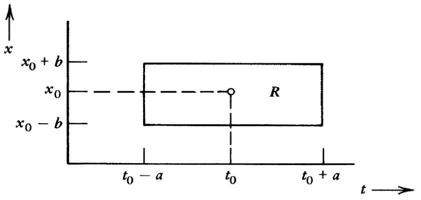
- We consider an explicit ordinary differential equation of the first order $x' = f(t,x)$, where $' = \frac{d}{dt}$.
- An **initial value problem** for such an equation consists of the equation and an initial condition $x(t_0) = x_0$, where t_0 and x_0 are given real numbers.
- We shall use Banach's Theorem to prove the famous Picard's Theorem:
    - The initial value problem will be converted to an integral equation, which defines a mapping T;
    - The conditions of the theorem will imply that T is a contraction such that its fixed point becomes the solution of our problem.

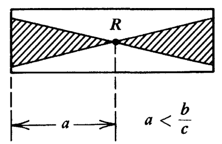
Picard's Existence and Uniqueness Theorem

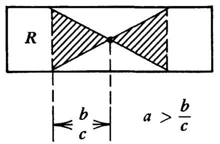
Picard's Existence and Uniqueness Theorem (ODEs)

Let f be continuous on a rectangle $R = \{(t,x) : |t - t_0| \le a, |x - x_0| \le b\}$ and, thus, bounded on R, say $|f(t,x)| \le c$, for all $(t,x) \in R$. Suppose that f satisfies a **Lipschitz condition** on R with respect to its second argument, i.e., there is a constant k (Lipschitz constant), such that for (t,x), $(t,v) \in R$,

$$|f(t,x) - f(t,v)| \le k|x - v|.$$







Then, the initial value problem has a unique solution. This solution exists on an interval $[t_0 - \beta, t_0 + \beta]$, where $\beta < \min\{a, \frac{b}{c}, \frac{1}{k}\}$.

Picard's Existence and Uniqueness Theorem (Cont'd)

- Let $C(J)$ be the metric space of all real-valued continuous functions on the interval $J=[t_0-\beta,t_0+\beta]$ with metric d defined by $d(x,y)=\max_{t\in J}|x(t)-y(t)|$.

  $C(J)$ is complete.

  Let $\widetilde{C}$ be the subspace of $C(J)$ consisting of all those functions $x\in C(J)$ that satisfy $|x(t)-x_0|\leq c\beta$. It is not difficult to see that $\widetilde{C}$ is closed in $C(J)$, so that $\widetilde{C}$ is complete.

  By integration we see that the equation can be written $x=Tx$, where $T:\widetilde{C}\to\widetilde{C}$ is defined by $Tx(t)=x_0+\int_{t_0}^t f(\tau,x(\tau))d\tau$. Indeed, T is defined for all $x\in\widetilde{C}$, because $c\beta<b$, so that if $x\in C$, then $\tau\in J$ and $(\tau,x(\tau))\in R$, and the integral exists since f is continuous on R. To see that T maps $\widetilde{C}$ into itself, we can use the integral form of $Tx(t)$ and boundedness to obtain

$$|Tx(t)-x_0|=\left|\int_{t_0}^t f(\tau,x(\tau))d\tau\right|\leq c|t-t_0|\leq c\beta.$$

Picard's Existence and Uniqueness Theorem (Cont'd)

- We show that T is a contraction on $\widetilde{C}$. By the Lipschitz condition,

$$\begin{array}{rcl} |Tx(t)-Tv(t)| & = & \left|\int_{t_0}^t [f(\tau,x(\tau))-f(\tau,v(\tau))]d\tau\right| \\ & \leq & |t-t_0|\max_{\tau\in J} k|x(\tau)-v(\tau)| \\ & \leq & k\beta d(x,v). \end{array}$$

Since the last expression does not depend on t, we can take the maximum on the left and have $d(Tx,Tv)\leq\alpha d(x,v)$, where $\alpha=k\beta$. From the assumption on β, we see that $\alpha=k\beta<1$, so that T is indeed a contraction on $\widetilde{C}$. Banach's Theorem thus implies that T has a unique fixed point $x\in\widetilde{C}$, that is, a continuous function x on J satisfying $x=Tx$. Writing $x=Tx$ out, $x(t)=x_0+\int_{t_0}^t f(\tau,x(\tau))d\tau$. Since $(\tau,x(\tau))\in R$, where f is continuous, this expression may be differentiated. Hence x is even differentiable and satisfies the given equation. Conversely, every solution of the equation must satisfy $x(t)=x_0+\int_{t_0}^t f(\tau,x(\tau))d\tau$.

Remarks on the Initial Value Problem

- Banach's Theorem also implies that the solution x of $\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$ is the limit of the sequence $(x_0, x_1, \ldots)$ obtained by the *Picard Iteration*

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(\tau, x_n(\tau)) d\tau, \quad n = 0, 1, \ldots.$$

- The practical usefulness of this way of obtaining approximations to the solution and corresponding error bounds is rather limited because of the integrations involved.
- It can be shown that continuity of f is sufficient (but not necessary) for the existence of a solution of the initial value problem, but not sufficient for uniqueness.
- A Lipschitz condition is sufficient (as Picard's Theorem shows), but not necessary.

Subsection 4

Application of Banach's Theorem to Integral Equations

Fredholm Equation of the Second Kind

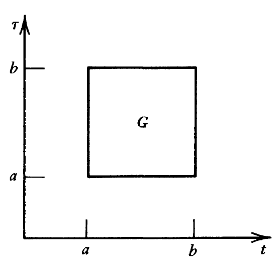
- An integral equation of the form

$$x(t) - \mu \int_a^b k(t,\tau)x(\tau)d\tau = v(t)$$

is called a **Fredholm equation** of the *second kind*.

Here:

- $[a,b]$ is a given interval;
- x is a function on $[a,b]$ which is unknown;
- μ is a parameter;
- The **kernel** k of the equation is a given function on the square $G = [a,b] \times [a,b]$;
- v is a given function on $[a,b]$.



Fredholm Integral Equation Theorem

Theorem (Fredholm Integral Equation)

Suppose k and v in the Fredholm Equation are continuous on $J \times J$ and $J = [a, b]$, respectively, and assume that μ satisfies $|\mu| < \frac{1}{c(b-a)}$ with c defined by $|k(t, \tau)| \leq c$, for all $(t, \tau) \in J \times J$. Then the Equation has a unique solution x on J. This function x is the limit of the iterative sequence $(x_0, x_1, \ldots)$, where x_0 is any continuous function on J and for $n = 0, 1, \ldots$,

$$x_{n+1} = v(t) + \mu \int_a^b k(t, \tau) x_n(\tau) d\tau.$$

- We consider the integral equation on $C[a, b]$, the space of all continuous functions defined on the interval $J = [a, b]$, with metric d given by $d(x, y) = \max_{t \in J} |x(t) - y(t)|$. Note that $C[a, b]$ is complete. Assume that $v \in C[a, b]$ and k is continuous on G. Then k is a bounded function on G, say, $|k(t, \tau)| \leq c$, for all $(t, \tau) \in G$.

Fredholm Integral Equation Theorem

- Obviously, $x(t) - \mu \int_a^b k(t,\tau)x(\tau)d\tau = v(t)$ can be written $x = Tx$, where
$$Tx(t) = v(t) + \mu \int_a^b k(t,\tau)x(\tau)d\tau.$$
Since v and k are continuous, this formula defines an operator $T : C[a,b] \to C[a,b]$.
We use the bound on μ to show that T is a contraction:
$$\begin{array}{rcl} d(Tx,Ty) & = & \max_{t\in J} |Tx(t) - Ty(t)| \\ & = & |\mu| \max_{t\in J} \left| \int_a^b k(t,\tau)[x(\tau) - y(\tau)]d\tau \right| \\ & \leq & |\mu| \max_{t\in J} \int_a^b |k(t,\tau)||x(\tau) - y(\tau)|d\tau \\ & \leq & |\mu| c \max_{\sigma\in J} |x(\sigma) - y(\sigma)| \int_a^b d\tau \\ & = & |\mu| c d(x,y)(b-a). \end{array}$$
This can be written $d(Tx,Ty) \leq \alpha d(x,y)$, where $\alpha = |\mu| c(b-a) < 1$.
Now we apply Banach's Fixed Point Theorem.

The Fixed Point Lemma

Lemma (Fixed Point)

Let $T : X \to X$ be a continuous mapping on a complete metric space $X = (X, d)$, and suppose that T^m is a contraction on X, for some positive integer m. Then T has a unique fixed point.

- $B = T^m$ is a contraction on X, i.e., $d(Bx, By) \leq \alpha d(x, y)$, for all $x, y \in X$, where $\alpha < 1$. Hence, for every $x_0 \in X$,

$$d(B^n Tx_0, B^n x_0) \leq \alpha d(B^{n-1} Tx_0, B^{n-1} x_0) \leq \cdots \leq \alpha^n d(Tx_0, x_0) \stackrel{n\to\infty}{\longrightarrow} 0.$$

Banach's theorem implies that B has a unique fixed point, call it x, and $B^n x_0 \to x$.

Since the mapping T is continuous, this implies $B^n Tx_0 = TB^n x_0 \to Tx$. Hence $d(B^n Tx_0, B^n x_0) \to d(Tx, x)$, so that $d(Tx, x) = 0$. This shows that x is a fixed point of T. Since every fixed point of T is also a fixed point of B, we see that T cannot have more than one fixed point.

The Volterra Integral Equation

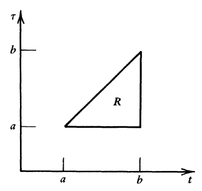
- We now consider the **Volterra integral equation**

$$x(t) - \mu \int_a^t k(t,\tau)x(\tau)d\tau = v(t).$$

- The difference with the Fredholm equation is that the upper limit of integration in the Volterra equation is variable.

Theorem (Volterra Integral Equation)

Suppose that v in the Volterra Equation is continuous on $[a,b]$ and the kernel k is continuous on the triangular region R in the $t\tau$-plane given by $a \le \tau \le t$, $a \le t \le b$. Then the equation has a unique solution x on $[a,b]$, for every μ.



- The equation can be written $x = Tx$, with $T: C[a,b] \to C[a,b]$ defined by $Tx(t) = v(t) + \mu \int_a^t k(t,\tau)x(\tau)d\tau$.

The Volterra Integral Equation (Cont'd)

- Since k is continuous on R and R is closed and bounded, k is a bounded function on R, say, $|k(t,\tau)| \le c$, for all $(t,\tau) \in R$.
  Using $d(x,y) = \max_{t\in J}|x(t)-y(t)|$, we have, for all $x,y \in C[a,b]$,

$$\begin{array}{rcl} |Tx(t)-Ty(t)| & = & |\mu|\left|\int_a^t k(t,\tau)[x(\tau)-y(\tau)]d\tau\right| \\ & \le & |\mu|cd(x,y)\int_a^t d\tau = |\mu|c(t-a)d(x,y). \end{array}$$

  We show by induction that

$$|T^m x(t) - T^m y(t)| \le |\mu|^m c^m \frac{(t-a)^m}{m!} d(x,y).$$

  - For $m=1$ this is the preceding inequality.
  - Assuming that it holds for any m, we obtain

$$\begin{array}{rcl} |T^{m+1}x(t) - T^{m+1}y(t)| & = & |\mu|\left|\int_a^t k(t,\tau)[T^m x(\tau) - T^m y(\tau)]d\tau\right| \\ & \le & |\mu|c\int_a^t |\mu|^m c^m \frac{(\tau-a)^m}{m!} d\tau d(x,y) \\ & = & |\mu|^{m+1}c^{m+1}\frac{(t-a)^{m+1}}{(m+1)!}d(x,y). \end{array}$$

The Volterra Integral Equation (Cont'd)

- Using $t-a \le b-a$ on the right-hand side and then taking the maximum over $t \in J$ on the left, we obtain $d(T^m x, T^m y) \le \alpha_m d(x,y)$, where

$$\alpha_m = |\mu|^m c^m \frac{(b-a)^m}{m!}.$$

  For any fixed μ and sufficiently large m, we have $\alpha_m < 1$. Hence, the corresponding T^m is a contraction on $C[a,b]$. Now we apply the Fixed Point Lemma.
- We finally note that a Volterra equation can be regarded as a special Fredholm equation whose kernel k is zero in the part of the square $[a,b] \times [a,b]$ where $\tau > t$ and may not be continuous at points on the diagonal $\tau = t$.