Introduction to Functional Analysis

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1. Approximation Theory
   - Approximation in Normed Spaces
   - Uniqueness, Strict Convexity
   - Uniform Approximation
   - Chebyshev Polynomials
   - Approximation in Hilbert Space

Subsection 1

Approximation in Normed Spaces

Introducing Approximation Theory

- **Approximation theory** is concerned with the approximation of functions of a certain kind (e.g., continuous functions on some interval) by other (probably simpler) functions (e.g., polynomials).
- E.g., in calculus, if a function has a Taylor series, we may regard and use the partial sums of the series as approximations.

  To get information about the quality of such approximations, we would have to estimate the corresponding remainders.
- More generally, one may want to set up practically useful **criteria for the quality of approximations**.
- Given a set X of functions to be approximated and a set Y of functions by which the elements of X are to be approximated, one may consider the problems of:
  - existence,
  - uniqueness, and
  - construction

  of a "best approximation" in the sense of such a criterion.

An Approximation Framework

- Let $X = (X, \|\cdot\|)$ be a normed space and suppose that any given $x \in X$ is to be approximated by a $y \in Y$, where Y is a fixed subspace of X.
- Let δ denote the distance from x to Y:

$$\delta = \delta(x, Y) = \inf_{y \in Y} \|x - y\|.$$

- Clearly, δ depends on both x and Y, which we keep fixed, so that the simple notation δ makes sense.
- If there exists a $y_0 \in Y$, such that $\|x - y_0\| = \delta$, then y_0 is called a **best approximation to x out of Y**.
- We see that a best approximation y_0 is an element of minimum distance from the given x.
  - Such a $y_0 \in Y$ may or may not exist; this raises the problem of **existence**.
  - The problem of **uniqueness** is of practical interest, too, since for given x and Y, there may be more than one best approximation.

Existence Theorem

Existence Theorem (Best Approximations)

If Y is a finite dimensional subspace of a normed space $X = (X, \|\cdot\|)$, then, for each $x \in X$, there exists a best approximation to x out of Y.

- Let $x \in X$ be given. Consider the closed ball $\widetilde{B} = \{y \in Y : \|y\| \le 2\|x\|\}$. Then $0 \in \widetilde{B}$, so that for the distance from x to $\widetilde{B}$ we obtain

$$\delta(x, \widetilde{B}) = \inf_{\widetilde{y} \in \widetilde{B}} \|x - \widetilde{y}\| \le \|x - 0\| = \|x\|.$$

Now if $y \notin \widetilde{B}$, then $\|y\| > 2\|x\|$ and $\|x - y\| \ge \|y\| - \|x\| > \|x\| \ge \delta(x, \widetilde{B})$. This shows that $\delta(x, \widetilde{B}) = \delta(x, Y) = \delta$, and this value cannot be assumed by a $y \in Y - \widetilde{B}$ because of the $>$. Hence if a best approximation to x exists, it must lie in $\widetilde{B}$.

Since $\widetilde{B}$ is closed and bounded and Y finite dimensional, $\widetilde{B}$ is compact. The norm is continuous, whence there is a $y_0 \in \widetilde{B}$, such that $\|x - y\|$ assumes a minimum at $y = y_0$. By definition, y_0 is a best approximation to x out of Y.

The Space $C[a,b]$

- A finite dimensional subspace of the space $C[a,b]$ is

$$Y = \text{span}\{x_0, \ldots, x_n\}, \quad x_j(t) = t^j, \quad n \text{ fixed.}$$

This is the set of all polynomials of degree at most n, together with $x = 0$.

The theorem implies that for a given continuous function x on $[a,b]$, there exists a polynomial p_n of degree at most n, such that for every $y \in Y$,

$$\max_{t \in J} |x(t) - p_n(t)| \leq \max_{t \in J} |x(t) - y(t)|,$$

where $J = [a,b]$.

Approximation in $C[a,b]$ is called **uniform approximation**.

Necessity of Finite Dimensionality of Y

- Finite dimensionality of Y in the theorem is essential.

  In fact, let Y be the set of all polynomials on $[0,\frac{1}{2}]$ of any degree, considered as a subspace of $C[0,\frac{1}{2}]$.

  Then $\dim Y = \infty$.

  Let $x(t) = \frac{1}{1-t}$. Then, for every $\varepsilon > 0$, there is an N, such that, setting

  $$y_n(t) = 1 + t + \cdots + t^n,$$

  we have $\|x - y_n\| < \varepsilon$, for all $n > N$. Hence $\delta(x, Y) = 0$.

  However, since x is not a polynomial, we see that there is no $y_0 \in Y$ satisfying

  $$\delta = \delta(x, Y) = \|x - y_0\| = 0.$$

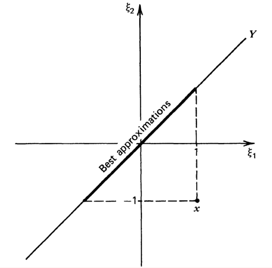
Subsection 2

Uniqueness, Strict Convexity

Examples

- If $X = \mathbb{R}^3$ and Y is the $\xi_1\xi_2$-plane ($\xi_3 = 0$), then we know that, for a given point $x_0 = (\xi_{10}, \xi_{20}, \xi_{30})$, a best approximation out of Y is the point $y_0 = (\xi_{10}, \xi_{20}, 0)$, the distance from x_0 to Y is $\delta = |\xi_{30}|$ and that the best approximation y_0 is unique.
- Let $X = (X, \|\cdot\|_1)$ be the vector space of ordered pairs $x = (\xi_1, \xi_2), \ldots$ of real numbers with norm defined by $\|x\|_1 = |\xi_1| + |\xi_2|$.

  Let us take the point $x = (1, -1)$ and the subspace Y shown in the figure, that is, $Y = \{y = (\eta, \eta) : \eta \text{ real}\}$. Then, for all $y \in Y$, we clearly have $\|x - y\|_1 = |1 - \eta| + |-1 - \eta| \geq 2$. The distance from x to Y is $\delta(x, Y) = 2$. All $y = (\eta, \eta)$ with $|\eta| \leq 1$, are best approximations to x out of Y.



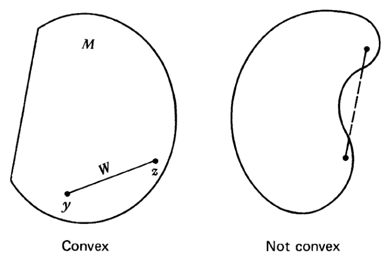
Convex Subsets

- A subset M of a vector space X is said to be **convex** if, for all $y, z \in M$, the set

$$W = \{v = \alpha y + (1-\alpha)z : 0 \leq \alpha \leq 1\}$$

is a subset of M.

  - This set W is called a **closed segment**.
  - y and z are called the **boundary points** of the segment W.
  - Any other point of W is called an **interior point** of W.



The Convexity Lemma

Lemma (Convexity)

In a normed space $(X,\|\cdot\|)$ the set M of best approximations to a given point x out of a subspace Y of X is convex.

- Let δ denote the distance from x to Y. The statement holds if M is empty or has just one point. Suppose that M has more than one point. Then for $y,z\in M$, we have, by definition, $\|x-y\|=\|x-z\|=\delta$.
  We show that $w=\alpha y+(1-\alpha)z\in M$.
  We have $\|x-w\|\geq\delta$, since $w\in Y$.
  Also $\|x-w\|\leq\delta$, since
  $$\begin{aligned}\|x-w\| &= \|\alpha(x-y)+(1-\alpha)(x-z)\|\\ &\leq \alpha\|x-y\|+(1-\alpha)\|x-z\|\\ &= \alpha\delta+(1-\alpha)\delta=\delta.\end{aligned}$$
  Here we used that $\alpha\geq 0$ as well as $1-\alpha\geq 0$.
  Together, $\|x-w\|=\delta$, whence $w\in M$.

Best Approximations and the Unit Sphere

- If there are several best approximations to x out of Y, then each of them lies in Y, of course, and has distance δ from x, by definition.
- Moreover, by the lemma, Y and the closed ball

$$\widetilde{B}(x;\delta) = \{v : \|v - x\| \le \delta\}$$

must have a segment W in common.

  - Obviously, W lies on the boundary sphere $S(x;\delta)$ of that closed ball, since every $w \in W$ has distance $\|w - x\| = \delta$ from x.
  - Furthermore, to each $w \in W$, there corresponds a unique $v = \frac{1}{\delta}(w - x)$ of norm $\|v\| = \frac{1}{\delta}\|w - x\| = 1$.

This means that to each best approximation $w \in W$, there corresponds a unique v on the unit sphere $\{x : \|x\| = 1\}$.

Uniqueness Theorem

- For obtaining uniqueness of best approximations, we must exclude norms for which the unit sphere contains segments of straight lines:

Definition (Strict Convexity)

A **strictly convex norm** is a norm such that, for all x, y of norm 1,

$$\|x+y\|<2, \quad x \neq y.$$

A normed space with such a norm is called a **strictly convex normed space**.

- Note that for $\|x\|=\|y\|=1$, the triangle inequality gives

$$\|x+y\| \leq \|x\|+\|y\|=2$$

and strict convexity excludes the equality sign, except when $x=y$.

Uniqueness Theorem (Best Approximation)

In a strictly convex normed space X, there is at most one best approximation to an $x \in X$ out of a given subspace Y.

The Strict Convexity Lemma

Lemma (Strict Convexity)

We have:

(a) Hilbert space is strictly convex.

(b) The space $C[a,b]$ is not strictly convex.

(a) Suppose $x \neq y$ of norm one. Let $\|x-y\| = \alpha > 0$. By the Parallelogram Equality, $\|x+y\|^2 = -\|x-y\|^2 + 2(\|x\|^2 + \|y\|^2) = -\alpha^2 + 2(1+1) < 4$. Hence, $\|x+y\| < 2$.

(b) We consider x_1 and x_2 defined by $x_1(t) = 1$ and $x_2(t) = \frac{t-a}{b-a}$, $t \in [a,b]$. Clearly, $x_1, x_2 \in C[a,b]$ and $x_1 \neq x_2$. Also $\|x_1\| = \|x_2\| = 1$ and

$$\|x_1 + x_2\| = \max_{t \in J} \left| 1 + \frac{t-a}{b-a} \right| = 2.$$

This shows that $C[a,b]$ is not strictly convex.

Consequences of the the Strict Convexity Lemma

- The first statement of this lemma had to be expected:

Theorem (Hilbert Space)

For every given x in a Hilbert space H and every given closed subspace Y of H, there is a unique best approximation to x out of Y (namely, $y = Px$, where P is the projection of H onto Y).

- From the second statement in the Strict Convexity Lemma, we see that in uniform approximation, additional effort will be necessary to guarantee uniqueness.

Subsection 3

Uniform Approximation

Introducing Uniform Approximation

- Depending on the choice of a norm, we get different types of approximations:
  (A) **Uniform approximation** uses the norm on $C[a,b]$ defined by
  $$\|x\| = \max_{t\in J} |x(t)|, \qquad J = [a,b].$$
  (B) **Least squares approximation** uses the norm on $L^2[a,b]$ defined by
  $$\|x\| = \langle x,x\rangle^{1/2} = \left(\int_a^b |x(t)|^2 dt\right)^{1/2}.$$
- We look now at uniform approximation (also known as **Chebyshev approximation**):
  Consider the real space $X = C[a,b]$ (of real-valued and continuous functions on $[a,b]$) and an n-dimensional subspace $Y \subseteq C[a,b]$.
  - For every function $x \in X$, we show the existence of a best approximation to x out of Y.
  - However, since $C[a,b]$ is not strictly convex, the problem of uniqueness requires a special investigation.

Extremal Points and the Haar Condition

Definition (Extremal Point)

An **extremal point** of an x in $C[a,b]$ is a $t_0 \in [a,b]$, such that $|x(t_0)| = \|x\|$.

- Hence at an extremal point t_0 of x we have either $x(t_0) = +\|x\|$ or $x(t_0) = -\|x\|$.
- The definition of the norm on $C[a,b]$ shows that such a point is a $t_0 \in [a,b]$ at which $|x(t)|$ has a maximum.

Definition (Haar Condition)

A finite dimensional subspace Y of the real space $C[a,b]$ is said to satisfy the **Haar condition** if every $y \in Y$, $y \neq 0$, has at most $n-1$ zeros in $[a,b]$, where $n = \dim Y$.

- For instance, an n-dimensional subspace Y of $C[a,b]$ satisfying the Haar Condition is given by the polynomial $y = 0$ and all polynomials of degree not exceeding $n-1$ and with real coefficients.

Characterization of the Haar Condition

- The Haar condition is equivalent to the condition that, for every basis $\{y_1,\ldots,y_n\} \subseteq Y$ and every n-tuple of distinct points $t_1,\ldots,t_n$ in the interval $J=[a,b]$, $\begin{vmatrix} y_1(t_1) & y_1(t_2) & \cdots & y_1(t_n) \\ y_2(t_1) & y_2(t_2) & \cdots & y_2(t_n) \\ \vdots & \vdots & & \vdots \\ y_n(t_1) & y_n(t_2) & \cdots & y_n(t_n) \end{vmatrix} \neq 0.$

  Every $y \in Y$ has a representation $y = \sum \alpha_k y_k$. The subspace Y satisfies the Haar condition if and only if every $y = \sum \alpha_k y_k \in Y$, with n or more zeros $t_1,t_2,\ldots,t_n,\ldots$ in $J=[a,b]$, is identically zero. This means that the n conditions

$$y(t_j) = \sum_{k=1}^{n} \alpha_k y_k(t_j) = 0, \qquad j=1,\ldots,n,$$

  should imply $\alpha_1 = \cdots = \alpha_n = 0$. But this happens if and only if the determinant is not zero.

The Extremal Points Lemma

Lemma (Extremal Points)

Suppose a subspace Y of the real space $C[a,b]$ satisfies the Haar condition. If, for a given $x \in C[a,b]$ and a $y \in Y$ the function $x-y$ has less than $n+1$ extremal points, then y is not a best approximation to x out of Y, where $n = \dim Y$.

- By assumption, the function $v = x-y$ has $m \le n$ extremal points $t_1, \ldots, t_m$. If $m < n$, we choose any additional points t_j in $J = [a,b]$ until we have n distinct points $t_1, \ldots, t_n$. Using these points and a basis $\{y_1, \ldots, y_n\}$ for Y, we consider the nonhomogeneous system of linear equations $\sum_{k=1}^{n} \beta_k y_k(t_j) = v(t_j)$, $j = 1, \ldots, n$, in the unknowns $\beta_1, \ldots, \beta_n$. Since Y satisfies the Haar condition, the determinant is nonzero. Hence, the system has a unique solution. We use this solution to define $y_0 = \beta_1 y_1 + \cdots + \beta_n y_n$ as well as $\widetilde{y} = y + \varepsilon y_0$.

The Extremal Points Lemma (Cont'd)

Claim: For a sufficiently small ε, the function $\widetilde{v} = x - \widetilde{y}$ satisfies $\|\widetilde{v}\| < \|v\|$, so that y cannot be a best approximation to x out of Y.

We estimate $\widetilde{v}$, breaking $J = [a, b]$ up into two sets N and $K = J - N$, where N contains the extremal points $t_1, \ldots, t_m$ of v. At the extremal points, $|v(t_i)| = \|v\|$ and $\|v\| > 0$ since $v = x - y \neq 0$. Also $y_0(t_i) = v(t_i)$ by the system and the definition of y_0. Hence, by continuity, for each t_i, there is a neighborhood N_i, such that in $N = N_1 \cup \cdots \cup N_m$ we have

$$\mu = \inf_{t \in N} |v(t)| > 0, \quad \inf_{t \in N} |y_0(t)| \geq \frac{1}{2} \|v\|.$$

Since $y_0(t_i) = v(t_i) \neq 0$, for all $t \in N$, we have $\frac{y_0(t)}{v(t)} > 0$ and $\frac{y_0(t)}{v(t)} = \frac{|y_0(t)|}{|v(t)|} \geq \frac{\inf |y_0(t)|}{\|v\|} \geq \frac{1}{2}$. Let $M_0 = \sup_{t \in N} |y_0(t)|$. Then, for every positive $\varepsilon < \frac{\mu}{M_0}$ and every $t \in N$, we obtain $\frac{\varepsilon y_0(t)}{v(t)} = \frac{\varepsilon |y_0(t)|}{|v(t)|} \leq \frac{\varepsilon M_0}{\mu} < 1$.

The Extremal Points Lemma (Conclusion)

- Recall $\widetilde{v} = x - \widetilde{y} = x - y - \varepsilon y_0 = v - \varepsilon y_0$. Using the inequalities, we see that, for all $t \in N$ and $0 < \varepsilon < \frac{\mu}{M_0}$,

$$|\widetilde{v}(t)| = |v(t) - \varepsilon y_0(t)| = |v(t)|\left(1 - \frac{\varepsilon y_0(t)}{v(t)}\right) \leq \|v\|\left(1 - \frac{\varepsilon}{2}\right) < \|v\|.$$

  We turn to the complement $K = J - N$ and define $M_1 = \sup_{t \in K} |y_0(t)|$ and $M_2 = \sup_{t \in K} |v(t)|$. Since N contains all the extremal points of v, we have $M_2 < \|v\|$. So $\|v\| = M_2 + \eta$, where $\eta > 0$. Choosing a positive $\varepsilon < \frac{\eta}{M_1}$, we have $\varepsilon M_1 < \eta$. Thus, for all $t \in K$,

$$|\widetilde{v}(t)| \leq |v(t)| + \varepsilon |y_0(t)| \leq M_2 + \varepsilon M_1 < M_2 + \eta = \|v\|.$$

  We see that $|\widetilde{v}(t)|$ does not exceed a bound which is independent of $t \in K$ and strictly less than $\|v\|$. Similarly in the first case, where $t \in N$ and $\varepsilon > 0$ is sufficiently small. Choosing $\varepsilon < \min\{\frac{\mu}{M_0}, \frac{\eta}{M_1}\}$ and taking the supremum, we thus have $\|\widetilde{v}\| < \|v\|$.

The Haar Uniqueness Theorem (Sufficiency)

Haar Uniqueness Theorem (Best Approximation)

Let Y be a finite dimensional subspace of the real space $C[a,b]$. Then the best approximation out of Y is unique for every $x \in C[a,b]$ if and only if Y satisfies the Haar condition.

(a) **Sufficiency**: Suppose Y satisfies the Haar condition, but both $y_1 \in Y$ and $y_2 \in Y$ are best approximations to some fixed $x \in C[a,b]$. Then, setting $v_1 = x - y_1$, $v_2 = x - y_2$, we have $\|v_1\| = \|v_2\| = \delta$, where δ is the distance from x to Y. Now $y = \frac{1}{2}(y_1 + y_2)$ is also a best approximation to x. Thus, $v = x - y = x - \frac{1}{2}(y_1 + y_2) = \frac{1}{2}(v_1 + v_2)$ has at least $n+1$ extremal points $t_1, \ldots, t_{n+1}$. At such a point, $|v(t_j)| = \|v\| = \delta$. We now get $2v(t_j) = v_1(t_j) + v_2(t_j) = \pm 2\delta$. Now $|v_1(t_j)| \le \|v_1\| = \delta$ and similarly for v_2. Hence, for the equation to hold, $v_1(t_j) = v_2(t_j) = \pm\delta$, where $j = 1, \ldots, n+1$. But this implies that $y_1 - y_2 = v_2 - v_1$ has $n+1$ zeros in $[a,b]$. Hence $y_1 - y_2 = 0$ by the Haar condition. Thus, $y_1 = y_2$.

The Haar Uniqueness Theorem (Necessity)

(b) **Necessity**: Assume that Y does not satisfy the Haar condition. We show that we do not have uniqueness of best approximations for all $x \in C[a,b]$. There is a basis for Y and n values t_i in $[a,b]$, such that the determinant is zero. Hence, the homogeneous system $\gamma_1 y_k(t_1) + \gamma_2 y_k(t_2) + \cdots + \gamma_n y_k(t_n) = 0$, $k = 1, \ldots, n$, has a nontrivial solution $\gamma_1, \ldots, \gamma_n$. Using this solution and any $y = \sum \alpha_k y_k \in Y$,

$$\sum_{j=1}^{n} \gamma_j y(t_j) = \sum_{k=1}^{n} \alpha_k \left[\sum_{j=1}^{n} \gamma_j y_k(t_j) \right] = 0.$$

The transposed system $\beta_1 y_1(t_j) + \beta_2 y_2(t_j) + \cdots + \beta_n y_n(t_j) = 0$, $j = 1, \ldots, n$, has a nontrivial solution $\beta_1, \ldots, \beta_n$. Define $y_0 = \sum \beta_k y_k$. Then $y_0 \neq 0$, and y_0 is zero at $t_1, \ldots, t_n$. Let λ be such that $\|\lambda y_0\| \leq 1$. Take $z \in C[a,b]$ such that $\|z\| = 1$, $z(t_j) = \mathrm{sgn}\gamma_j = \begin{cases} -1, & \text{if } \gamma_j < 0 \\ 1, & \text{if } \gamma_j \geq 0 \end{cases}$.

The Haar Uniqueness Theorem (Necessity Cont'd)

- Define $x \in C[a,b]$ by $x(t) = z(t)(1 - |\lambda y_0(t)|)$.
  Then $x(t_j) = z(t_j) = \text{sgn}\gamma_j$, since $y_0(t_j) = 0$. Also $\|x\| = 1$.
  We show that x has infinitely many best approximations out of Y.
  Using $|z(t)| \le \|z\| = 1$ and $|\lambda y_0(t)| \le \|\lambda y_0\| \le 1$, for every $\varepsilon \in [-1,1]$,

$$\begin{array}{rcl} |x(t) - \varepsilon\lambda y_0(t)| & \le & |x(t)| + |\varepsilon\lambda y_0(t)| \\ & = & |z(t)|(1 - |\lambda y_0(t)|) + |\varepsilon\lambda y_0(t)| \\ & \le & 1 - |\lambda y_0(t)| + |\varepsilon\lambda y_0(t)| \\ & = & 1 - (1 - |\varepsilon|)|\lambda y_0(t)| \le 1. \end{array}$$

  Hence every $\varepsilon\lambda y_0$, $-1 \le \varepsilon \le 1$, is a best approximation to x, provided $\|x - y\| \ge 1$, for all $y \in Y$. We prove this for arbitrary $y = \sum \alpha_k y_k \in Y$.
  Suppose that $\|x - \widetilde{y}\| < 1$, for a $\widetilde{y} \in Y$. Then $x(t_j) = \text{sgn}\gamma_j = \pm 1$, and $|x(t_j) - \widetilde{y}(t_j)| \le \|x - \widetilde{y}\| < 1$ imply, for all $\gamma_j \ne 0$, $\text{sgn}\widetilde{y}(t_j) = \text{sgn}x(t_j) = \text{sgn}\gamma_j$. This contradicts $\sum_{j=1}^n \gamma_j y(t_j) = 0$, with $y = \widetilde{y}$: Since $\gamma_j \ne 0$, for some j, $\sum_{j=1}^n \gamma_j \widetilde{y}(t_j) = \sum_{j=1}^n \gamma_j \text{sgn}\gamma_j = \sum_{j=1}^n |\gamma_j| \ne 0$.

Polynomial Approximations

- Note that if Y is the set of all real polynomials of degree $\leq n$, together with the polynomial $y = 0$ (for which a degree is not defined in the usual discussion of degree), then $\dim Y = n + 1$ and Y satisfies the Haar condition.

Theorem (Polynomials)

The best approximation to an x in the real space $C[a, b]$ out of Y_n is unique, where Y_n is the subspace consisting of $y = 0$ and all polynomials of degree not exceeding a fixed given n.

- In this theorem, it is worthwhile to compare the approximations for various n and see what happens as $n \to \infty$:

  Let $\delta_n = \|x - p_n\|$, where $p_n \in Y_n$ is the best approximation to a fixed given x. Since $Y_0 \subseteq Y_1 \subseteq \cdots$, we have $\delta_0 \geq \delta_1 \geq \delta_2 \geq \cdots$. The Weierstraß approximation theorem implies that $\lim_{n \to \infty} \delta_n = 0$.

Subsection 4

Chebyshev Polynomials

Alternating Set

Definition (Alternating Set)

Let $x \in C[a,b]$ and $y \in Y$, where Y is a subspace of the real space $C[a,b]$. A set of points $t_0, \ldots, t_k$ in $[a,b]$, where $t_0 < t_1 < \cdots < t_k$, is called an **alternating set** for $x - y$ if $x(t_j) - y(t_j)$ has alternately the values $+\|x-y\|$ and $-\|x-y\|$ at consecutive points t_j.

- These $k+1$ points in the definition are extremal points of $x-y$ and the values of $x-y$ at these points are alternating positive and negative.
- The following lemma states that the existence of a sufficiently large alternating set for $x-y$ implies that y is the best approximation to x.
- This condition is also necessary for y to be the best approximation to x, but this will not be proven.

The Best Approximation Lemma

Lemma (Best Approximation)

Let Y be a subspace of the real space $C[a,b]$ satisfying the Haar condition. Given $x \in C[a,b]$, let $y \in Y$ such that for $x-y$, there exists an alternating set of $n+1$ points, where $n = \dim Y$. Then y is the best uniform approximation to x out of Y.

- We know that there is a unique best approximation to x out of Y. If this is not y, it is some other $y_0 \in Y$ and then $\|x-y\| > \|x-y_0\|$. At the $n+1$ extremal points the function $y_0 - y = (x-y)-(x-y_0)$ has the same sign as $x-y$:
  - $x-y$ equals $\pm\|x-y\|$ at such a point;
  - the other term on the right, $x-y_0$, can never exceed $\|x-y_0\|$ in absolute value, which is strictly less than $\|x-y\|$.

  Thus, $y_0 - y$ is alternating positive and negative at those $n+1$ points. So it must have at least n zeros in $[a,b]$. But this is impossible unless $y_0 - y = 0$, since $y_0 - y \in Y$ and Y satisfies the Haar condition. Hence y must be the best approximation to x out of Y.

A Classical Approximation Problem

- An application of this lemma is the approximation of $x \in C[-1,1]$ defined by $x(t) = t^n$, $n \in \mathbb{N}$ fixed, out of $Y = \text{span}\{y_0, \ldots, y_{n-1}\}$, where $y_j(t) = t^j$, $j = 0, \ldots, n-1$.

  Obviously, this means that we want to approximate x on $[-1,1]$ by a real polynomial y of degree less than n.

  Such a polynomial is of form $y(t) = \alpha_{n-1} t^{n-1} + \alpha_{n-2} t^{n-2} + \cdots + \alpha_0$.

  For $z = x - y$, we have $z(t) = t^n - (\alpha_{n-1} t^{n-1} + \alpha_{n-2} t^{n-2} + \cdots + \alpha_0)$.

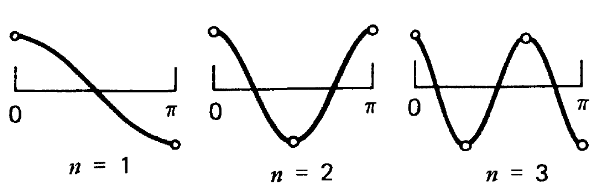
  We want to find y such that $\|z\|$ becomes as small as possible.

  Since z is a polynomial of degree n with leading coefficient 1, our original problem is equivalent to:

  > Find the polynomial z which, among all polynomials of degree n and with leading coefficient 1, has the smallest maximum deviation from 0 on the interval $[-1,1]$ under consideration.

Introducing Trigonometric Expressions

- Set $t = \cos\theta$ and let θ vary from 0 to π.
- Then t varies on the interval $[-1,1]$.
- On $[0,\pi]$ the function defined by $\cos n\theta$ has $n+1$ extremal points, the values being ± 1 in alternating order.



- Because of the lemma, we hope that $\cos n\theta$ will help to solve our problem, provided we are able to write $\cos n\theta$ as a polynomial in $t = \cos\theta$.

Expression for $\cos n\theta$

Claim: There is a representation of the form

$$\cos n\theta = 2^{n-1}\cos^n\theta + \sum_{j=0}^{n-1}\beta_{nj}\cos^j\theta, \quad n = 1, 2, \ldots,$$

where the β_{nj}'s are constants.

This is true for $n = 1$ (take $\beta_{10} = 0$). Assuming it to be true for any n, we show that it holds for $n+1$. We use The addition formula for the cosine $\cos(n\pm 1)\theta = \cos n\theta\cos\theta \mp \sin n\theta\sin\theta$. Adding, we have $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos n\theta\cos\theta$. Consequently, by the induction hypothesis,

$$\begin{array}{rcl}\cos(n+1)\theta & = & 2\cos\theta\cos n\theta - \cos(n-1)\theta \\ & = & 2\cos\theta(2^{n-1}\cos^n\theta + \sum_{j=0}^{n-1}\beta_{nj}cos^j\theta) \\ & & -2^{n-2}\cos^{n-1}\theta - \sum_{j=0}^{n-2}\beta_{n-1,j}\cos^j\theta.\end{array}$$

This yields $\cos(n+1)\theta = 2^n\cos^{n+1}\theta + \sum_{j=0}^{n}\beta_{n+1,j}\cos^j\theta$.

The Chebyshev Polynomials

- The functions defined by
$$T_n(t) = \cos n\theta, \qquad \theta = \arccos t, \qquad n = 0, 1, \ldots,$$
are called **Chebyshev polynomials** *of the first kind of order* n.
- The leading coefficient in $\cos n\theta = 2^{n-1}\cos^n\theta + \sum_{j=0}^{n-1}\beta_{nj}\cos^j\theta$ is not 1, as we want it, but 2^{n-1}.
Hence, we obtain the following formulation of our result, which expresses the minimum property of the Chebyshev polynomials:

Theorem (Chebyshev Polynomials)

The polynomial defined by
$$\widetilde{T}_n(t) = \frac{1}{2^{n-1}} T_n(t) = \frac{1}{2^{n-1}} \cos(n \arccos t), \qquad n \geq 1,$$
has the smallest maximum deviation from 0 on the interval $[-1, 1]$, among all real polynomials considered on $[-1, 1]$ which have degree n and leading coefficient 1.

An Alternative Formulation

- The best uniform approximation to the function $x \in C[-1,1]$ defined by

$$x(t) = t^n,$$

out of $Y = \text{span}\{y_0, \ldots, y_{n-1}\}$ with $y_j(t) = t^j$, $j = 0, \ldots, n-1$, (that is, the approximation by a real polynomial of degree less than n) is y defined by

$$y(t) = x(t) - \frac{1}{2^{n-1}} T_n(t), \qquad n \geq 1.$$

A Generalization

- Let $\widetilde{x}$ be a real polynomial of degree n with leading term $\beta_n t^n$.
- We are looking for the best approximation $\widetilde{y}$ to $\widetilde{x}$ on $[-1,1]$, where $\widetilde{y}$ is a polynomial of lower degree, at most $n-1$.
- We may write $\widetilde{x} = \beta_n x$.
- We see that x has the leading term t^n.
- From the theorem we conclude that $\widetilde{y}$ must satisfy $\frac{1}{\beta_n}(\widetilde{x} - \widetilde{y}) = \widetilde{T}_n$.
- The solution is

$$\widetilde{y}(t) = \widetilde{x} - \frac{\beta_n}{2^{n-1}} T_n(t), \qquad n \geq 1.$$

The Formula for Chebyshev Polynomials

- Explicit expressions of the first few Chebyshev polynomials can be readily obtained as follows:
  - $T_0(t) = \cos 0 = 1$;
  - $T_1(t) = \cos\theta = t$;
  - Since $T_n(t) = \cos n\theta$ and $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos n\theta\cos\theta$, we get $T_{n+1}(t) + T_{n-1}(t) = 2tT_n(t)$.
    Hence, we get the recursion formula
    $$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t), \qquad n = 1, 2, \ldots.$$
    Therefore, we have
    $$\begin{array}{ll} T_0(t) = 1, & T_1(t) = t, \\ T_2(t) = 2t^2 - 1, & T_3(t) = 4t^3 - 3t, \\ T_4(t) = 8t^4 - 8t^2 + 1, & T_5(t) = 16t^5 - 20t^3 + 5t. \end{array}$$
    In general,
    $$T_n(t) = \frac{n}{2}\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{(n-j-1)!}{j!(n-2j)!}(2t)^{n-2j}, \quad n = 1, 2, \ldots.$$

Subsection 5

Approximation in Hilbert Space

The Case of Hilbert Space

- For any given x in a Hilbert space H and a closed subspace $Y \subseteq H$, there exists a unique best approximation to x out of Y:
  We have
  $$H = Y \oplus Z, \qquad Z = Y^{\perp},$$
  so that, for each $x \in H$,
  $$x = y + z,$$
  where $z = x - y \perp y$, hence $\langle x - y, y \rangle = 0$.
- Suppose Y is finite dimensional, say, $\dim Y = n$, with basis $\{y_1, \ldots, y_n\}$.
  We have a unique representation $y = \alpha_1 y_1 + \cdots + \alpha_n y_n$.
  Then $x - y \perp Y$ gives the n conditions
  $$\begin{array}{rcl} 0 & = & \langle y_j, x - y \rangle \\ & = & \langle y_j, x - \sum_{k=1}^{n} \alpha_k y_k \rangle \\ & = & \langle y_j, x \rangle - \overline{\alpha}_1 \langle y_j, y_1 \rangle - \cdots - \overline{\alpha}_n \langle y_j, y_n \rangle. \end{array}$$

The Gram Determinant

- $\langle y_j, x\rangle - \overline{\alpha}_1\langle y_j, y_1\rangle - \cdots - \overline{\alpha}_n\langle y_j, y_n\rangle = 0$, where $j = 1, \ldots, n$, is a nonhomogeneous system of n linear equations in n unknowns $\overline{\alpha}_1, \ldots, \overline{\alpha}_n$.

  The determinant of the coefficients is

  $$G(y_1, \ldots, y_n) = \begin{vmatrix} \langle y_1, y_1\rangle & \langle y_1, y_2\rangle & \cdots & \langle y_1, y_n\rangle \\ \langle y_2, y_1\rangle & \langle y_2, y_2\rangle & \cdots & \langle y_2, y_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle y_n, y_1\rangle & \langle y_n, y_2\rangle & \cdots & \langle y_n, y_n\rangle \end{vmatrix}.$$

  Since y exists and is unique, that system has a unique solution. Hence, $G(y_1, \ldots, y_n)$ must be different from 0. The determinant $G := G(y_1, \ldots, y_n)$ is called the **Gram determinant** of $y_1, \ldots, y_n$.

  Cramer's rule now yields $\alpha_j = \frac{\overline{G_j}}{\overline{G}}$, where G_j is obtained from G by replacing the j-th column of G by the column with elements $\langle y_1, x\rangle, \ldots, \langle y_n, x\rangle$.

Linear Independence and Gram Determinants

Theorem (Linear Independence)

Elements $y_1, \ldots, y_n$ of a Hilbert space H constitute a linearly independent set in H if and only if

$$G(y_1, \ldots, y_n) \neq 0.$$

- The preceding discussion shows that in the case of linear independence, $G \neq 0$.

  If $\{y_1, \ldots, y_n\}$ is linearly dependent, one of the vectors, say y_j, is a linear combination of the others. Then the j-th column of G is a linear combination of the other columns, whence $G = 0$.

Distance and Gram Determinants

- The distance $\|z\| = \|x - y\|$ between x and the best approximation y to x can also be expressed by Gram determinants:

Theorem (Distance)

If $\dim Y < \infty$ and $\{y_1, \ldots, y_n\}$ is any basis for Y, then

$$\|z\|^2 = \frac{G(x, y_1, \ldots, y_n)}{G(y_1, \ldots, y_n)}, \text{ where } G(x, y_1, \ldots, y_n) = \begin{vmatrix} \langle x, x\rangle & \langle x, y_1\rangle & \cdots & \langle x, y_n\rangle \\ \langle y_1, x\rangle & \langle y_1, y_1\rangle & \cdots & \langle y_1, y_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle y_n, x\rangle & \langle y_n, y_1\rangle & \cdots & \langle y_n, y_n\rangle \end{vmatrix}.$$

- We have $\langle y, z\rangle = 0$, where $z = x - y$, so that $\|z\|^2 = \langle z, z\rangle + \langle y, z\rangle = \langle x, x - y\rangle = \langle x, x\rangle - \langle x, \sum \alpha_k y_k\rangle$. This can be written $-\|z\|^2 + \langle x, x\rangle - \overline{\alpha}_1 \langle x, y_1\rangle - \cdots - \overline{\alpha}_n \langle x, y_n\rangle = 0$. We now remember the n equations
$$\langle y_j, x\rangle - \overline{\alpha}_1 \langle y_j, y_1\rangle - \cdots - \overline{\alpha}_n \langle y_j, y_n\rangle = 0, \quad j = 1, \ldots, n.$$

Distance and Gram Determinants (Cont'd)

- The equations

$$\begin{array}{rcl} -\|z\|^2+\langle x,x\rangle-\overline{\alpha}_1\langle x,y_1\rangle-\cdots-\overline{\alpha}_n\langle x,y_n\rangle & = & 0 \\ \langle y_j,x\rangle-\overline{\alpha}_1\langle y_j,y_1\rangle-\cdots-\overline{\alpha}_n\langle y_j,y_n\rangle & = & 0, \quad j=1,\ldots,n, \end{array}$$

form a homogeneous system of $n+1$ linear equations in the $n+1$ "unknowns" $1,-\overline{\alpha}_1,\ldots,-\overline{\alpha}_n$.
Since the system has a nontrivial solution, the determinant of its coefficients must be zero, i.e., $\begin{vmatrix} \langle x,x\rangle-\|z\|^2 & \langle x,y_1\rangle & \cdots & \langle x,y_n\rangle \\ \langle y_1,x\rangle+0 & \langle y_1,y_1\rangle & \cdots & \langle y_1,y_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle y_n,x\rangle+0 & \langle y_n,y_1\rangle & \cdots & \langle y_n,y_n\rangle \end{vmatrix}=0.$
We can write this determinant as the sum of:
  - $G(x,y_1,\ldots,y_n)$;
  - the determinant that has elements $-\|z\|^2,0,\ldots,0$ in its first column.

Developing by the first column, $G(x,y_1,\ldots,y_n)-\|z\|^2G(y_1,\ldots,y_n)=0$.
This concludes the proof, since $G(y_1,\ldots,y_n)\neq 0$.

The Case of Orthonormal Basis

- If the basis $\{y_1,\ldots,y_n\}$ in

$$\|z\|^2 = \frac{G(x,y_1,\ldots,y_n)}{G(y_1,\ldots,y_n)}$$

  is orthonormal, then $G(y_1,\ldots,y_n) = 1$.

  By developing $G(x,y_1,\ldots,y_n)$ by its first row and noting that $\langle x,y_1\rangle\langle y_1,x\rangle = |\langle x,y_1\rangle|^2$, etc., we obtain

$$\|z\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x,y_k\rangle|^2.$$

- This agrees with

$$\|z\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x,e_k\rangle|^2,$$

  where $Y = \mathrm{span}\{e_1,\ldots,e_n\}$.