Introduction to Game Theory

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Subsection 1

Game Theory

Introducing Game Theory

- Game theory is concerned with the interaction of decision makers.
- It assumes that decision makers:
  - Pursue well-defined objectives (are rational);
  - Take into account their knowledge or expectations of other decision makers' behavior (they reason strategically).
- The models are highly abstract to ensure wide real-life applicability.
  - Nash equilibria have been used in economic and political competition.
  - Mixed strategy equilibria explain the distributions of biological features.
  - Repeated games illuminate social phenomena like threats and promises.

Introducing Game Theory (Cont'd)

- Game theory uses mathematics to express its ideas formally so that:
  - It is precise;
  - It is logically consistent;
  - It can deduce formal conclusions based on solid assumptions.
- Game theory is also a social science whose aim is to understand the behavior of interacting decision-makers.
  - Mathematical results should be confirmed by intuition;
  - Mathematical results should support and enhance the intuition.

Subsection 2

Games and Solutions

Games, Solutions and Classification

- A game is described by:
  - The constraints on the actions or moves that the players can make;
  - The players' interests or goals.
- A solution:
  - Specifies the moves that achieve the goals;
  - Describes the outcomes that may emerge.
- Game theory:
  - Discovers reasonable solutions for classes of games;
  - Examines their properties.
- Games are classified as:
  - Cooperative and Noncooperative Games;
  - Strategic and Extensive Games;
  - Games with Perfect or Imperfect Information.

Classification of Games

- In game theory, a player may be interpreted as an individual or as a group of individuals making a decision.
  - In noncooperative games we focus on the actions of individual players;
  - In cooperative games we focus on joint actions of groups of players.
- During gaming, a plan of actions can be decided either in advance or as the game develops.
  - In a strategic game each player chooses his plan of action in advance, simultaneously with all other players and independent of the plan of action chosen by the other players.
  - In an extensive game each player can consider his plan of action whenever he has to make a decision throughout the game.
- Depended on how informed players are, we distinguish between games:
  - With perfect information, in which players are fully informed about each others' moves;
  - With imperfect information, where some information about the other participants' actions is lacking.

Subsection 3

Rational Behavior

Deterministic Model of Rational Choice

- Given a deterministic environment, a **model of rational choice** consists of:
  - A set A of **actions** from which the decision maker makes a choice;
  - A set C of possible **consequences** of these actions;
  - A **consequence function**

    $$g : A \to C$$

    that associates a consequence with each action;
  - A **preference relation** (a complete transitive reflexive binary relation) $\succsim$ on the set C.

Deterministic Model of Rational Choice (Cont'd)

- Alternatively, the decision maker's preferences are specified by giving a **utility function** $U : C \to \mathbb{R}$, which defines a preference relation by the condition

$$x \succsim y \quad \text{if and only if} \quad U(x) \geq U(y).$$

- Given feasible $B \subseteq A$, a **rational decision maker** chooses an action $a^* \in B$ that is **optimal**, in the sense that

$$g(a^*) \succsim g(a), \quad \text{for all } a \in B.$$

- I.e., the rational decision maker solves the problem

$$\max_{a \in B} U(g(a)).$$

- The preference relation is independent of which $B \subseteq A$ is considered.

Nondeterministic Environments

- Nondeterministic environments are created when:
  - The players are uncertain about parameters of the environment;
  - Imperfectly informed about events that happen in the game;
  - Uncertain about actions of other players that are not deterministic;
  - Uncertain about the reasoning of the other players.

Nondeterministic Model of Rational Choice

- Consider the probabilities of the consequences of an action.
  - If they are known, players behave as if they maximize the expected value of a utility function that attaches a number to each consequence.
  - If they are not known, players behave as if they:
    - Subjectively create a "state space" Ω;
    - Evaluate a probability measure over Ω;
    - Attach consequences to pairs of actions and states using some function

$$g : A \times \Omega \to C;$$

    - Associate a utility function $u : C \to \mathbb{R}$.

  The choice then maximizes the expected value of $u(g(a, \omega))$ with respect to the probability measure.

Subsection 4

The Steady State and Deductive Interpretations

Steady State versus Deductive Interpretation

- There are two conflicting interpretations of solutions for strategic and extensive games.
  - The steady state interpretation treats a game as a model designed to explain regularities observed in similar situations.
    - Each participant recognizes, based on experience, the equilibrium.
    - He tests the optimality of his behavior given this knowledge.
  - The deductive interpretation treats a game in isolation and attempts to infer the restrictions that rationality imposes on the outcome.
    - It assumes that each player deduces how the other players will behave simply from principles of rationality.

Subsection 5

Bounded Rationaliy

Rationality

- Game theory assumes that:
  - The players' knowledge of the rules of the game is perfect;
  - Their ability to analyze it is ideal.
- Game theoretic results imply, e.g., that chess is a trivial game in the sense that an algorithm exists that can be used to "solve" the game.
- The algorithm defines a strategy for each player, that leads to an "equilibrium" outcome.
- The outcome for a player who follows the strategy will be at least as good as the equilibrium outcome.

Bounded Rationality

- Despite these results, in reality, chess remains a very popular and interesting game.
  - The reason is that its equilibrium outcome is yet to be calculated, since it is still impossible to do so using the algorithm.
  - While the abstract model of chess allows us to deduce a significant fact about the game, it does not factor in the players' "abilities".
- Modeling asymmetries in abilities and in perceptions of a situation by different players requires models of "bounded rationality", a newer area of game theory.

Subsection 6

Terminology and Notation

Sets and Inequalities

- The set of real numbers is denoted $\mathbb{R}$.
- The set of nonnegative real numbers by $\mathbb{R}_+$.
- The set of vectors of n real numbers by $\mathbb{R}^n$.
- The set of vectors of n nonnegative real numbers by $\mathbb{R}^n_+$.
- For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we use:
  - $x \geq y$ to mean $x_i \geq y_i$, for $i = 1, \ldots, n$;
  - $x > y$ to mean $x_i > y_i$, for $i = 1, \ldots, n$.

Monotonicity, Concavity, Maximizers and Images

- A function $f : \mathbb{R} \to \mathbb{R}$ is **increasing** if

$$x > y \quad \text{implies} \quad f(x) > f(y).$$

- A function $f : \mathbb{R} \to \mathbb{R}$ is **nondecreasing** if

$$x > y \quad \text{implies} \quad f(x) \geq f(y).$$

- A function $f : \mathbb{R} \to \mathbb{R}$ is **concave** if, for all $x, x' \in \mathbb{R}$, and all $\alpha \in [0, 1]$,

$$f(\alpha x + (1 - \alpha)x') \geq \alpha f(x) + (1 - \alpha)f(x').$$

- Given a function $f : X \to \mathbb{R}$ we denote by $\operatorname{argmax}_{x \in X} f(x)$ the **set of maximizers** of f.
- For any $Y \subseteq X$ we write $f(Y) := \{f(x) : x \in Y\}$.

Profiles

- Throughout we use N to denote the set of players.
- A **profile** is a collection of values of some variable, one for each player.
- Such a profile is written $(x_i)_{i\in N}$, or, if the qualifier $i \in N$ is clear, (x_i).
- For any profile $x = (x_j)_{j\in N}$ and any $i \in N$, we let

$$x_{-i} := (x_j)_{j\in N-\{i\}},$$

  the list of elements of the profile x for all players except player i.
- Given a list $x_{-i} = (x_j)_{j\in N-\{i\}}$ and an element x_i, we denote by (x_{-i}, x_i) the profile $(x_i)_{i\in N}$.
- If X_i is a set, for each $i \in N$, then we denote by

$$X_{-i} := \bigtimes_{j\in N-\{i\}} X_j.$$

Preference Relations

- A binary relation $\succsim$ on a set A is:
  - **Complete** if $a \succsim b$ or $b \succsim a$, for every $a \in A$ and $b \in A$;
  - **Reflexive** if $a \succsim a$, for every $a \in A$;
  - **Transitive** if $a \succsim c$ whenever $a \succsim b$ and $b \succsim c$.
- A **preference relation** is a complete reflexive transitive binary relation.
- If $a \succsim b$, but not $b \succsim a$, then we write $a \succ b$.
- If $a \succsim b$ and $b \succsim a$, then we write $a \sim b$.

Continuous and Quasi-Concave Preference Relations

- A preference relation $\succsim$ on A is **continuous** if for all sequences $(a^k)_k$ and $(b^k)_k$ in A that converge to a and b, respectively,

$$a^k \succsim b^k, \text{ for all } k, \quad \text{imply} \quad a \succsim b.$$

- A preference relation $\succsim$ on $\mathbb{R}^n$ is **quasi-concave** if for every $b \in \mathbb{R}^n$, the set $\{a \in \mathbb{R}^n : a \succsim b\}$ is convex.
- A preference relation $\succsim$ on $\mathbb{R}^n$ is **strictly quasi-concave** if every such set is strictly convex.

Partitions and Pareto Efficiency

- Let X be a set.
- We denote by $|X|$ the number of members of X.
- A **partition** of X is a collection of disjoint subsets of X whose union is X.
- Let N be a finite set and let $X \subseteq \mathbb{R}^N$ be a set.
- $x \in X$ is **Pareto efficient** if there is no $y \in X$ for which

$$y_i > x_i, \quad \text{for all } i \in N.$$

- $x \in X$ is **strongly Pareto efficient** if there is no $y \in X$ for which:
  - $y_i \geq x_i$, for all $i \in N$;
  - $y_i > x_i$, for some $i \in N$.

Probability Measures

- A **probability measure** μ on a finite (or countable) set X is a function
$$\mu : 2^X \to \mathbb{R}$$
that satisfies:
  - $\mu(A) \geq 0$, for every $A \subseteq X$;
  - If $B, C \subseteq X$, with $B \cap C = \emptyset$,
  $$\mu(B \cup C) = \mu(B) + \mu(C);$$
  - $\mu(X) = 1$.
- We do occasionally work with probability measures over spaces that are not necessarily finite.