Introduction to Game Theory

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 500

1. Extensive Games with Imperfect Information
   - Extensive Games with Imperfect Information
   - Principles for the Equivalence of Extensive Games
   - Mixed and Behavioral Strategies
   - Nash Equilibrium

Subsection 1

Extensive Games with Imperfect Information

Imperfect Information

- In each of the models we studied previously there is a sense in which the players are not perfectly informed when making their choices.
  - In a strategic game, a player, when taking an action, does not know the actions that the other players take.
  - In an extensive game with perfect information, a player does not know the future moves planned by the other players.
- In extensive games with imperfect information, the players may, in addition, be imperfectly informed about some (or all) of the choices that have already been made.
- Each player, when choosing an action, forms an expectation about the unknowns.
  - The expectations are not deduced solely from the equilibrium behavior and the exogenous information about the moves of chance.
  - The expectations relate not only to the other players' future behavior but also to events that happened in the past.

Extensive Games

- We generalize an extensive game with perfect information to allow:
  - Players to be imperfectly informed about past events;
  - Some moves to be made by "chance".
- The model does not, however, incorporate the situation in which more than one player may move after any history.

Definition (Extensive Game)

An **extensive game** has the following components:

- A finite set N of **players**;
- A set H of sequences that satisfies the following three properties:
  - The empty sequence $\emptyset$ is a member of H;
  - If $(a^k)_{k=1,\ldots,K} \in H$ (K may be infinite) and $L < K$ then $(a^k)_{k=1,\ldots,L} \in H$;
  - If an infinite sequence $(a^k)_{k=1}^{\infty}$ satisfies $(a^k)_{k=1,\ldots,L} \in H$, for every positive integer L, then $(a^k)_{k=1}^{\infty} \in H$.

  Each member of H is a **history**;

Extensive Games (Cont'd)

Definition (Extensive Game Continued)

Each component of a history is an **action** taken by a player;
A history $(a^k)_{k=1,\ldots,K} \in H$ is **terminal** if it is infinite or if there is no a^{K+1}, such that $(a^k)_{k=1,\ldots,K+1} \in H$;
The set of actions available after the nonterminal history h is denoted $A(h) = \{a : (h, a) \in H\}$;
The set of terminal histories is denoted Z.

- A function P that assigns to each nonterminal history (each member of $H - Z$) a member of $N \cup \{c\}$;
  P is the **player function**, $P(h)$ being the player who takes an action after the history h; If $P(h) = c$, then chance determines the action taken after the history h.
- A function f_c that associates with every history h, for which $P(h) = c$, a probability measure $f_c(\bullet \mid h)$ on $A(h)$, where each such probability measure is independent of every other such measure;
  $f_c(a \mid h)$ is the probability that a occurs after the history h.

Extensive Games (Cont'd)

Definition (Extensive Game Ending)

- For each player $i \in N$ a partition $\mathcal{I}_i$ of $\{h \in H : P(h) = i\}$, with the property that $A(h) = A(h')$ whenever h and h' are in the same member of the partition;
  For $I_i \in \mathcal{I}_i$, we denote by:
  - $A(I_i)$ the set $A(h)$;
  - $P(I_i)$ the Player $P(h)$, for any $h \in I_i$;

  $\mathcal{I}_i$ is the **information partition** of player i;
  A set $I_i \in \mathcal{I}_i$ is an **information set** of Player i.
- For each player $i \in N$ a preference relation $\succsim_i$ on lotteries over Z (the **preference relation** of Player i) that can be represented as the expected value of a payoff function defined on Z.

- We refer to a tuple $\langle N, H, P, f_c, (\mathcal{I}_i)_{i \in N} \rangle$ (which excludes the players' preferences) whose components satisfy the conditions in the definition as an **extensive game form**.

Interpretation of Information Partitions

- Relative to the definition of an extensive game with perfect information and chance moves, the new element is the collection $(\mathcal{I}_i)_{i\in N}$ of information partitions.
- We interpret the histories in any given member of $\mathcal{I}_i$ to be indistinguishable to Player i.
- Thus, the game models a situation in which, after any history $h \in I_i \in \mathcal{I}_i$, Player i is informed that some history in I_i has occurred but is not informed that the history h has occurred.
- The condition that $A(h) = A(h')$ whenever h and h' are in the same member of $\mathcal{I}_i$ captures the idea that if $A(h) \neq A(h')$ then Player i could deduce, when he faced $A(h)$, that the history was not h', contrary to our interpretation of $\mathcal{I}_i$.
- Note that the definition does not rule out the possibility that an information set contains two histories h and h' where $h' = (h, a^1, \ldots, a^K)$ for some sequence of actions $(a^1, \ldots, a^K)$.

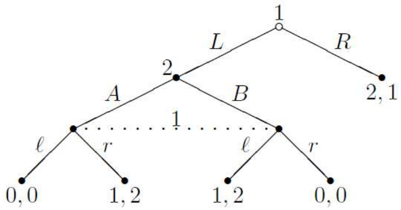
Refinement of Information Partitions

- Each player's information partition is a primitive of the game.
- A player can distinguish between histories in different members of his partition without having to make any inferences from observed actions.
- A participant may be able, by analyzing the other players' behavior, to make inferences that refine this information.

Example

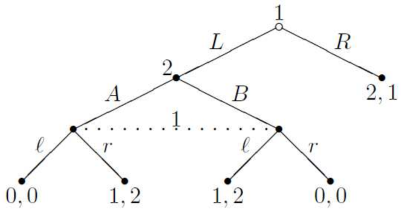
- Consider the following game.
  - Player 1 moves first choosing between a and b;
  - Player 2 moves next and has information sets $\{a, b\}$.
- We interpret this game to model a situation in which Player 2 does not observe the choice of Player 1.
- When making his move, he is not informed whether Player 1 chose a or b.
- Nevertheless, when making his move, Player 2 may infer (from his knowledge of a steady state or from introspection about Player 1) that the history is a, even though he does not observe the action chosen by Player 1.

Example of an Extensive Game with Imperfect Information



- Player 1 moves first, choosing between L and R.
  - If she chooses R, the game ends.
  - If she chooses L, Player 2 moves;
- Player 2 is informed that Player 1 chose L, and chooses A or B.
- In either case it is Player 1's turn to move, without being informed whether Player 2 chose A or B (dotted line), choosing an action from the set $\{\ell, r\}$.

Example (Cont'd)



- The player function P is specified by

$$\begin{array}{l} P(\emptyset) = P(L,A) = P(L,B) = 1; \\ P(L) = 2; \end{array}$$

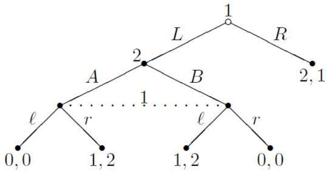
- The information partitions are

$$\begin{array}{lcl} \mathcal{I}_1 & = & \{\{\emptyset\},\{(L,A),(L,B)\}\}; \\ \mathcal{I}_2 & = & \{\{L\}\}. \end{array}$$

- The numbers under the terminal histories are the players' payoffs.

Simultaneous Moves

- Only one player is allowed to move after any history.
- However, there is a sense in which an extensive game, as we have defined it, also captures simultaneous moves.
- Consider the preceding example



After player 1 chooses L, the situation in which Players 1 and 2 are involved is essentially the same as that captured by a game with perfect information in which they choose actions simultaneously.

- Because of this, often the definition of an extensive game with perfect information does not include the possibility of simultaneous moves.

Pure Strategies

- A player's strategy in an extensive game with perfect information is a function that specifies an action for every history after which the player chooses an action.
- To extend to a general extensive game, we add the qualifier "pure" to be able to accommodate later the possibility that the players may randomize.

Definition (Pure Strategy)

A **pure strategy** of player $i \in N$ in an extensive game $\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i) \rangle$ is a function that assigns an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}_i$.

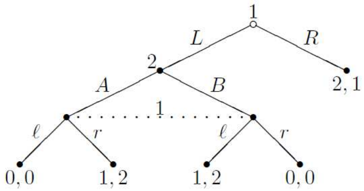
- As for an extensive game with perfect information, we can associate with any extensive game a strategic game.
- Note that the outcome of a strategy profile here may be a lottery over the terminal histories, since we allow moves of chance.

Player Record Along a History

- The model of an extensive game is capable of capturing situations in which at some points players forget what they knew earlier.
- We refer to games in which at every point every player remembers whatever he knew in the past as games with perfect recall.
- To define such games formally, let $\langle N, H, P, f_c, (\mathcal{I}_i)\rangle$ be an extensive game form.
- $X_i(h)$, the **record of Player i's experience along the history h**, is the sequence consisting of:
  - The information sets that the player encounters in the history h;
  - The actions that he takes at them, in the order that these events occur.

Example

- Consider again the game



- In this game

$$X_1((L,A)) = (\emptyset, L, \{(L,A),(L,B)\}).$$

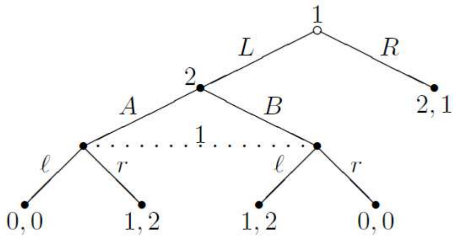
Extensive Games With Perfect Recall

Definition (Extensive Game with Perfect Recall)

An extensive game form has **perfect recall** if, for each player i and for all histories h and h' in the same information set of Player i,

$$X_i(h) = X_i(h').$$

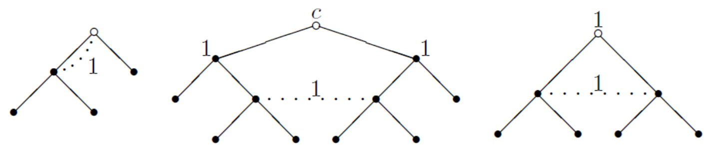
- Consider again the game



It is a game with perfect recall.

Examples of Games Without Perfect Recall

- We consider three (one player) game forms without perfect recall.



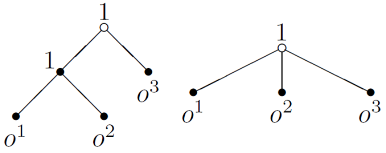
  - In the left-hand game a player does not know if she has made a choice or not. When choosing an action she does not know whether she is at the beginning of the game or has already chosen her left-hand action.
  - In the middle game the player forgets something that she previously knew. When making a choice at her last information set she is not informed of the action of chance, though she was so informed when she made her previous choice.
  - In the right-hand game she does not remember the action she took in the past.

Subsection 2

Principles for the Equivalence of Extensive Games

Identical Strategic Situations

- Some extensive games appear to represent the same strategic situation as others.
- Example: Consider the two one-player games:



- We associate letters with terminal histories.
- If two terminal histories are assigned the same letter then they represent the same event.
- Formally, the two games are different, since in the left-hand game Player 1 makes two decisions, while in the right-hand game she makes only one.
- However, principles of rationality suggest that the two games model the same situation.

Equivalence Principles and Reduced Strategic Form

- We give examples of pairs of games that arguably represent the same situation and discuss principles generalizing the examples.
- The four principles that we consider all preserve the reduced strategic form of the game.

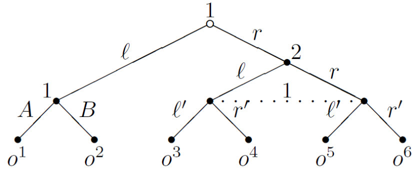
  If one extensive game is equivalent to another according to the principles, then the reduced strategic forms of the two games are the same.
- A solution concept that does not depend solely on the reduced strategic form may assign different outcomes to games that are equivalent according to the principles.
- To justify such a solution concept one has to argue that at least one of the principles is inappropriate.

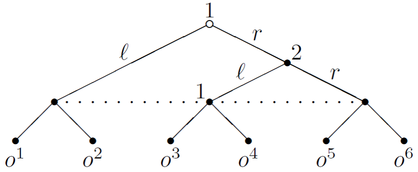
The Four Equivalence Principles

- Each of the principles that we discuss asserts that the game is equivalent to another extensive game.
- These principles are:
  - Inflation-Deflation Principle;
  - Addition of a Superfluous Move;
  - Coalescing of Moves;
  - Interchange of Moves.

The Inflation-Deflation Principle

- Let Γ_1 be the game





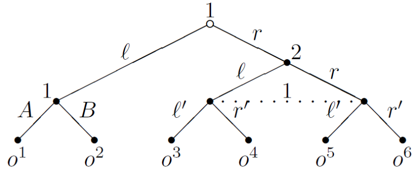
- According to the Inflation-Deflation Principle Γ_1 is equivalent to the game Γ_2 on the right.
- In Γ_2, player 1 has imperfect recall. At her second information set, she is not informed whether she chose r or ℓ at the start of the game.
- The three histories ℓ, (r, ℓ), and (r, r) are all in the same information set in Γ_2, while in Γ_1 the history ℓ lies in one information set and the histories (r, ℓ) and (r, r) lie in another.

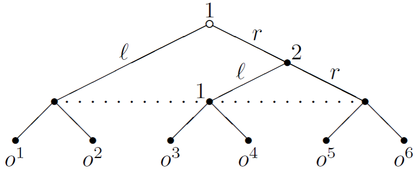
The Inflation-Deflation Principle Formalism

- According to the **Inflation-Deflation Principle** the extensive game Γ is **equivalent** to the extensive game Γ' if Γ' differs from Γ only in that, there is an information set of some player i in Γ that is a union of information sets of Player i in Γ', with the following property:

  > Any two histories h and h' in different members of the union have subhistories that are in the same information set of Player i and Player i's action at this information set is different in h and h'.

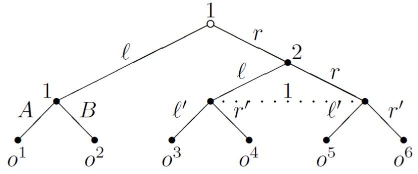
- To relate this to the example, let $\Gamma = \Gamma_2$, $\Gamma' = \Gamma_1$, and $i = 1$.

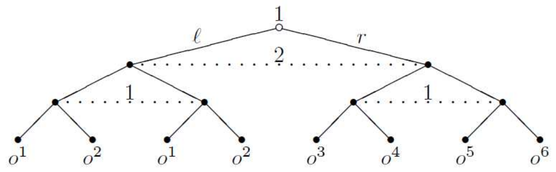




Addition of a Superfluous Move

- According to this principle, Γ_1 is equivalent to the game Γ_3:





- If in the game Γ_3 Player 1 chooses ℓ at the start of the game, then the action of Player 2 is irrelevant, since it has no effect on the outcome.
- Thus, in Γ_3 whether Player 2 is informed of Player 1's choice at the start of the game should make no difference to his choice.

Addition of a Superfluous Move (Hypotheses)

- The **Principle of Addition of a Superfluous Move** is the following:

  Let Γ be an extensive game and h be a history, such that $P(h) = i$ and $a \in A(h)$.

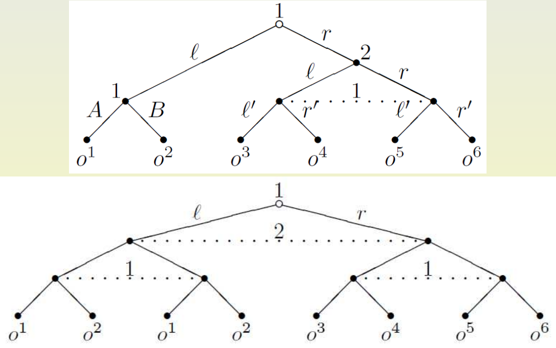
  Suppose, for any sequence h' of actions following history (h, a) and for any $b \in A(h)$:
  - $(h, a, h') \in H$ if and only if $(h, b, h') \in H$ and (h, a, h') is terminal if and only if (h, b, h') is terminal;
  - If both (h, a, h') and (h, b, h') are terminal, then $(h, a, h') \sim_i (h, b, h')$, for all $i \in N$;
  - If both (h, a, h') and (h, b, h') are nonterminal then they are in the same information set.

Addition of a Superfluous Move (Equivalence)

- Then Γ is **equivalent** to the game Γ' that differs from Γ only in that:
  (i) All histories of the form (h, c, h'), for $c \in A(h)$, are replaced by (h, h');
  (ii) If the information set I_i that contains the history h in Γ is not a singleton then h is excluded from I_i in Γ';
  (iii) The player who is assigned to the history (h, h') in Γ' is the one who is assigned to (h, a, h') in Γ;
  (iv) (h, h') and (h, h'') are in the same information set of Γ' if and only if (h, a, h') and (h, a, h'') are in the same information set of Γ;
  (v) The players' preferences are modified accordingly.

Illustration of Addition of a Superfluous Move

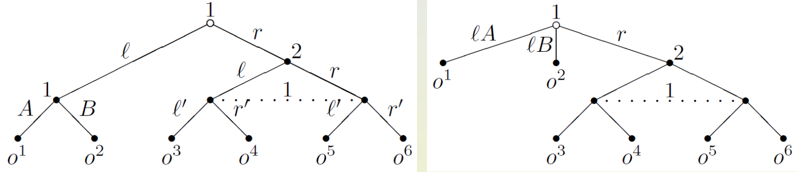
- The superfluous move in Γ is removed to create Γ'.
- We relate the definition to Γ_1 and Γ_3, seen previously.



- Take $\Gamma = \Gamma_3$, $\Gamma' = \Gamma_1$, $i = 2$, and $h = \ell$.
- Let a be one of the actions of Player 2.

Coalescing of Moves

- According to this principle, Γ_1 is equivalent to the game Γ_4.



- In Γ_1, Player 1:
  - First chooses between ℓ and r;
  - Then chooses between A and B, in the event she chose ℓ.
- The idea is that this decision problem is equivalent to that of deciding between $\ell A, \ell B$, and r, as in Γ_4.
- The argument is that if Player 1 is rational, then her choice at the start of Γ_1 requires her to compare the outcomes of choosing ℓ and r. To determine the outcome of choosing ℓ requires her to plan at the start of the game whether to choose A or B.

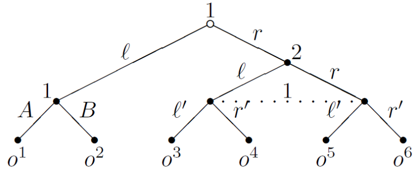
Coalescing of Moves Formalism

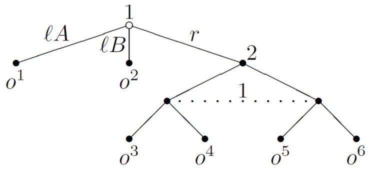
- The **Principle of Coalescing of Moves**: Let Γ be an extensive game and let h be a history, such that $P(h) = i$, with $h \in I_i$.

  Assume that:
  - $a \in A(I_i)$;
  - $\{(h', a) : h' \in I_i\} = I_i'$ is an information set of player i.
- The Γ is **equivalent** to Γ', the game that differs from Γ only in that:
  - The information set I_i' is deleted;
  - For all $h' \in I_i$, the history (h', a) is deleted and every history (h', a, b, h''), where $b \in A(h', a)$, is replaced by the history (h', ab, h''), where ab is a new action (that is not a member of $A(h')$);
  - The information sets, player function, and players' preferences are changed accordingly.

Illustration of Coalescing of Moves

- Consider

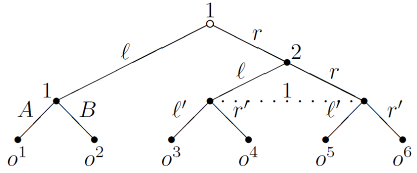


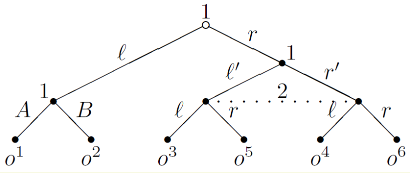


- Take $\Gamma = \Gamma_1$, $\Gamma' = \Gamma_4$, $h = \emptyset$, $i = 1$ and $a = \ell$.

Interchange of Moves

- According to this principle Γ_1 is equivalent to the game Γ_5.





- The idea is that the order of play is immaterial if one player does not have any information about the other player's action when making his choice.

Interchange of Moves Formalism

- The **Principle of Interchange of Moves**: Let Γ be an extensive game and let $h \in I_i$, an information set of player i.

  Suppose that, for all histories h' in some subset H' of I_i, the player who takes an action after i is j, who is not informed of the action that i takes at h'.

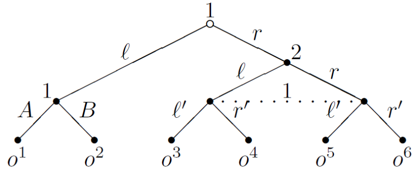
  I.e., suppose that $(h', a) \in I_j$, for all $h' \in H'$ and all $a \in A(h')$, where I_j is an information set of player j.

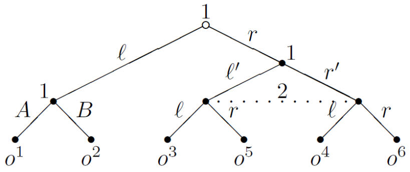
  The information set I_j may contain other histories.

  Let H'' be the subset of I_j consisting of histories of the form (h', a), for some $h' \in H'$.
- Then Γ is **equivalent** to the extensive game in which:
  - Every history of the type (h', a, b) for $h' \in H'$ is replaced by (h', b, a);
  - The information set I_i of player i is replaced by the union of $I_i - H'$ and all histories of the form (h', b) for $h' \in H'$ and $b \in A(h', a)$;
  - The information set I_j of player j is replaced by $(I_j - H'') \cup H'$.

Illustration of Interchange of Moves

- Consider again the example





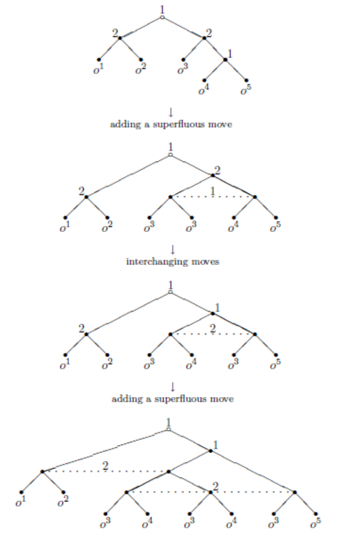
- We have $\Gamma = \Gamma_1$, $\Gamma' = \Gamma_5$, $h = r$, $i = 2$, $j = 1$ and

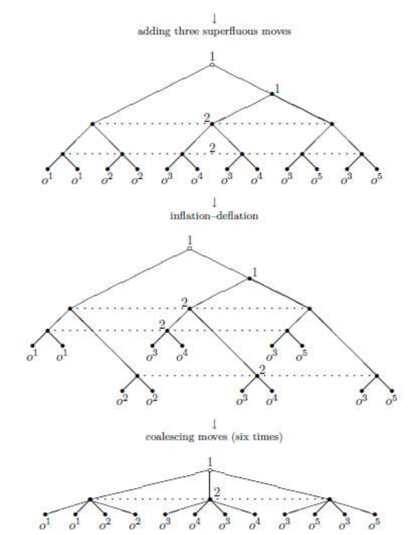
$$\begin{array}{rcl} H' & = & I_2 = \{r\}; \\ H'' & = & I_1 = \{(r, \ell), (r, r)\}. \end{array}$$

Thompson's Result

- These four transformations preserve the reduced strategic form.
- Conversely, restrict attention to finite extensive games in which no information set contains both a history h and some subhistory of h.
- Then, if any two such games have the same reduced strategic form, then one can be obtained from the other by a sequence of the four transformations.

Example


1
2
2
1
o^1 o^2 o^3 o^4 o^5
adding a superfluous move
interchanging moves
adding a superfluous move


adding three superfluous moves
inflation–deflation
coalescing moves (six times)

Subsection 3

Mixed and Behavioral Strategies

Mixed Strategy and Behavioral Strategy

- We have defined the notion of a pure strategy in an extensive game.
- There are two ways to model the possibility that a player's actions in such a game depend upon random factors.

Definition (Mixed Strategy and Behavioral Strategy)

Let $\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i)\rangle$ be an extensive game.

- A **mixed strategy of Player** i is a probability measure over the set of player i's pure strategies.
- A **behavioral strategy of Player** i is a collection $(\beta_i(I_i))_{I_i \in \mathcal{I}_i}$ of independent probability measures, where $\beta_i(I_i)$ is a probability measure over $A(I_i)$.

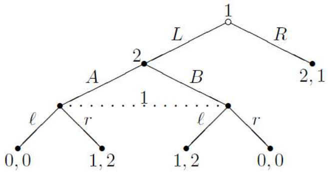
- For any history $h \in I_i \in \mathcal{I}_i$ and action $a \in A(h)$, we denote by $\beta_i(h)(a)$ the probability $\beta_i(I_i)(a)$ assigned by $\beta_i(I_i)$ to the action a.

Mixed Strategies versus Behavioral Strategies

- Let $\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i) \rangle$ be an extensive game.
- As in a strategic game, a mixed strategy of Player i is a probability measure over Player i's set of pure strategies.
- By contrast, a behavioral strategy specifies a probability measure over the actions available to Player i at each of his information sets.

Illustrating the Difference

- Consider again the game

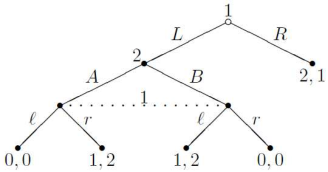


- Player 1 has two information sets $\{\emptyset\}$ and $\{(L,A),(L,B)\}$.
- At each information set, she has two possible actions.
- So Player 1 has four pure strategies, which assign to the information sets $\{\emptyset\}$ and $\{(L,A),(L,B)\}$ respectively, the actions shown:

Info Set	s_1	s_2	s_3	s_4
$\{\emptyset\}$	L	L	R	R
$\{(L,A),(L,B)\}$	ℓ	r	ℓ	r

Illustrating the Difference (Cont'd)

- We are looking at strategies for Player 1 in



- A mixed strategy of Player 1 is a probability distribution over his four pure strategies.
- By contrast, a behavioral strategy of Player 1 is a pair of probability distributions, one for each information set:
  - The first is a distribution over $\{L, R\}$;
  - The second is a distribution over $\{\ell, r\}$.

Outcome of a Strategy Profile

- Let $\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i)\rangle$ be an extensive game.
- Let $\sigma = (\sigma_i)_{i \in N}$ be a profile of either mixed or behavioral strategies.
- We define the **outcome** $O(\sigma)$ **of** σ to be the probability distribution over the terminal histories that results when each player i follows the precepts of σ_i.
- We look at finite games more closely.
- For any history $h = (a^1, \ldots, a^k)$, define a pure strategy s_i of Player i to be **consistent with** h if, for every subhistory $(a^1, \ldots, a^\ell)$ of h, for which $P(a^1, \ldots, a^\ell) = i$, we have
$$s_i(a^1, \ldots, a^\ell) = a^{\ell+1}.$$
- For any history h, let $\pi_i(h)$ be the sum of the probabilities, according to σ_i, of all the pure strategies of player i that are consistent with h.
  E.g., if h is a history in which player i never moves, then $\pi_i(h) = 1$.

Outcome of a Strategy Profile (Cont'd)

- Then for any profile σ of mixed strategies, the probability that $O(\sigma)$ assigns to any terminal history h is

$$O(\sigma)(h) = \prod_{i \in N \cup \{c\}} \pi_i(h).$$

- For any profile σ of behavioral strategies, the probability that $O(\sigma)$ assigns to the terminal history $h = (a^1, \ldots, a^K)$ is

$$O(\sigma)(h) = \prod_{k=0}^{K-1} \beta_{P(a^1,\ldots,a^k)}(a^1, \ldots, a^k)(a^{k+1}),$$

where, for $k = 0$, the history $(a^1, \ldots, a^k)$ is the initial history.

Outcome Equivalent Strategies

- Two (mixed or behavioral) strategies of any player are **outcome equivalent** if, for every collection of pure strategies of the other players, the two strategies induce the same outcome.
- We now examine the conditions under which, for any mixed strategy, there is an outcome equivalent behavioral strategy and vice versa.
- In particular, this is the case in any game with perfect recall.

Outcome Equivalent Strategies

- Claim: Consider an extensive game in which no information set contains both some history h and a history of the form (h, h'), for some $h' \neq \emptyset$.

  This condition is satisfied, e.g., by any game with perfect recall.

  For any behavioral strategy, there is an outcome equivalent mixed strategy.

  Let σ_i be a behavioral strategy of Player i.

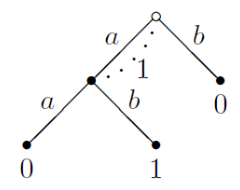
  Define a mixed strategy for Player i by assigning to any pure strategy s_i (which specifies an action $s_i(I_i)$ for every information set $I_i \in \mathcal{I}_i$) probability

$$\prod_{I_i \in \mathcal{I}_i} \beta_i(I_i)(s_i(I_i)).$$

  This mixed strategy is outcome equivalent to the given behavioral strategy.

Necessity of the Condition

- In a game in which some information set contains histories of the form h and (h, h'), with $h' \neq \emptyset$, there may be a behavioral strategy for which there is no equivalent mixed strategy.
- Consider the game



Take the behavioral strategy that assigns probability $p \in (0, 1)$ to a.

It generates the outcomes $(a, a), (a, b)$, and b with probabilities $p^2, p(1 - p)$, and $1 - p$ respectively.

This distribution cannot be duplicated by any mixed strategy.

The Converse Implication

Proposition

For any mixed strategy of a player in a finite extensive game with perfect recall, there is an outcome equivalent behavioral strategy.

- Let σ_i be a mixed strategy of player i.

  For any history h, let $\pi_i(h)$ be the sum of the probabilities, according to σ_i, of all the pure strategies of Player i that are consistent with h.

  Let h and h' be two histories in the same information set I_i of Player i.

  Consider an action $a \in A(h)$.

  By hypothesis, the game has perfect recall.

  So the sets of actions of Player i in h and h' are the same.

  Thus $\pi_i(h) = \pi_i(h')$.

The Converse Implication (Cont'd)

- In any pure strategy of player i the action a is taken after h if and only if it is taken after h'.

  So we also have $\pi_i(h, a) = \pi_i(h', a)$.

  Thus, we can define a behavioral strategy β_i of Player i by

  $$\beta_i(I_i)(a) = \pi_i(h, a) = \pi_i(h),$$

  for any $h \in I_i$ for which $\pi_i(h) > 0$.

  Clearly,

  $$\sum_{a \in A(h)} \beta_i(I_i)(a) = 1.$$

  If $\pi_i(h) = 0$, how we define $\beta_i(I_i)(a)$ is immaterial.

The Converse Implication (Cont'd)

Claim: β_i is outcome equivalent to σ_i.

Let s_{-i} be a collection of pure strategies for the players other than i.

Let h be a terminal history.

If h includes moves that are inconsistent with s_{-i}, then the probability of h is zero under both σ_i and β_i.

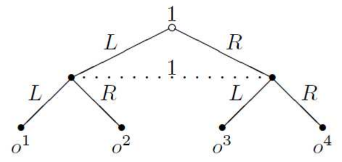
Now assume that all the moves of players other than i in h are consistent with s_{-i}.

- If h includes a move after a subhistory $h' \in I_i$ of h that is inconsistent with σ_i, then $\beta_i(I_i)$ assigns probability zero to this move. Hence, the probability of h according to β_i is zero.
- If h is consistent with σ_i, then:
  - $\pi_i(h') > 0$, for all subhistories h' of h;
  - The probability of h according to β_i is the product of $\pi_i(h', a) = \pi_i(h')$ over all (h', a) that are subhistories of h.

  This product is $\pi_i(h)$, the probability of h according to σ_i.

The Case of Games with Imperfect Recall

- In a game with imperfect recall there may be a mixed strategy for which there is no outcome-equivalent behavioral strategy
- Example: Consider the one player game with imperfect recall.



Consider the mixed strategy in which player 1 chooses:
  - LL with probability $\frac{1}{2}$;
  - RR with probability $\frac{1}{2}$.

It has as its outcome the following probability distribution over terminal histories:

h	o^1	o^2	o^3	o^4
$p(h)$	$\frac{1}{2}$	0	0	$\frac{1}{2}$

The Case of Games with Imperfect Recall (Cont'd)

- The outcome

h	o^1	o^2	o^3	o^4
$p(h)$	$\frac{1}{2}$	0	0	$\frac{1}{2}$

  cannot be achieved by any behavioral strategy.

  In fact consider the behavioral strategy

$$((p, 1 - p), (q, 1 - q)).$$

  It induces a distribution over the terminal histories in which LR has zero probability only if either $p = 0$ or $q = 1$.

  But, then, the probability of either LL or RR is zero.

Subsection 4

Nash Equilibrium

Nash Equilibria in Mixed and in Behavioral Strategies

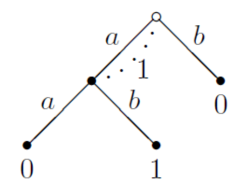
- Let $\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i)\rangle$ be an extensive game.
- A **Nash equilibrium in mixed strategies** is a profile σ^* of mixed strategies with the property that, for every player $i \in N$, we have

$$O(\sigma^*_{-i}, \sigma^*_i) \succsim_i O(\sigma^*_{-i}, \sigma_i),$$

  for every mixed strategy σ_i of player i.
- For finite games an equivalent definition of a mixed strategy equilibrium is that every pure strategy in the support of each player's mixed strategy is a best response to the strategies of the other players.
- A **Nash equilibrium in behavioral strategies** is defined analogously.
- Given the preceding proposition, the two definitions are equivalent for games with perfect recall.

The Case of Imperfect Recall

- For games with imperfect recall they are not equivalent.
- Example: Consider, again, the game



In this game the player is indifferent among all her mixed strategies, which yield her a payoff of 0.

The behavioral strategy that assigns probability p to a yields her a payoff of

$$p^2 \cdot 0 + p \cdot (1-p) \cdot 1 + (1-p) \cdot 0 = p(1-p).$$

Thus, the best behavioral strategy has $p = \frac{1}{2}$, with payoff of $\frac{1}{4}$.