Introduction to Game Theory

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Subsection 1

Coalitional Games with Transferable Payoff

Coalitional Games with Transferable Payoff

- In the simplest version of a coalitional game, each group of players is associated with a single number, the payoff available to the group.
- There are no restrictions on how this payoff may be divided among the members of the group.

Definition (Coalitional Game with Transferable Payoff)

A **coalitional game with transferable payoff** consists of:

- A finite set N (of **players**);
- A function v that associates with every nonempty subset S of N (a **coalition**) a real number $v(S)$ (the **worth** of S).

- For each coalition S the number $v(S)$ is the total payoff that is available for division among the members of S.
- The set of joint actions that the coalition S can take consists of all possible divisions of $v(S)$ among the members of S.

Suitability of the Model

- In many situations the payoff that a coalition can achieve depends on the actions taken by the other players.
- A coalitional game models best a situation in which the actions of the players who are not part of S do not influence $v(S)$.
- Another interpretation for $v(S)$ is as the most payoff that the coalition S can guarantee independently of the behavior of the coalition $N - S$.
- The interpretation of the solution concepts defined depend on how the game is interpreted.

Cohesive Coalitional Games with Transferable Payoff

- In the coalitional games with transferable payoff studied here, the worth of the coalition N of all players is at least as large as the sum of the worths of the members of any partition of N.

Definition (Cohesive Coalitional Game)

A coalitional game $\langle N, v\rangle$ with transferable payoff is **cohesive** if, for every partition $\{S_1, \ldots, S_K\}$ of N,

$$v(N) \geq \sum_{k=1}^{K} v(S_k).$$

This is a special case of the condition of **superadditivity**, which requires that, for all coalitions S and T, with $S \cap T = \emptyset$,

$$v(S \cup T) \geq v(S) + v(T).$$

Subsection 2

The Core

Idea Behind the Core

- The idea behind the core is analogous to that behind a Nash equilibrium of a noncooperative game:

  An outcome is stable if no deviation is profitable.
- In the case of the core, an outcome is stable if no coalition can deviate and obtain an outcome better for all its members.
- For a coalitional game with transferable payoff, the stability condition is that no coalition can obtain a payoff that exceeds the sum of its members' current payoffs.
- Given our assumption that the game is cohesive, we confine ourselves to outcomes in which the coalition N of all players forms.

Feasible Payoff Vectors and Profiles

- Let $\langle N, v\rangle$ be a coalitional game with transferable payoff.
- For any profile $(x_i)_{i\in N}$ of real numbers and any coalition S, we define

$$x(S) = \sum_{i\in S} x_i.$$

- A vector $(x_i)_{i\in S}$ of real numbers is an S-**feasible payoff vector** if

$$x(S) = v(S).$$

- We refer to an N-feasible payoff vector as a **feasible payoff profile**.

The Core

Definition (The Core)

The **core** of the coalitional game with transferable payoff $\langle N, v \rangle$ is the set of feasible payoff profiles $(x_i)_{i \in N}$ for which there is no coalition S and S-feasible payoff vector $(y_i)_{i \in S}$, with $y_i > x_i$, for all $i \in S$.

- Equivalently, the core is the set of feasible payoff profiles $(x_i)_{i \in N}$, such that, for every coalition S,

$$v(S) \leq x(S).$$

- Thus, the core is a set of payoff profiles satisfying a system of weak linear inequalities.
- Consequently, the core is closed and convex.

Example: A Three-Player Majority Game

- Consider the following scenario.
  - Three players can obtain one unit of payoff;
  - Any two of them can obtain $\alpha \in [0,1]$ independently of the actions of the third;
  - Each player alone can obtain nothing, independently of the actions of the remaining two players.
- We can model this situation as the coalitional game $\langle N, v\rangle$ in which:
  - $N = \{1,2,3\}$;
  - $v : \mathcal{P}(N)\backslash\{\emptyset\} \to \mathbb{R}$ is defined by:
    - $v(N) = 1$;
    - $v(S) = \alpha$, whenever $|S| = 2$;
    - $v(\{i\}) = 0$, for all $i \in N$.
- The core of this game is the set of all nonnegative payoff profiles (x_1, x_2, x_3), for which:
  - $x(N) = 1$;
  - $x(S) \geq \alpha$, for every two-player coalition S.
- The core is nonempty if and only if $\alpha \leq \frac{2}{3}$.

Example: Sharing a Treasure

- An expedition of n people has discovered treasure in the mountains. Each pair of them can carry out one piece.
- A coalitional game that models this situation is $\langle N, v\rangle$, where:
  - $N = \{1, 2, \ldots, n\}$;
  - $v(S) = \begin{cases} \frac{|S|}{2}, & \text{if } |S| \text{ is even} \\ \frac{|S|-1}{2}, & \text{if } |S| \text{ is odd} \end{cases}$
- If $|N| \geq 4$ is even, then the core consists of the single payoff profile $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.
- If $|N| \geq 3$ is odd, then the core is empty.

Example: A Market for an Indivisible Good

- We consider a market for an indivisible good.
  The set of buyers is B and the set of sellers is L.
  - Each seller holds one unit of the good and has a reservation price of 0.
  - Each buyer wants one unit and has a reservation price of 1.
- We model this marketplace as a coalitional game with transferable payoff $\langle N, v \rangle$.
  - $N = B \cup L$;
  - $v(S) = \min\{|S \cap B|, |S \cap L|\}$, for each coalition S.

Example: A Market for an Indivisible Good (Cont'd)

- If $|B| > |L|$, then the core consists of the single payoff profile in which every seller receives 1 and every buyer receives 0.

  Suppose that the payoff profile x is in the core.

  Let b be a buyer whose payoff is minimal among all the buyers.

  Let ℓ be a seller whose payoff is minimal among all the sellers.

  Since x is in the core, we have:

  $$x_b + x_\ell \geq v(\{b, \ell\}) = 1.$$

  Therefore,

  $$|L| = v(N) = x(N) \geq |B|x_b + |L|x_\ell \geq (|B| - |L|)x_b + |L|.$$

  This implies that $x_b = 0$ and $x_\ell \geq 1$.

  Hence, (using $v(N) = |L|$ and the fact that ℓ is the worst-off seller) $x_i = 1$, for every seller i.

Example: A Majority Game

- A group of n players, where $n \geq 3$ is odd, has one unit to divide among its members.
  - A coalition consisting of a majority of the players can divide the unit among its members as it wishes.
- This situation is modeled by the coalitional game $\langle N, v \rangle$, with:
  - $|N| = n$;
  - $v(S) = \begin{cases} 1, & \text{if } |S| \geq \frac{n}{2} \\ 0, & \text{otherwise} \end{cases}$
- We claim that this game has an empty core.

Example: A Majority Game (Cont'd)

- The game has an empty core.

  Suppose, to the contrary, that x is in the core.

  If $|S| = n-1$, then $v(S) = 1$. So

  $$\sum_{i \in S} x_i \geq 1.$$

  - There are n coalitions of size $n-1$.
    So we have

    $$\sum_{\{S:|S|=n-1\}} \sum_{i \in S} x_i \geq n.$$

  - On the other hand,

    $$\sum_{\{S:|S|=n-1\}} \sum_{i \in S} x_i = \sum_{i \in N} \sum_{\{S:|S|=n-1, S \ni i\}} x_i = \sum_{i \in N} (n-1) x_i = n-1.$$

  These contradict each other.

Subsection 3

Nonemptiness of the Core

Notation

- We now derive a condition under which the core of a coalitional game is nonempty.
- Recall that the core is defined by a system of linear inequalities.
- So such a condition could be derived from the conditions for the existence of a solution to a general system of inequalities.
- But the special structure of the system of inequalities that defines the core yields a more specific condition.
- Denote:
  - By $\mathcal{C}$ the set of all coalitions;
  - For any coalition S, by $\mathbb{R}^S$ the $|S|$-dimensional Euclidian space in which the dimensions are indexed by the members of S;
  - By $1_S \in \mathbb{R}^N$ the characteristic vector of S, given by

$$(1_S)_i = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}.$$

Balanced Games

- A collection $(\lambda_S)_{S \in \mathcal{C}}$ of numbers in $[0, 1]$ is a **balanced collection of weights** if, for every player i, the sum of λ_S over all the coalitions that contain i is 1:
$$\sum_{S \in \mathcal{C}} \lambda_S 1_S = 1_N.$$
Example: Let $|N| = 3$.
  - The collection (λ_S) in which $\lambda_S = \frac{1}{2}$, if $|S| = 2$, and $\lambda_S = 0$, otherwise, is a balanced collection of weights.
  - The collection (λ_S) in which $\lambda_S = 1$, if $|S| = 1$, and $\lambda_S = 0$, otherwise, is also a balanced collection of weights.
- A game $\langle N, v \rangle$ is **balanced** if
$$\sum_{S \in \mathcal{C}} \lambda_S v(S) \leq v(N),$$
for every balanced collection of weights $(\lambda_S)_{S \in \mathcal{C}}$.

Interpretation of a Balanced Game

- Each player has one unit of time, which he must distribute among all the coalitions of which he is a member.
- In order for a coalition S to be active for the fraction of time λ_S, all its members must be active in S for this fraction of time, in which case the coalition yields the payoff $\lambda_S v(S)$.
- In this interpretation the condition that the collection of weights be balanced is a feasibility condition on the players' allocation of time.
- A game is balanced if there is no feasible allocation of time that yields the players more than $v(N)$.

The Bondareva-Shapley Theorem

The Bondareva-Shapley Theorem

A coalitional game with transferable payoff has a nonempty core if and only if it is balanced.

- Let $\langle N, v\rangle$ be a coalitional game with transferable payoff.

  First, let x be a payoff profile in the core of $\langle N, v\rangle$.

  Let $(\lambda_S)_{S\in\mathcal{C}}$ be a balanced collection of weights. Then

$$\begin{array}{rcl}\sum_{S\in\mathcal{C}}\lambda_S v(S) & \leq & \sum_{S\in\mathcal{C}}\lambda_S x(S)\\ & = & \sum_{i\in N}x_i\sum_{S\ni i}\lambda_S\\ & = & \sum_{i\in N}x_i\\ & = & v(N).\end{array}$$

  So $\langle N, v\rangle$ is balanced.

Proving the Converse

- Now assume that $\langle N, v\rangle$ is balanced. Then, there is no balanced collection $(\lambda_S)_{S\in\mathcal{C}}$ of weights for which $\sum_{S\in\mathcal{C}}\lambda_S v(S) > v(N)$.
  We show that the convex set
  $$\{(1_N, v(N)+\epsilon)\in\mathbb{R}^{|N|+1} : \epsilon > 0\}$$
  is disjoint from the convex cone
  $$\left\{y\in\mathbb{R}^{|N|+1} : y=\sum_{S\in\mathcal{C}}\lambda_S(1_S, v(S)),\ \text{where } \lambda_S\geq 0, \text{ for all } S\in\mathcal{C}\right\}.$$
  Assume that this is not the case. Then $1_N=\sum_{S\in\mathcal{C}}\lambda_S 1_S$.
  So $(\lambda_S)_{S\in\mathcal{C}}$ is a balanced collection of weights, with
  $$\sum_{S\in\mathcal{C}}\lambda_S v(S) > v(N).$$

Proving the Converse (Cont'd)

- By the Separating Hyperplane Theorem, there is a nonzero vector

$$(\alpha_N, \alpha) \in \mathbb{R}^{|N|} \times \mathbb{R},$$

such that:

  - $(\alpha_N, \alpha) \cdot y \geq 0$, for all y in the cone;
  - $(\alpha_N, \alpha) \cdot (1_N, v(N) + \epsilon) < 0$, for all $\epsilon > 0$.

Now $(1_N, v(N))$ is in the cone. So we have $\alpha < 0$.

Let $x = \frac{\alpha_N}{-\alpha}$.

Note that $(1_S, v(S))$ is in the cone, for all $S \in \mathcal{C}$.

Hence, $x(S) = x \cdot 1_S \geq v(S)$, for all $S \in \mathcal{C}$, by the first inequality.

Moreover, $v(N) \geq 1_N x = x(N)$ by the second inequality.

Thus, $v(N) = x(N)$ and the payoff profile x is in the core of $\langle N, v \rangle$.

Example

- Let $N = \{1, 2, 3, 4\}$.
  Consider the game $\langle N, v \rangle$ in which
  $$v(S) = \begin{cases} 1, & \text{if } S = N \\ \frac{3}{4}, & \text{if } S = \{1,2\}, \{1,3\}, \{1,4\}, \text{ or } \{2,3,4\} \\ 0, & \text{otherwise} \end{cases}$$
  We show that $\langle N, v \rangle$ has an empty core.
  It suffices to show that the game is not balanced.
  Consider the collection $(\lambda_S)_{S \in \mathcal{C}}$ of weights defined by
  $$\lambda_S = \begin{cases} \frac{1}{3}, & \text{if } S = \{1,2\}, \{1,3\} \text{ or } \{1,4\} \\ \frac{2}{3}, & \text{if } S = \{2,3,4\} \\ 0, & \text{otherwise} \end{cases}.$$
  It is easy to see that $(\lambda_S)_{S \in \mathcal{C}}$ is balanced.
  Moreover, $\sum_{S \in \mathcal{C}} \lambda_S v(S) = 3 \cdot \frac{1}{3} \cdot \frac{3}{4} + \frac{2}{3} \cdot \frac{3}{4} = \frac{5}{4} > V(N)$.
  Therefore, the game is not balanced.

Subsection 4

Markets with Transferable Payoff

A Production Process

- We apply the concept of the core to a classical model of an economy.
- Each of the agents is endowed with a bundle of goods.
- Goods can be used as inputs in a production process that the agent can operate.
- All production processes produce the same output.
- The output can be transferred between the agents.

Markets with Transferable Payoff

- Formally, a **market with transferable payoff**

$$\langle N, \ell, (\omega_i), (f_i) \rangle$$

  consists of:
  - A finite set N (of **agents**);
  - A positive integer ℓ (the number of **input goods**);
  - For each agent $i \in N$, a vector $\omega_i \in \mathbb{R}_+^\ell$ (the **endowment** of agent i);
  - For each agent $i \in N$, a continuous, nondecreasing and concave function $f_i : \mathbb{R}_+^\ell \to \mathbb{R}_+$ (the **production function** of agent i).
- An **input vector** is a member of $\mathbb{R}_+^\ell$.
- An **allocation** is a profile $(z_i)_{i \in N}$ of input vectors, such that

$$\sum_{i \in N} z_i = \sum_{i \in N} \omega_i.$$

Cooperation and Conflict

- The agents may gain by cooperating.

  If their endowments are complementary, then in order to maximize total output they may need to exchange inputs.

- On the other hand, the agents' interests conflict.

  They need to distribute the benefits of cooperation.

From a Market to a Coalitional Game

- Let $\langle N, \ell, (\omega_i), (f_i) \rangle$ be a market with transferable payoff.
- Let $\langle N, v \rangle$ be the following coalitional game with transferable payoff:
  - N is the set of agents;
  - For each coalition S, we have

$$v(S) = \max_{(z_i)_{i \in S}} \left\{ \sum_{i \in S} f_i(z_i) : z_i \in \mathbb{R}_+^\ell \text{ and } \sum_{i \in S} z_i = \sum_{i \in S} \omega_i \right\}.$$

- Note $v(S)$ is the maximal total output that the members of S can produce by themselves.
- The **core of a market** is the core of the associated coalitional game.
- Note, also, the importance of the following assumptions:
  (a) All agents produce the same good;
  (b) The production of any coalition S is independent of the behavior of $N - S$.

Nonemptiness of the Core

Proposition

Every market with transferable payoff has a nonempty core.

- Let $\langle N, \ell, (\omega_i), (f_i)\rangle$ be a market with transferable payoff.

  Let $\langle N, v\rangle$ be the corresponding coalitional game.

  By the Bondareva -Shapley Theorem, it suffices to show that $\langle N, v\rangle$ is balanced.

  Let $(\lambda_S)_{S\in\mathcal{C}}$ be a balanced collection of weights.

  We must show that $\sum_{S\in\mathcal{C}}\lambda_S v(S) \leq v(N)$.

  For each coalition S, let $(z_i^S)_{i\in S}$ be a solution of the max problem defining $v(S)$. Define

$$z_i^* = \sum_{S\in\mathcal{C}, S\ni i} \lambda_S z_i^S.$$

Nonemptiness of the Core (Cont'd)

- We have

$$\begin{array}{rcl}\sum_{i\in N} z_i^* & = & \sum_{i\in N}\sum_{S\in\mathcal{C},S\ni i}\lambda_S z_i^S \\ & = & \sum_{S\in\mathcal{C}}\sum_{i\in S}\lambda_S z_i^S \\ & = & \sum_{S\in\mathcal{C}}\lambda_S\sum_{i\in S} z_i^S \\ & = & \sum_{S\in\mathcal{C}}\lambda_S\sum_{i\in S}\omega_i \\ & = & \sum_{i\in N}\omega_i\sum_{S\in\mathcal{C},S\ni i}\lambda_S \\ & = & \sum_{i\in N}\omega_i \quad ((\lambda_S)_{S\in\mathcal{C}} \text{ balanced})\end{array}$$

It follows from the definition of $v(N)$ that $v(N) \geq \sum_{i\in N} f_i(z_i^*)$.

The concavity of each function f_i and the fact that the collection of weights is balanced imply that

$$\begin{array}{rcl}\sum_{i\in N} f_i(z_i^*) & \geq & \sum_{i\in N}\sum_{S\in\mathcal{C},S\ni i}\lambda_S f_i(z_i^S) \\ & = & \sum_{S\in\mathcal{C}}\lambda_S\sum_{i\in S} f_i(z_i^S) \\ & = & \sum_{S\in\mathcal{C}}\lambda_S v(S).\end{array}$$

An Example

- Consider the market with transferable payoff in which:
  - $N = K \cup M$;
  - There are two input goods ($\ell = 2$);
  - $\omega_i = \begin{cases} (1,0), & \text{if } i \in K \\ (0,1), & \text{if } i \in M \end{cases}$.
  - $f_i(a,b) = \min\{a,b\}$, for every $i \in N$.
- Then $v(S) = \min\{|K \cap S|, |M \cap S|\}$.
- By the preceding proposition, the core is nonempty.
- If $|K| < |M|$, it consists of a single point, in which:
  - Each agent in K receives the payoff of 1;
  - Each agent in M receives the payoff of 0.
- The proof is identical to that for the market with an indivisible good.

The Core and the Competitive Equilibria

- Classical economic theory defines the solution of "competitive equilibrium" for a market.
- We show that the core of a market contains its competitive equilibria.
- We begin with the simple case in which:
  - All agents have the same production function f;
  - There is only one input.
- Define the average endowment

$$\omega^* = \frac{\sum_{i \in N} \omega_i}{|N|}.$$

By hypothesis, f is concave.

It follows that the allocation in which each agent receives the amount ω^* of the input maximizes the total output.

The Core and the Competitive Equilibria (Cont'd)

- Let p^* be the slope of a tangent to the production function at ω^*. Let g be the affine function with slope p^* for which $g(\omega^*) = f(\omega^*)$. Then $(g(\omega_i))_{i \in N}$ is in the core.

$$\begin{array}{rcl} v(S) & = & |S| f\left(\frac{\sum_{i \in S} \omega_i}{|S|}\right) \leq |S| g\left(\frac{\sum_{i \in S} \omega_i}{|S|}\right) = \sum_{i \in S} g(\omega_i); \\ v(N) & = & |N| f\left(\frac{\sum_{i \in N} \omega_i}{|N|}\right) = |N| f(\omega^*) = |N| g(\omega^*) = \sum_{i \in N} g(\omega_i). \end{array}$$

  The payoff profile $(g(\omega_i))_{i \in N}$ can be achieved by each agent trading input for output at the price p^* (each unit of input costs p^* units of output): If trade at p^* is possible, i maximizes his payoff by choosing the amount z of input to solve $\max_z (f(z) - p^*(z - \omega_i))$, the solution of which is ω^*.

Competitive Equilibria

- We define a **competitive equilibrium** of a market with transferable payoff as a pair $(p^*, (z_i^*)_{i \in N})$ consisting of:
  - A vector $p^* \in \mathbb{R}_+^\ell$ (the vector of **input prices**);
  - An allocation $(z_i^*)_{i \in N}$, such that for each agent i the vector z_i^* solves the problem
$$\max_{z_i \in \mathbb{R}_+^\ell} (f_i(z_i) - p^*(z_i - \omega_i)).$$
- If $(p^*, (z_i^*)_{i \in N})$ is a competitive equilibrium, then the value of the maximum
$$f_i(z_i^*) - p^*(z_i^* - \omega_i)$$
is referred to as a **competitive payoff of agent** i.

The Idea Behind Competitive Equilibria

- The idea is that the agents can trade inputs at fixed prices, which are expressed in terms of units of output.
- Suppose after buying and selling inputs, agent i holds the bundle z_i.
- Then his net expenditure, in units of output, is

$$p^*(z_i - \omega_i).$$

- Agent i can produce $f_i(z_i)$ units of output.
- So his net payoff is

$$f_i(z_i) - p^*(z_i - \omega_i).$$

- A price vector p^* generates a competitive equilibrium if, when each agent chooses his trades to maximize his payoff, the resulting profile $(z_i^*)_{i \in N}$ of input vectors is feasible in the sense that it is an allocation.

Competitive Payoffs and Core

Proposition

Every profile of competitive payoffs in a market with transferable payoff is in the core of the market.

- Let $\langle N, \ell, (\omega_i), (f_i) \rangle$ be a market with transferable payoff.

  Let $\langle N, v \rangle$ the associated coalitional game.

  Suppose $(p^*, (z_i^*)_{i \in N})$ is a competitive equilibrium of the market.

  Suppose, for the sake of obtaining a contradiction, that the profile of associated competitive payoffs is not in the core.

  Then, there is a coalition S and a vector $(z_i)_{i \in S}$, such that:
  - $\sum_{i \in S} z_i = \sum_{i \in S} \omega_i$;
  - $\sum_{i \in S} f_i(z_i) > \sum_{i \in S} (f_i(z_i^*) - p^* z_i^* + p^* \omega_i)$.

Competitive Payoffs and Core (Cont'd)

- By the preceding hypotheses,

$$\sum_{i \in S}(f_i(z_i) - p^* z_i) > \sum_{i \in S}(f_i(z_i^*) - p^* z_i^*).$$

Hence, for at least one agent $i \in S$,

$$f_i(z_i) - p^* z_i > f_i(z_i^*) - p^* z_i^*.$$

This contradicts the fact that z_i^* is a max problem solution.

Now let $(z_i)_{i \in N}$ be such that $\sum_{i \in N} z_i = \sum_{i \in N} \omega_i$.

We have

$$\begin{array}{rcl} \sum_{i \in N} f_i(z_i) & \leq & \sum_{i \in N}(f_i(z_i^*) - p^* z_i^* + p^* \omega_i) \\ & = & \sum_{i \in N} f_i(z_i^*). \end{array}$$

Therefore, $v(N) = \sum_{i \in N} f_i(z_i^*)$.

Subsection 5

Coalitional Games Without Transferable Payoff

Coalitional Games and Transferable Payoff

- In a coalitional game with transferable payoff each coalition S is characterized by a single number $v(S)$.
- The interpretation is that $v(S)$ is a payoff that may be distributed in any way among the members of S.
- We now switch to games in which each coalition S:
  - Cannot necessarily achieve all distributions of some fixed payoff;
  - Is characterized, instead, by a set $V(S)$ of consequences.

Coalitional Games Without Transferable Payoff

Definition (Coalitional Game Without Transferable Payoff)

A **coalitional game** (without transferable payoff)

$$\langle N, X, V, (\succsim_i)_{i \in N} \rangle$$

consists of:

- A finite set N (of **players**);
- A set X (of **consequences**);
- A function V that assigns to every nonempty subset S of N (a **coalition**) a set $V(S) \subseteq X$;
- For each player $i \in N$, a **preference relation** $\succsim_i$ on X.

Relation Between Coalitional Games

- Let $\langle N, v\rangle$ be a coalitional game with transferable payoff;
- The associated coalitional game

$$\langle N, X, V, (\succsim_i)_{i \in N}\rangle$$

  is defined as follows:
  - $X = \mathbb{R}^N$;
  - $V(S) = \{x \in \mathbb{R}^N : \sum_{i \in S} x_i = v(S) \text{ and } x_j = 0, \text{ if } j \in N - S\}$;
  - $x \succsim_i y$ if and only if $x_i \geq y_i$.
- Under this association the set of coalitional games with transferable payoff is a subset of the set of all coalitional games.

The Core of a Coalitional Game

- The definition of the core of a general coalitional game is a natural extension of our definition for the core of a game with transferable payoff.

Definition (Core)

The **core** of the coalitional game $\langle N, V, X, (\succsim_i)_{i\in N}\rangle$ is the set of all $x \in V(N)$ for which there is no coalition S and $y \in V(S)$, such that

$$y \succ_i x, \quad \text{for all } i \in S.$$

- Under suitable conditions (similar to that of balancedness for a coalitional game with transferable payoff) the core of a general coalitional game is nonempty.

Subsection 6

Exchange Economies

Exchange Economies

- A generalization of the notion of a market with transferable payoff is an exchange economy.
- An **exchange economy** $\langle N, \ell, (\omega_i), (\succsim_i)\rangle$ consists of:
  - A finite set N (of **agents**);
  - A positive integer ℓ (the number of **goods**);
  - For each agent $i \in N$, a vector $\omega_i \in \mathbb{R}^\ell_+$ (the **endowment** of agent i), such that every component of $\sum_{i \in N} \omega_i$ is positive;
  - For each agent $i \in N$ a nondecreasing, continuous and quasi-concave preference relation $\succsim_i$ over the set $\mathbb{R}^\ell_+$ of bundles of goods.
- ω_i represents the bundle of goods that agent i owns initially.
- The requirement that $\sum_{i \in N} \omega_i$ be positive means that there is a positive quantity of every good.
- Goods may be transferred between the agents, but there is no payoff that is freely transferable.

Allocations

- An **allocation** of an exchange economy $\langle N, \ell, (\omega_i), (\succsim_i) \rangle$ is a distribution of the total endowment in the economy among the agents.
- That is, an allocation is a profile $(x_i)_{i \in N}$, with $x_i \in \mathbb{R}_+^\ell$, for all $i \in N$, such that

$$\sum_{i \in N} x_i = \sum_{i \in N} \omega_i.$$

Competitive Equilibria

- A **competitive equilibrium** of an exchange economy is a pair

$$(p^*, (x_i^*)_{i \in N})$$

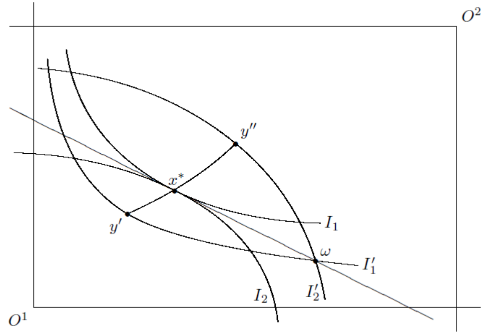
  consisting of
  - A vector $p^* \in \mathbb{R}_+^\ell$ with $p^* \neq 0$ (the **price vector**);
  - An allocation $(x_i^*)_{i \in N}$, such that, for each agent i, we have:
    - $p^* x_i^* \leq p^* \omega_i$;
    - $x_i^* \succsim_i x_i$, for any x_i for which $p^* x_i \leq p^* \omega_i$.
- If $(p^*, (x_i^*)_{i \in N})$ is a competitive equilibrium, then $(x_i^*)_{i \in N}$ is referred to as a **competitive allocation**.

Interpretation of Competitive Equilibria

- The main idea is that the agents can trade goods at fixed prices.
- We can think of p_j^* as the "money" price of good j.
- Given any price vector p, each agent i chooses a bundle that is most desirable (according to his preferences) among all those that are affordable (i.e., satisfy $px_i \leq p\omega_i$).
- Typically an agent chooses a bundle that contains more of some goods and less of others than he initially owns.
- This is interpreted as "demanding" some goods, while "supplying" others.
- The requirement that the profile of chosen bundles be an allocation means that, for every good, the sum of the individuals' demands is equal to the sum of their supplies.
- A standard result in economic theory is that an exchange economy, in which every agent's preference relation is increasing, has a competitive equilibrium and an economy may possess many such equilibria.

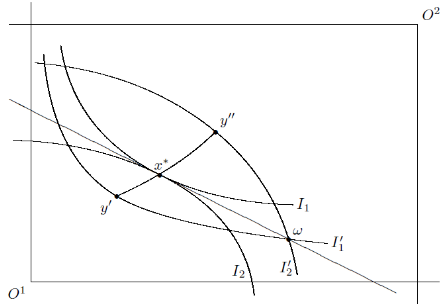
Edgeworth Boxes

- An exchange economy that contains two agents ($|N| = 2$) and two goods ($\ell = 2$) can be represented in an **Edgeworth box**.



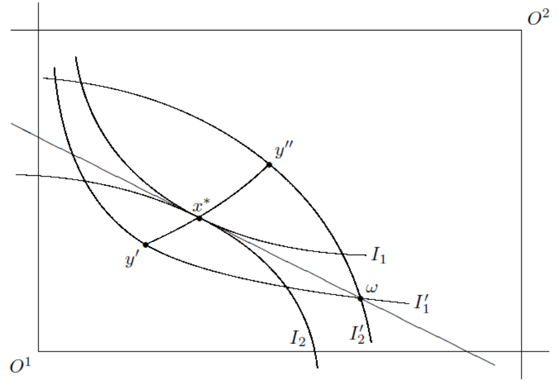
- Bundles of goods consumed by Agent 1 are measured from O^1.
- Bundles of goods consumed by Agent 2 are measured from O^2.
- The width of the box is the total endowment of Good 1.
- The height of the box is the total endowment of Good 2.

Edgeworth Boxes (Cont'd)



- Each point x corresponds to an allocation in which Agent i receives the bundle x measured from O^i.
- The point labeled ω corresponds to the pair of endowments.
- The curved lines labeled I_i and I'_i are indifference curves of Agent i: If x and y are points on one of these curves then $x \sim_i y$.
- The straight line passing through ω and x^* is (relative to O^i) the set of all bundles x^i for which $px_i = p\omega_i$.

Edgeworth Boxes (Cont'd)



- The point x^* corresponds to a competitive allocation.
  The most preferred bundle of agent i in the set $\{x_i : px_i \leq p\omega_i\}$ is x^* when measured from origin O^i.
- The ratio of the competitive prices is the negative of the slope of the straight line through ω and x^*.

Exchange Economies and Coalitional Games

- An exchange economy is closely related to a market.
- In a market, payoff can be directly transferred between agents.
- In an exchange economy only goods can be directly transferred.
- Let $\langle N, \ell, (\omega_i), (\succsim_i)\rangle$ be an exchange economy.
- The associated coalitional game $\langle N, X, V, (\succsim_i)\rangle$ is defined by:
  - $X = \{(x_i)_{i \in N} : x_i \in \mathbb{R}^{\ell}_{+},$ for all $i \in N\}$;
  - $V(S) = \{(x_i)_{i\in N} \in X : \sum_{i\in S} x_i = \sum_{i \in S} \omega_i$ and $x_j = \omega_j,$ for all $j \in N - S\}$, for each coalition S;
  - Each preference relation $\succsim_i$ is defined by

$$(x_j)_{j\in N} \succsim_i (y_j)_{j \in N} \quad \text{if and only if} \quad x_i \succsim_i y_i.$$

- The third condition expresses the assumption that each agent cares only about his own consumption.

The Core

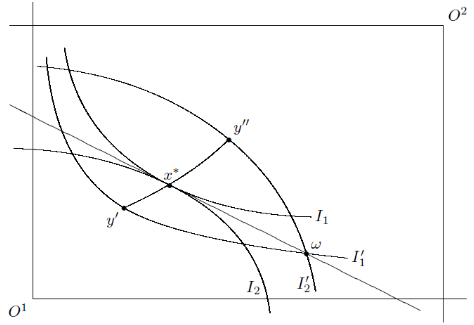
- We define the **core** of an exchange economy $\langle N, \ell, (\omega_i), (\succsim_i)\rangle$ to be the core of the associated coalitional game $\langle N, X, V, (\succsim_i)\rangle$.
  - The set $V(N)$ is the set of all allocations.
  - For each $j \in N$, we have

$$V(\{j\}) = \{(\omega_i)_{i\in N}\}.$$

- The core of a two-agent economy is the set of all allocations $(x_i)_{i\in N}$, such that:
  - $x_j \succsim_j \omega_j$, for each agent j;
  - There is no allocation $(x_i')_{i\in N}$, such that

$$x_j' \succ_j x_j, \quad \text{for both agents } j.$$

The Core of a Two-Agent Economy



- The core corresponds to the locus of points in the area bounded by I_1' and I_2' for which an indifference curve of Agent 1 and an indifference curve of Agent 2 share a common tangent.
- I.e., it is the curved line passing through y', x^*, and y''.
- In particular, the core contains the competitive allocation.

Competitive Allocations and the Core

Proposition

Every competitive allocation in an exchange economy is in the core.

- Let $E = \langle N, \ell, (\omega_i), (\succsim_i) \rangle$ be an exchange economy.

  Let $(p^*, (x_i^*)_{i \in N})$ be a competitive equilibrium of E.

  Assume that $(x_i^*)_{i \in N}$ is not in the core of E.

  Then there is a coalition S and $(y_i)_{i \in S}$, such that:

  - $\sum_{i \in S} y_i = \sum_{i \in S} \omega_i$;
  - $y_i \succ_i x_i^*$, for all $i \in S$.

  Thus, we get $p^* y_i > p^* \omega_i$, for all $i \in S$.

  Hence, $p^* \sum_{i \in S} y_i > p^* \sum_{i \in S} \omega_i$.

  This contradicts $\sum_{i \in S} y_i = \sum_{i \in S} \omega_i$.
- It follows from this result that an economy that has a competitive equilibrium has a nonempty core.

Competitive Equilibria and the Core in Large Economies

- We now show that as the number of agents increases, the core shrinks to the set of competitive allocations.
- This shows that, in a large enough economy, the following kinds of predictions are tightly connected.
  - Those relying on the competitive equilibrium, which is based on agents who trade at fixed prices;
  - Those relying on the core, which is based on the ability of a group of agents to improve its lot by forming an autonomous subeconomy, without reference to prices.
- Put differently, in a large enough economy, the only outcomes that are immune to deviations by groups of agents are competitive equilibrium allocations.

Derived Economies and Types

- To state the announced result precisely, let

$$E = \langle N, \ell, (\omega_i), (\succsim_i) \rangle$$

  be an exchange economy in which there are n agents.
- For any positive integer k let kE be the economy derived from E in which there are kn agents - k copies of each agent in E.
- An agent j in kE who is a copy of Agent i in E is of **type** $i = \iota(j)$.

Equal Treatment in the Core

Lemma (Equal Treatment in the Core)

Let E be an exchange economy in which the preference relation of every agent is increasing and strictly quasi-concave, and let k be a positive integer. In any allocation in the core of kE, all agents of the same type obtain the same bundle.

- Let $E = \langle N, \ell, (\omega_i), (\succsim_i) \rangle$ and let x be an allocation in the core of kE in which there are two agents of type t^* whose bundles are different.

  We show that there is a distribution of the endowment of the coalition consisting of the worst-off agent of each type that makes every member of the coalition better off than he is in x.

  For each type t, select one agent, i_t, in kE who is least well off (according to $\succsim_t$) in x among all agents of type t.

  Let S be the coalition (of size $|N|$) of these agents.

Equal Treatment in the Core (Cont'd)

- For each type t, let z_t be the average bundle of the agents of type t in the allocation x,

$$z_t = \frac{\sum_{\{j:\iota(j)=t\}} x_j}{k}.$$

- Then we have:
  - $\sum_{t \in N} z_t = \sum_{t \in N} \omega_t$;
  - $z_t \succsim_t x_{i_t}$;
    If not, for every j, such that $\iota(j) = t$, $z_t \prec_t x_j$.
    So, by the quasi-concavity of $\succsim_t$, we have $z_t \prec_t z_t$.
    This yields a contradiction.
  - $z_{t^*} \succ_{t^*} x_{i_{t^*}}$;
    Preference relations are strictly quasi-concave.

Equal Treatment in the Core (Cont'd)

- We showed that:
  - (i) It is feasible for the coalition S to assign to each agent $j \in S$ the bundle $z_{\iota(j)}$, since $\sum_{j \in S} z_{\iota(j)} = \sum_{t \in N} z_t = \sum_{j \in S} \omega_j$;
  - (ii) For every agent $j \in S$, the bundle $z_{\iota(j)}$ is at least as desirable as x_j;
  - (iii) For the agent $j \in S$ of type t^*, the bundle $z_{\iota(j)}$ is preferable to x_j.

  By hypothesis, each agent's preference relation is increasing.

  So we can modify the allocation $(z_t)_{t \in N}$ by reducing t^*'s bundle by a small amount and distributing this amount equally among the other members of S so that we have a profile $(z'_t)_{t \in N}$ with:
  - $\sum_{t \in N} z'_t = \sum_{t \in N} \omega_t$;
  - $z'_{\iota(j)} \succ_{\iota(j)} x_j$, for all $j \in S$.

  This contradicts the fact that x is in the core of kE.

Core of kE Shrinking to Competitive Allocations of E

- For any positive integer k we can identify the core of kE with a profile of $|N|$ bundles, one for each type.
- Under this identification, it is clear that the core of kE is a subset of the core of E.
- We now show that the core of kE shrinks to the set of competitive allocations of E as k increases.

Proposition (Core of kE Shrinking to Competitive Allocations of E)

Let E be an exchange economy in which every agent's preference relation is increasing and strictly quasi-concave and every agent's endowment of every good is positive. Let x be an allocation in E. If, for every positive integer k, the allocation in kE in which every agent of each type t receives the bundle x_t is in the core of kE, then x is a competitive allocation of E.

Core Shrinking to Competitive Allocations: Proof

- Let $E = \langle N, \ell, (\omega_i), (\succsim_i)\rangle$. Let

$$Q = \left\{ \sum_{t\in N} \alpha_t z_t : \sum_{t\in N} \alpha_t = 1, \alpha_t \geq 0 \text{ and } z_t + \omega_t \succ_t x_t \text{ for all } t \right\}.$$

Under our assumptions, Q is convex.

Claim: $0 \notin Q$.

Suppose $0 = \sum_{t\in N} \alpha_t z_t$, for some (α_t) and (z_t), with:

  - $\sum_{t\in N} \alpha_t = 1$, $\alpha_t \geq 0$;
  - $z_t + \omega_t \succ_t x_t$, for all t.

Suppose that every α_t is a rational number (otherwise, approximate).

Choose an integer K large enough that $K\alpha_t$ is an integer for all t.

Let S be a coalition in KE that consists of $K\alpha_t$ agents of each type t.

Let $x'_i = z_{\iota(i)} + \omega_i$, for each $i \in S$. We have:

  - $\sum_{i\in S} x'_i = \sum_{t\in N} K\alpha_t z_t + \sum_{i\in S} \omega_i = \sum_{i\in S} \omega_i$;
  - $x'_i \succ_i x_i$, for all $i \in S$.

This contradicts the fact that x is in the core of KE.

Core Shrinking to Competitive Allocations (Cont'd)

- The Separating Hyperplane Theorem, yields a $0 \neq p \in \mathbb{R}^\ell$, such that

$$pz \geq 0, \text{ for all } z \in Q.$$

Since all agents' preferences are increasing, each unit vector is in Q.

Indeed, let $1_{\{m\}}$ be the mth unit vector in $\mathbb{R}^\ell$. Take:

  - $z_t = x_t - \omega_t + 1_{\{m\}}$;
  - $\alpha_t = \frac{1}{|N|}$ for each t.

Thus, $p \geq 0$.

Now, for every agent i, $x_i - \omega_i + \epsilon \in Q$, for every $\epsilon > 0$.

So $p(x_i - \omega_i + \epsilon) \geq 0$.

Taking ϵ small, we conclude that $px_i \geq p\omega_i$, for all i.

But x is an allocation, so $px_i = p\omega_i$, for all i.

Core Shrinking to Competitive Allocations (Cont'd)

- Finally, we argue that, if $y_i \succ_i x_i$, for some $i \in N$, then $py_i > p\omega_i$, so that x is a competitive allocation of E.

  Suppose that $y_i \succ_i x_i$. Then $y_i - \omega_i \in Q$.

  So, by the choice of p, we have $py_i \geq p\omega_i$.

  Furthermore, $\theta y_i \succ_i x_i$, for some $\theta < 1$.

  So $\theta y_i - \omega_i \in Q$. Hence, $\theta py_i \geq p\omega_i$.

  Also, $p\omega_i > 0$, since every component of ω_i is positive.

  Thus, $py_i > p\omega_i$.

- In any competitive equilibrium of kE all agents of the same type consume the same bundle, so that any such equilibrium is naturally associated with a competitive equilibrium of E.

  Thus, the result shows a sense in which the larger k is, the closer are the core and the set of competitive allocations of kE.