Introduction to Game Theory

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Subsection 1

Two Approaches

Shortcomings of the Core

- The definition of the core does not restrict a coalition's credible deviations, beyond imposing a feasibility constraint.
- In particular, it assumes that any deviation is the end of the story and ignores the fact that a deviation may trigger a reaction that leads to a different final outcome.
- The solution concepts we study in this set consider various restrictions on deviations that are motivated by these considerations.

Restrictions on Possible Deviations

- In our first approach, an objection by a coalition to an outcome consists of an alternative outcome that is itself constrained to be stable.

  A stable outcome has the property that no coalition can achieve some other stable outcome that improves the lot of all its members.
- In our second approach, the chain of events that a deviation unleashes is cut short after two stages.

  The stability condition is that for every objection to an outcome there is a balancing counterobjection.

  Different notions of objection and counterobjection give rise to a number of different solution concepts.
- We restrict attention to coalitional games with transferable payoff.

Subsection 2

The Stable Sets of von Neumann and Morgenstern

Idea Behind Stability

- Suppose that a coalition S:
  - Is unsatisfied with the current division of $v(N)$;
  - Can credibly object by suggesting a stable division x of $v(N)$ that is better for all the members of S;
  - The objection is backed up by a threat to implement $(x_i)_{i \in S}$ on its own (by dividing the worth $v(S)$ among its members).
- The logic behind the requirement that an objection itself be stable is that:
  - Otherwise, the objection may unleash a process involving further objections by other coalitions;
  - At the end of this process some of the members of the deviating coalition may be worse off.

Idea Behind the Definition

- This idea leads to a definition in which a set of stable outcomes satisfies two conditions:
  - (i) For every outcome that is not stable some coalition has a credible objection;
  - (ii) No coalition has a credible objection to any stable outcome.
- Note that this definition is self-referential and admits the possibility that there may be many stable sets.

Imputations and Objections

- Let $\langle N, v \rangle$ be a cohesive coalitional game with transferable payoff.
- An **imputation** of $\langle N, v \rangle$ is a feasible payoff profile x for which

$$x_i \geq v(\{i\}), \quad \text{for all } i \in N.$$

- Let X be the set of all imputations of $\langle N, v \rangle$.
- An imputation x is an **objection of the coalition S to the imputation y** if:
  - $x_i > y_i$, for all $i \in S$;
  - $x(S) \leq v(S)$.

  If this is the case, we write $x \succ_S y$.
- The expression "x dominates y via S" also means that x is an objection of S to y.

Remarks on Objections to Imputations

- Recall that a coalitional game $\langle N, v\rangle$ is cohesive if $v(N) \geq \sum_{k=1}^{K} v(S_k)$, for every partition $\{S_1, \ldots, S_K\}$ of N.
- Since $\langle N, v\rangle$ is cohesive, we have $x \succ_S y$ if and only if, there is an S-feasible payoff vector $(x_i)_{i\in S}$, for which

$$x_i > y_i, \quad \text{for all } i \in S.$$

- The core of the game $\langle N, v\rangle$ is the set of all imputations to which there is no objection,

$$\{y \in X : \text{there is no } S \in \mathcal{C} \text{ and } x \in X, \text{ for which } x \succ_S y\}.$$

Stable Sets

Definition (Stable Set)

Let $\langle N, v \rangle$ be a coalitional game with transferable payoff. A subset Y of the set X of imputations is a **stable set** if it satisfies:

- **Internal Stability**: If $y \in Y$, then for no $z \in Y$ does there exist a coalition S for which $z \succ_S y$.
- **External Stability**: If $z \in X - Y$, then there exists $y \in Y$, such that $y \succ_S z$, for some coalition S.

Stable Sets (Alternative Formulation)

- The definition of a stable set has an alternative formulation.
- Let Y be a set of imputations.
- Let $\mathcal{D}(Y)$ be the set of imputations z, for which there is a coalition S and an imputation $y \in Y$, such that $y \succ_S z$.
- Then:
  - Internal stability is equivalent to the condition
    $$Y \subseteq X - \mathcal{D}(Y);$$
  - External stability is equivalent to the condition
    $$Y \supseteq X - \mathcal{D}(Y).$$
- So a set Y of imputations is a stable set if and only if
  $$Y = X - \mathcal{D}(Y).$$
- The core is a single set of imputations.
- A game, however, may have more than one stable sets.

Stable Sets and the Core

Proposition

Let $\langle N, v\rangle$ be a coalitional game with transferable payoff.

a. The core is a subset of every stable set.
b. No stable set is a proper subset of any other.
c. If the core is a stable set then it is the only stable set.

a. Every member of the core is an imputation.

   Let Y be stable and $x \notin Y$.

   By external stability, there exist $y \in Y$ and $S \in \mathcal{C}$, such that $x \prec_S y$.

   Hence, x cannot be in the core.

b. This also follows by external stability.
c. This follows from Statements (a) and (b).

Example: The Three-Player Majority Game

- Consider the game $\langle\{1,2,3\},v\rangle$ in which

$$v(S)=\begin{cases}1, & \text{if } |S|\geq 2\\ 0, & \text{otherwise}\end{cases}.$$

One stable set of this game is $Y=\{(\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},0,\frac{1}{2}),(0,\frac{1}{2},\frac{1}{2})\}$.
A pair of players shares equally the single unit of payoff.
  - For all x and y in Y only one player prefers x to y.
    Therefore, Y is internally stable.
  - Let z be an imputation outside Y.
    Then, there are two players i and j for whom $z_i<\frac{1}{2}$ and $z_j<\frac{1}{2}$.
    So there is an imputation in Y that is an objection of $\{i,j\}$ to z.
    This shows that Y is externally stable.

Example (Cont'd)

- For $c \in \left[0, \frac{1}{2}\right)$ and $i \in \{1, 2, 3\}$, define

$$Y_{i,c} = \{x \in X : x_i = c\}.$$

$Y_{i,c}$ is also stable, for all $i = 1, 2, 3$ and all $c \in \left[0, \frac{1}{2}\right)$.

  - For any x and y in the set, there is only one player who prefers x to y. This proves internal stability of $Y_{i,c}$.
  - Let $i = 3$ and let z be an imputation outside $Y_{3,c}$.
    - Suppose, first, $z_3 > c$.
      Then $z_1 + z_2 < 1 - c$.
      So, there exists $x \in Y_{3,c}$, such that $x_1 > z_1$ and $x_2 > z_2$.
      Thus, $x \succ_{\{1,2\}} z$.
    - Suppose, next, $z_3 < c$.
      Assume, say, $z_1 \leq z_2$.
      Then $(1 - c, 0, c) \succ_{\{1,3\}} z$.

Subsection 3

The Bargaining Set, Kernel and Nucleolus

Concepts Based on Objections and Counterobjections

- We regard an objection by a coalition to be convincing if no other coalition has a “balancing” counterobjection.
- Neither the objection nor the counterobjection are required to be stable.
- There are three solution concepts that differ in the nature of the objections and counterobjections:
  - The Bargaining Set;
  - The Kernel;
  - The Nucleolus.

Objections and Counterobjections

- Let $\langle N, v \rangle$ be a coalitional game with transferable payoff.
- Let x be an imputation.
- Consider a pair (y, S), where:
  - S is a coalition;
  - y is an S-feasible payoff vector.

  Such a pair is an **objection of i against j to x** if:
  - S includes i but not j;
  - $y_k > x_k$, for all $k \in S$.
- Consider a pair (z, T), where:
  - T is a coalition;
  - z is a T-feasible payoff vector.

  Such a pair is a **counterobjection to the objection (y, S) of i against j** if:
  - T includes j but not i;
  - $z_k \geq x_k$, for all $k \in T - S$;
  - $z_k \geq y_k$, for all $k \in T \cap S$.

Objections and Counterobjections (Comments)

- An objection is an argument by one player against another.
- An objection of i against j to x specifies:
  - A coalition S that includes i but not j;
  - A division y of $v(S)$ that is preferred by all members of S to x.
- A counterobjection to (y, S) by j specifies:
  - An alternative coalition T that contains j but not i;
  - A division of $v(T)$ that is:
    - At least as good as y for all the members of T who are also in S;
    - At least as good as x for the other members of T.

The Bargaining Set

Definition (The Bargaining Set)

The **bargaining set** of a coalitional game with transferable payoff is the set of all imputations x with the property that, for every objection (y, S) of any player i against any other player j to x, there is a counterobjection to (y, S) by j.

- The bargaining set models the stable arrangements in a society in which:
  - Any argument that i makes against x takes the form:

    "I get too little in the imputation x and j gets too much; I can form a coalition that excludes j in which everybody is better off than in x";
  - Such an argument is ineffective, if player j can respond:

    "Your demand is not justified; I can form a coalition that excludes you in which everybody is at least as well off as they are in x and the players who participate in your coalition obtain at least what you offer them".

The Bargaining Set as a Solution Concept

- The bargaining set assumes that the argument underlying an objection for which there is no counterobjection undermines the stability of an outcome.
- The appropriateness of the solution in a particular situation depends on the extent to which the participants in that situation regard the existence of an objection for which there is no counterobjection as a reason to change the outcome.
- Note that an imputation is in the core if and only if no player has an objection against any other player, whence the core is a subset of the bargaining set.
- We show later that the bargaining set of every game is nonempty.

Example: The Three-Player Majority Game

- Recall the three-player majority game $\langle \{1, 2, 3\}, v \rangle$, with

$$v(S) = \begin{cases} 1, & \text{if } |S| \geq 2 \\ 0, & \text{otherwise} \end{cases}.$$

- The core of this game is empty and the game has many stable sets.
- The bargaining set of the game is the singleton $\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$.

  Let x be an imputation.

  Suppose that (y, S) is an objection of i against j to x.

  Then we must have:
  - $S = \{i, h\}$, where h is the third player;
  - $y_h < 1 - x_i$, since $y_i > x_i$ and $y(S) = v(S) = 1$.

Example: The Three-Player Majority Game (Cont'd)

- For j to have a counterobjection to (y, S), we need $y_h + x_j \leq 1$.

  Thus, for x to be in the bargaining set we require that for all players i, j and h we have

  $$y_h < 1 - x_i \quad \text{implies} \quad y_h \leq 1 - x_j.$$

  This implies that

  $$1 - x_i \leq 1 - x_j, \quad \text{for all } i \text{ and } j.$$

  Equivalently,

  $$x_j \leq x_i, \quad \text{for all } i \text{ and } j.$$

  So $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

  Obviously this imputation is in the bargaining set.

Example: My Aunt and I

- Let $\langle\{1,2,3,4\},v\rangle$ be the game with

$$v(S)=\begin{cases}1, & \text{if } S \text{ contains } \{2,3,4\} \text{ or } \{1,i\} \text{ for } i\in\{2,3,4\}\\ 0, & \text{otherwise}\end{cases}$$

  Here Player 2 is "I" and Player 1 is his aunt.

  Player 1 appears to be in a stronger position than the other players.

  Suppose x is an imputation for which $x_2 < x_3$.

  Then Player 2 has an objection against Player 3 to x (via $\{1,2\}$) for which there is no counterobjection.

  Thus, if x is in the bargaining set, then $x_2 = x_3 = x_4 = \alpha$, say.

Example: My Aunt and I (Cont'd)

- Any objection of:
  - Player 1 against Player 2 to x takes the form $(y, \{1, j\})$, where $j = 3$ or 4 and $y_j = 1 - y_1 < 3\alpha$.
    There is no counterobjection if and only if $\alpha + 3\alpha + \alpha > 1$.
    Equivalently, $\alpha > \frac{1}{5}$.
  - Player 2 against Player 1 to x must use the coalition $\{2, 3, 4\}$ and give one of Players 3 or 4 less than $\frac{1-\alpha}{2}$.
    Player 1 does not have a counterobjection if and only if
    $1 - 3\alpha + \frac{1-\alpha}{2} > 1$.
    Equivalently, $\alpha < \frac{1}{7}$.

  Hence, the bargaining set is

$$\left\{(1 - 3\alpha, \alpha, \alpha, \alpha) : \frac{1}{7} \leq \alpha \leq \frac{1}{5}\right\}.$$

- Note that by contrast the core is empty.

Excess of a Coalition

- We now describe another solution that is defined by the condition that to every objection there is a counterobjection.
- It differs from the bargaining set in the nature of objections and counterobjections that are considered effective.
- Let $\langle N, v \rangle$ be a coalitional game with transferable payoff.
- Let x be an imputation.
- For a coalition S, the **excess of** S is defined by

$$e(S, x) = v(S) - x(S).$$

  - If the excess of the coalition S is positive, then it measures the amount that S has to forgo in order for the imputation x to be implemented. I.e., it is the sacrifice that S makes to maintain the social order.
  - If the excess of S is negative, then its absolute value measures the amount over and above the worth of S that S obtains when the imputation x is implemented.
    I.e., it is S's surplus in the social order.

Objections and Counterobjections (Ideas)

- A player i objects to an imputation x by:
  - Forming a coalition S that excludes some player j for whom $x_j > v(\{j\})$;
  - Pointing out that he is dissatisfied with the sacrifice or gain of this coalition.
- Player j counterobjects by pointing to the existence of a coalition that:
  - Contains j but not i;
  - Sacrifices more (if $e(S, x) > 0$) or gains less (if $e(S, x) < 0$).

Objections and Counterobjections (Formalism)

- A coalition S is an **objection of** i **against** j **to** x if:
  - S includes i but not j;
  - $x_j > v(\{j\})$.
- A coalition T is a **counterobjection to the objection** S of i against j if:
  - T includes j but not i;
  - $e(T, x) \geq e(S, x)$.

The Kernel

Definition (The Kernel)

The **kernel** of a coalitional game with transferable payoff is the set of all imputations x with the property that, for every objection S of any player i against any other player j to x, there is a counterobjection of j to S.

- For any two players i and j and any imputation x, define $s_{ij}(x)$ to be the maximum excess of any coalition that contains i but not j:

$$s_{ij}(x) = \max_{S \in \mathcal{C}} \{e(S, x) : i \in S \text{ and } j \in N - S\}.$$

- The kernel is, equivalently, the set of imputations $x \in X$, such that for every pair (i, j) of players either $s_{ji}(x) \geq s_{ij}(x)$ or $x_j = v(\{j\})$.

Remarks on the Kernel

- The kernel models the stable arrangements in a society in which:
  - A player makes arguments of the following type against an imputation x:

    > "Here is a coalition to which I belong that excludes player j and sacrifices too much (or gains too little)".

  - Such an argument is ineffective as far as the kernel is concerned if player j can respond by saying:

    > "Your demand is not justified, since I can name a coalition to which I belong that excludes you and sacrifices even more (or gains even less) than the coalition that you name".

The Kernel and the Bargaining Set

Lemma

The kernel of a coalitional game with transferable payoff is a subset of the bargaining set.

- Let $\langle N, v\rangle$ be a coalitional game with transferable payoff.

  Let x be an imputation in the kernel.

  Let (y, S) be an objection, in the sense of the bargaining set, of player i against j to x:
  - $i \in S$ and $j \in N - S$;
  - $y(S) = v(S)$;
  - $y_k > x_k$, for all $k \in S$.

  Suppose, first, that $x_j = v(\{j\})$.

  Then $(z, \{j\})$, with $z_j = v(\{j\})$, is a counterobjection to (y, S).

The Kernel and the Bargaining Set

- Suppose, on the other hand, that $x_j > v(\{j\})$.
  Then, since x is in the kernel,
  $$s_{ji}(x) \geq s_{ij}(x) \geq v(S) - x(S) = y(S) - x(S).$$
  Let T be a coalition that contains j but not i for which
  $$s_{ji}(x) = v(T) - x(T).$$
  Then $v(T) - x(T) \geq y(S) - x(S)$. Thus,
  $$\begin{array}{rcl} v(T) & \geq & y(S \cap T) + y(S - T) + x(T - S) - x(S - T) \\ & > & y(S \cap T) + x(T - S). \quad (y(S - T) > x(S - T)) \end{array}$$
  Thus, there exists a T-feasible payoff vector z with:
  - $z_k \geq x_k$, for all $k \in T - S$;
  - $z_k \geq y_k$, for all $k \in T \cap S$.

  So (z, T) is a counterobjection to (y, S).

Three-Player Majority Game Revisited

- It follows from our calculation of the bargaining set, the previous lemma and the nonemptiness of the kernel that the kernel of the three-player majority game is $\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$.

  We show this directly.

  Assume that $x_1 \geq x_2 \geq x_3$, with at least one strict inequality.

  Then

  $$s_{31}(x) = 1 - x_2 - x_3 > 1 - x_2 - x_1 = s_{13}(x).$$

  Moreover,

  $$x_1 > 0 = v(\{1\}).$$

  So x is not in the kernel.

My Aunt and I Revisited

- The kernel of the game is $\{(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})\}$.

  Let x be in the kernel.

  By the preceding lemma and the calculation of the bargaining set of the game we have

  $$x = (1 - 3\alpha, \alpha, \alpha, \alpha), \quad \text{for some } \frac{1}{7} \leq \alpha \leq \frac{1}{5}.$$

  So we have $s_{12}(x) = 2\alpha$ and $s_{21}(x) = 1 - 3\alpha$.

  But $x_1 = 1 - 3\alpha > 0 = v(\{1\})$.

  So we need

  $$2\alpha = s_{12}(x) \geq s_{21}(x) = 1 - 3\alpha.$$

  Equivalently, $\alpha \geq \frac{1}{5}$.

  Hence, $\alpha = \frac{1}{5}$.

Objections and Counterobjections by Coalitions

- A solution that is closely related to the kernel is the nucleolus.
- Let $\langle N, v\rangle$ be a coalitional game with transferable payoff.
- Let x be an imputation.
- A pair (S, y), consisting of a coalition S and an imputation y, is an **objection to** x if
$$e(S, x) > e(S, y),$$
i.e., if $y(S) > x(S)$.
- A coalition T is a **counterobjection to the objection** (S, y) if:
  - $e(T, y) > e(T, x)$ (i.e., $x(T) > y(T)$);
  - $e(T, y) \geq e(S, x)$.

The Nucleolus

Definition (The Nucleolus)

The **nucleolus** of a coalitional game with transferable payoff is the set of all imputations x with the property that, for every objection (S, y) to x, there is a counterobjection to (S, y).

- The excess of S is a measure of S's dissatisfaction with x.
- It is the price that S pays to tolerate x rather than secede from N.
- In the definition of the kernel an objection is made by a single player, while here an objection is made by a coalition.

Idea of Nucleolus

- An objection (S, y) may be interpreted as a statement by S of the form:

  > "Our excess is too large in x; we suggest the alternative imputation y in which it is smaller".

- In the nucleolus such objections cause unstable outcomes if no coalition T can respond by saying:

  > "Your demand is not justified since our excess under y is larger than it was under x and, furthermore, exceeds under y what yours was under x".

- An imputation fails to be stable according to the nucleolus if the excess of some coalition S can be reduced without increasing the excess of some coalition to a level at least as large as that of the original excess of S.

Notation

- Let x be an imputation.
- Let $S_1, \ldots, S_{2^{|N|}-1}$ be an ordering of the coalitions for which

$$e(S_\ell, x) \geq e(S_{\ell+1}, x), \quad \text{for } \ell = 1, \ldots, 2^{|N|} - 2.$$

- Let $E(x)$ be the vector of excesses defined by

$$E_\ell(x) = e(S_\ell, x), \quad \text{for all } \ell = 1, \ldots, 2^{|N|} - 1.$$

- Let $B_1(x), \ldots, B_K(x)$ be the partition of the set of all coalitions in which

$$S \text{ and } S' \text{ are in the same cell if and only if } e(S, x) = e(S', x).$$

- For $S \in B_k(x)$, let $e(S, x) = e_k(x)$.
- So we have $e_1(x) > e_2(x) > \cdots > e_K(x)$.

Characterization of the Nucleolus

- We say that $E(x)$ is **lexicographically less than** $E(y)$ if $E_\ell(x) < E_\ell(y)$ for the smallest ℓ for which $E_\ell(x) \neq E_\ell(y)$.
- Equivalently, if there exists k^*, such that:
  - For all $k < k^*$, we have $|B_k(x)| = |B_k(y)|$ and $e_k(x) = e_k(y)$;
  - One of the following holds:
    - (i) $e_{k^*}(x) < e_{k^*}(y)$;
    - (ii) $e_{k^*}(x) = e_{k^*}(y)$ and $|B_{k^*}(x)| < |B_{k^*}(y)|$.

Lemma

The nucleolus of a coalitional game with transferable payoff is the set of imputations x for which the vector $E(x)$ is lexicographically minimal.

Proof of the Characterization

- Let $\langle N, v \rangle$ be a coalitional game with transferable payoff.

  Let x be an imputation for which $E(x)$ is lexicographically minimal.

  We show that x is in the nucleolus.

  Let (S, y) is an objection to x, so that $e(S, y) < e(S, x)$.

  Let k^* be the maximal value of k, such that, for all $k < k^*$,

  - $e_k(x) = e_k(y)$;
  - $B_k(x) = B_k(y)$ (not just $|B_k(x)| = |B_k(y)|$).

  Now $E(y)$ is not lexicographically less than $E(x)$.

  Hence, one of the following holds:

  (i) $e_{k^*}(y) > e_{k^*}(x)$;

  (ii) $e_{k^*}(x) = e_{k^*}(y)$ and $|B_{k^*}(x)| \leq |B_{k^*}(y)|$.

  In either case, there is a coalition $T \in B_{k^*}(y)$, with

$$e_{k^*}(y) = e(T, y) > e(T, x).$$

Proof of the Characterization (Cont'd)

Claim: $e(T,y) \geq e(S,x)$, so that T is a counterobjection to (S,y).

Since $e(S,y) < e(S,x)$, we have

$$S \notin \bigcup_{k=1}^{k^*-1} B_k(x).$$

Hence, $e_{k^*}(x) \geq e(S,x)$.

But $e_{k^*}(y) \geq e_{k^*}(x)$.

Therefore, $e(T,y) \geq e(S,x)$.

Proof of the Characterization (Converse)

- Suppose x is in the nucleolus.

  Assume that $E(y)$ is lexicographically less than $E(x)$.

  Let k^* be the smallest value of k, such that:
  - $B_k(x) = B_k(y)$, for all $k < k^*$;
  - One of the following holds:
    - (i) $e_{k^*}(y) < e_{k^*}(x)$;
    - (ii) $e_{k^*}(y) = e_{k^*}(x)$ and $B_{k^*}(y) \neq B_{k^*}(x)$ (so $|B_{k^*}(y)| \neq |B_{k^*}(x)|$).

  In either case, there exists $S \in B_{k^*}(x)$, for which $e(S, y) < e(S, x)$.

  Let $\lambda \in (0, 1)$ and let $z(\lambda) = \lambda x + (1 - \lambda)y$.

  We have, for any coalition R,

$$e(R, z(\lambda)) = \lambda e(R, x) + (1 - \lambda)e(R, y).$$

Proof of the Characterization (Converse Cont'd)

Claim: $(S, z(\lambda))$ is an objection to x with no counterobjection.

Clearly, it is an objection, since $e(S, z(\lambda)) < e(S, x)$.

For T to be a counterobjection we need:

- $e(T, z(\lambda)) > e(T, x)$;
- $e(T, z(\lambda)) \geq e(S, x)$.

However, if $e(T, z(\lambda)) > e(T, x)$, then $e(T, y) > e(T, x)$.

Thus, $T \notin \bigcup_{k=1}^{k^*} B_k(x)$.

Hence, $e(S, x) = e_{k^*}(x) > e(T, x)$.

But $T \notin \bigcup_{k=1}^{k^*-1} B_k(y)$.

So $e(S, x) = e_{k^*}(x) \geq e_{k^*}(y) \geq e(T, y)$.

Thus, $e(S, x) > \lambda e(T, x) + (1 - \lambda) e(T, y) = e(T, z(\lambda))$.

So no counterobjection to $(S, z(\lambda))$ exists.

The Nucleolus and the Kernel

Lemma

The nucleolus of a coalitional game with transferable payoff is a subset of the kernel.

- Let $\langle N, v\rangle$ be a coalitional game with transferable payoff.

  Let x be an imputation that is not in the kernel of $\langle N, v\rangle$.

  We show that x is not in the nucleolus of $\langle N, v\rangle$.

  Since x is not in the kernel, there are players i and j for which:
  - $s_{ij}(x) > s_{ji}(x)$;
  - $x_j > v(\{j\})$.

  Since $x_j > v(\{j\})$, there exists $\epsilon > 0$, such that $y = x + \epsilon 1_{\{i\}} - \epsilon 1_{\{j\}}$ is an imputation (where $1_{\{k\}}$ is the kth unit vector).

  Choose ϵ small enough that $s_{ij}(y) > s_{ji}(y)$.

The Nucleolus and the Kernel (Cont'd)

- Note that:
  - $e(S,x) < e(S,y)$ if and only if S contains i but not j;
  - $e(S,x) > e(S,y)$ if and only if S contains j but not i.

  Let k^* be the minimal value of k for which there is a coalition $S \in B_{k^*}(x)$ with $e(S,x) \neq e(S,y)$.

  Since $s_{ij}(x) > s_{ji}(x)$, the set $B_{k^*}(x)$ contains:
  - At least one coalition that contains i but not j;
  - No coalition that contains j but not i.

  Further, for all $k < k^*$, we have $B_k(y) = B_k(x)$ and $e_k(y) = e_k(x)$.
  - If $B_{k^*}(x)$ contains coalitions that contain both i and j or neither of them, then $e_{k^*}(y) = e_{k^*}(x)$ and $B_{k^*}(y)$ is a strict subset of $B_{k^*}(x)$.
  - If not, then, since $s_{ij}(y) > s_{ji}(y)$, we have $e_{k^*}(y) < e_{k^*}(x)$.

  In both cases $E(y)$ is lexicographically less than $E(x)$.

  Therefore, x is not in the nucleolus of $\langle N, v \rangle$.

Nonemptiness of the Nucleolus

Proposition

The nucleolus of any coalitional game with transferable payoff is nonempty.

- For all k, E_k is continuous.

  This can be seen by expressing E_k in the form

  $$E_k(x) = \min_{\mathcal{T} \in \mathcal{C}^{k-1}} \max_{S \in \mathcal{C} - \mathcal{T}} e(S, x),$$

  where $\mathcal{C}_0 = \{\emptyset\}$ and $\mathcal{C}_k$ for $k \geq 1$ is the set of all collections of k coalitions.

  - Now E_1 is continuous.
    So $X_1 = \text{argmin}_{x \in X} E_1(x)$ is nonempty and compact.
  - For $k \geq 1$, define $X_{k+1} = \text{argmin}_{x \in X_k} E_{k+1}(x)$.
    By induction, every such set is nonempty and compact.

  But, by the Characterization Theorem, $X_{2^{|N|}-1}$ is the nucleolus.

  Hence, the nucleolus is nonempty.

Nonemptiness of the Kernel and of the Bargaining Set

Corollary

The bargaining set and kernel of any coalitional game with transferable payoff are nonempty.

- Recall that:
  - The nucleolus is a subset of the kernel;
  - The kernel is a subset of the bargaining set.

  So both statements follow from the proposition.

Nucleolus is a Singleton

- In contrast with the bargaining set and the kernel of a game, which may contain many imputations, we have

Proposition

The nucleolus of a coalitional game with transferable payoff is a singleton.

- Let $\langle N, v \rangle$ be a coalitional game with transferable payoff.

  Suppose that imputations x and y are both in the nucleolus.

  Then we have $E(x) = E(y)$.

  We show that, for any coalition S, $e(S, x) = e(S, y)$.

  In particular, for all i, $e(\{i\}, x) = e(\{i\}, y)$, so that $x = y$.

Nucleolus is a Singleton (Cont'd)

- Assume there exists S^*, with $e(S^*, x) \neq e(S^*, y)$.

  Consider the imputation

  $$z = \frac{1}{2}(x + y).$$

  Since $E_k(x) = E_k(y)$, for all k, we have, for all k:
  - $e_k(x) = e_k(y)$;
  - $|B_k(x)| = |B_k(y)|$.

  But $e(S^*, x) \neq e(S^*, y)$.

  So, there exists a minimal value k^* of k, for which $B_{k^*}(x) \neq B_{k^*}(y)$.
  - If $B_{k^*}(x) \cap B_{k^*}(y) \neq \emptyset$, then $B_{k^*}(z) = B_{k^*}(x) \cap B_{k^*}(y) \subset B_{k^*}(x)$.
  - If $B_{k^*}(x) \cap B_{k^*}(y) = \emptyset$, then $e_{k^*}(z) < e_{k^*}(x) = e_{k^*}(y)$.

  In both cases, $E(z)$ is lexicographically less than $E(x)$.

  This contradicts the fact that x is in the nucleolus.

Subsection 4

The Shapley Value

Values

- The solution concepts for coalitional games that we have studied so far are defined with reference to single games in isolation.
- The Shapley value of a game is defined with reference to other games.
- It is an example of a *value*.
- A payoff profile of a coalitional game with transferable payoff $\langle N, v \rangle$ is **feasible** if the sum of its components is $v(N)$.
- A **value** is a function that assigns a unique feasible payoff profile to every coalitional game with transferable payoff.
- The requirement that the payoff profile be feasible is called **efficiency**.

Subgames

- Let $\langle N, v \rangle$ be a coalitional game with transferable payoff.
- Let S be a coalition.
- Define the **subgame** $\langle S, v^S \rangle$ of $\langle N, v \rangle$ to be the coalitional game with transferable payoff in which

$$v^S(T) = v(T), \quad \text{for any } T \subseteq S.$$

Objections

- Our first presentation of the Shapley value is in terms of certain types of objections and counterobjections.
- Let ψ be a value.
- An **objection of player** i **against player** j **to the division** x of $v(N)$ may take one of the following two forms:
  - "Give me more, since, otherwise, I will leave the game, causing you to obtain only $\psi_j(N-\{i\}, v^{N-\{i\}})$ rather than the larger payoff x_j, so that you will lose the positive amount $x_j - \psi_j(N-\{i\}, v^{N-\{i\}})$".
  - "Give me more, since, otherwise, I will persuade the other players to exclude you from the game, causing me to obtain $\psi_i(N-\{j\}, v^{N-\{j\}})$ rather than the smaller payoff x_i, so that I will gain the positive amount $\psi_i(N-\{j\}, v^{N-\{j\}}) - x_i$".

Counterobjections

- A **counterobjection by player j to an objection of the first type** is an assertion:

  "It is true that if you leave then I will lose, but, if I leave, then you will lose at least as much:

  $$x_i - \psi_i(N - \{j\}, v^{N-\{j\}}) \geq x_j - \psi_j(N - \{i\}, v^{N-\{i\}})\text{''}.$$

- A **counterobjection by player j to an objection of the second type** is an assertion:

  "It is true that if you exclude me, then you will gain, but, if I exclude you, then I will gain at least as much:

  $$\psi_j(N - \{i\}, v^{N-\{i\}}) - x_j \geq \psi_i(N - \{j\}, v^{N-\{j\}}) - x_i\text{''}.$$

- The Shapley value is required to satisfy the property that, for every objection of any player i against any other player j, there is a counterobjection of player j.

Comparing With Bargaining Set, Kernel and Nucleolus

- These objections and counterobjections differ from those used to define the bargaining set, kernel, and nucleolus in that they refer to the outcomes of smaller games.
- It is assumed that these outcomes are derived from the same logic as the payoff of the game itself.
- The outcomes of the smaller games, like the outcome of the game itself, are given by the value.

The Balanced Contributions Property

- The requirement that a value assign to every game a payoff profile with the property that every objection is balanced by a counter-objection is equivalent to the following condition:

Definition (Balanced Contributions Property)

A value ψ satisfies the **balanced contributions property** if, for every coalitional game with transferable payoff $\langle N, v \rangle$, we have, for all $i, j \in N$,

$$\psi_i(N, v) - \psi_i(N - \{j\}, v^{N-\{j\}}) = \psi_j(N, v) - \psi_j(N - \{i\}, v^{N-\{i\}}).$$

The Shapley Value

- Define the **marginal contribution of player i to a coalition S**, with $i \notin S$, in the game $\langle N, v \rangle$ to be

$$\Delta_i(S) = v(S \cup \{i\}) - v(S).$$

Definition (The Shapley Value)

The **Shapley value** φ is defined by the condition

$$\varphi_i(N, v) = \frac{1}{|N|!} \sum_{R \in \mathcal{R}} \Delta_i(S_i(R)), \text{ for all } i \in N,$$

where:

- $\mathcal{R}$ is the set of all $|N|!$ orderings of N;
- $S_i(R)$ is the set of players preceding i in the ordering R.

Remarks on the Shapley Value

- If all players are arranged in some order and all orders are equally likely, $\varphi_i(N, v)$ is the expected marginal contribution over all orders of player i to the set of players who precede him.
- The Shapley value is a value.

$$\begin{aligned}
\sum_{i \in N} \varphi_i(N, v) &= \sum_{i \in N} \frac{1}{|N|!} \sum_{R \in \mathcal{R}} \Delta_i(S_i(R)) \\
&= \frac{1}{|N|!} \sum_{R \in \mathcal{R}} \sum_{i \in N} [v(S_i(R) \cup \{i\}) - v(S_i(R))] \\
&= \frac{1}{|N|!} \sum_{R \in \mathcal{R}} v(N) \\
&= \frac{1}{|N|!} |N|! v(N) \\
&= v(N).
\end{aligned}$$

Uniqueness of Value with Balanced Contributions

Proposition

The unique value that satisfies the balanced contributions property is the Shapley value.

- First, we show there is at most one value satisfying the property.

  Let ψ and ψ' be any two values that satisfy the condition.

  We prove by induction on $n = |N|$ that ψ and ψ' are identical.

  Suppose they are identical for all games with less than n players.

  Let $\langle N, v\rangle$ be a game with n players.

  By the induction hypothesis, For all $i, j \in N$,

$$\psi_i(N - \{j\}, v^{N-\{j\}}) = \psi'_i(N - \{j\}, v^{N-\{j\}}).$$

Uniqueness of Value with Balanced Contributions (Cont'd)

- By the Balanced Contributions Property, for all $i, j \in N$,

$$\begin{array}{l}\psi_i(N,v) - \psi_i'(N,v) \\ = \psi_j(N,v) - \psi_j(N-\{i\}, v^{N-\{i\}}) + \psi_i(N-\{j\}, v^{N-\{j\}}) \\ \quad - \psi_j'(N,v) + \psi_j'(N-\{i\}, v^{N-\{i\}}) - \psi_i'(N-\{j\}, v^{N-\{j\}}) \\ = \psi_j(N,v) - \psi_j'(N,v).\end{array}$$

Now fix i and sum over $j \in N$, using

$$\sum_{j\in N} \psi_j(N,v) = \sum_{j\in N} \psi_j'(N,v) = v(N).$$

We get, for all $i \in N$,

$$\begin{array}{c}\sum_{j\in N}(\psi_i(N,v) - \psi_i'(N,v)) = \sum_{j\in N} \psi_j(N,v) - \sum_{j\in N} \psi_j'(N,v) \\ |N|(\psi_i(N,v) - \psi_i'(N,v)) = v(N) - v(N) \\ \psi_i(N,v) = \psi_i'(N,v).\end{array}$$

The Shapley Value Has Balanced Contributions

- We now verify that the Shapley value φ satisfies the balanced contributions property. Fix a game $\langle N, v\rangle$.
  We show that, for all $i, j \in N$,

$$\varphi_i(N, v) - \varphi_j(N, v) = \varphi_i(N - \{j\}, v^{N-\{j\}}) - \varphi_j(N - \{i\}, v^{N-\{i\}}).$$

  - The left-hand side is

$$\sum_{S \subseteq N - \{i,j\}} \alpha_S[\Delta_i(S) - \Delta_j(S)] + \beta_S[\Delta_i(S \cup \{j\}) - \Delta_j(S \cup \{i\})],$$

    where $\alpha_S = \frac{|S|!(|N|-|S|-1)!}{|N|!}$ and $\beta_S = \frac{(|S|+1)!(|N|-|S|-2)!}{|N|!}$;
  - The right-hand side is

$$\sum_{S \subseteq N - \{i,j\}} \gamma_S[\Delta_i(S) - \Delta_j(S)],$$

    where $\gamma_S = \frac{|S|!(|N|-|S|-2)!}{(|N|-1)!}$.

The Shapley Value Has Balanced Contributions (Cont'd)

- The result follows from:
  - $\Delta_i(S) - \Delta_j(S) = \Delta_i(S \cup \{j\}) - \Delta_j(S \cup \{i\})$;
  - $\alpha_S + \beta_S = \gamma_S$.
- The Balanced Contributions Property links a game only with its subgames.

  So, in the derivation of the Shapley value of a game $\langle N, v \rangle$, we could restrict attention to the subgames of $\langle N, v \rangle$, rather than work with the set of all possible games.

Dummy Players and Interchangeable Players

- We formulate an axiomatic characterization of the Shapley value, restricting attention to the set of games with a given set of players.
- Throughout we fix this set to be N and denote a game simply by its worth function v.
- Player i is a **dummy in** v if, for every coalition S that excludes i,
$$\Delta_i(S) = v(\{i\}).$$
- Players i and j are **interchangeable in** v if, for every coalition S that contains neither i nor j,
$$\Delta_i(S) = \Delta_j(S).$$
- Equivalently, i and j are **interchangeable in** v if, for every coalition S that includes i but not j,
$$v((S - \{i\}) \cup \{j\}) = v(S).$$

Axioms Characterizing of the Shapley Value

- The axioms we impose are the following.

SYM (**Symmetry**) If i and j are interchangeable in v, then

$$\psi_i(v) = \psi_j(v).$$

DUM (**Dummy player**) If i is a dummy in v, then

$$\psi_i(v) = v(\{i\}).$$

ADD (**Additivity**) For any two games v and w, we have, for all $i \in N$,

$$\psi_i(v+w) = \psi_i(v) + \psi_i(w),$$

where $v+w$ is the game defined by

$$(v+w)(S) = v(S) + w(S), \quad \text{for every coalition } S.$$

- The first two axioms impose conditions on single games, while the last axiom links the outcomes of different games.

The Axiomatic Characterization

Proposition

The Shapley value is the only value that satisfies SYM, DUM and ADD.

- We first verify that the Shapley value satisfies the axioms.

SYM: Assume that i and j are interchangeable.
For every ordering $R \in \mathcal{R}$, let $R' \in \mathcal{R}$ differ from R only in that the positions of i and j are interchanged.

- If i precedes j in R, then we have $\Delta_i(S_i(R)) = \Delta_j(S_j(R'))$.
- If j precedes i, then $\Delta_i(S_i(R)) - \Delta_j(S_j(R')) = v(S \cup \{i\}) - v(S \cup \{j\})$, where $S = S_i(R) - \{j\}$. Since i and j are interchangeable, we have $v(S \cup \{i\}) = v(S \cup \{j\})$. So $\Delta_i(S_i(R)) = \Delta_j(S_j(R'))$.

It follows that φ satisfies SYM.

DUM: It is immediate that φ satisfies this condition.

ADD: This follows from the fact that, if $u = v + w$, then

$$u(S \cup \{i\}) - u(S) = v(S \cup \{i\}) - v(S) + w(S \cup \{i\}) - w(S).$$

The Axiomatic Characterization (Cont'd)

- We show the Shapley value is the only value satisfying the axioms.

  Let ψ be a value that satisfies the axioms.

  For any coalition S, define the game v_S by

$$v_S(T) = \begin{cases} 1, & \text{if } T \supseteq S \\ 0, & \text{otherwise} \end{cases}.$$

  Regard a game v as a collection of $2^{|N|} - 1$ numbers $(v(S))_{S \in \mathcal{C}}$.

  Claim: $(v_T)_{T \in \mathcal{C}}$ is an algebraic basis for the space of games.

  That is, for any v, there is a unique collection $(\alpha_T)_{T \in \mathcal{C}}$ of real numbers such that $v = \sum_{T \in \mathcal{C}} \alpha_T v_T$.

  The collection $(v_T)_{T \in \mathcal{C}}$ contains $2^{|N|} - 1$ members.

  It suffices to show that these games are linearly independent.

The Axiomatic Characterization (Cont'd)

- Suppose $\sum_{S \in \mathcal{C}} \beta_S v_S = 0$.
  We must show $\beta_S = 0$, for all S.
  Suppose there exists some T with $\beta_T \neq 0$.
  Then we can choose such a T, for which $\beta_S = 0$, for all $S \subset T$.
  Then $\sum_{S \in \mathcal{C}} \beta_S v_S(T) = \beta_T \neq 0$, a contradiction.
  Suppose a game has the form αv_T, for $\alpha \geq 0$.
  By SYM and DUM, its value is given uniquely by

$$\psi_i(\alpha v_T) = \begin{cases} \frac{\alpha}{|T|}, & \text{if } i \in T \\ 0, & \text{otherwise} \end{cases}.$$

  Next, suppose $v = \sum_{T \in \mathcal{C}} \alpha_T v_T$.
  Then

$$v = \sum_{\{T \in \mathcal{C} : \alpha_T > 0\}} \alpha_T v_T - \sum_{\{T \in \mathcal{C} : \alpha_T < 0\}} (-\alpha_T v_T).$$

  So, by ADD, the value of v is determined uniquely.

Example: Weighted Majority Games

- Consider the weighted majority game v, with:
  - Weights $w = (1, 1, 1, 2)$;
  - Quota $q = 3$.
- Regarding Player 4, we observe the following facts.
  - In all orderings in which Player 4 is first or last his marginal contribution is 0.
  - In all other orderings his marginal contribution is 1.

  Thus, $\varphi(v) = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2})$.
- Note that the unique expression in terms of the basis is

$$v = v_{\{1,4\}} + v_{\{2,4\}} + v_{\{3,4\}} + v_{\{1,2,3\}} - v_{\{1,2,4\}} - v_{\{1,3,4\}} - v_{\{2,3,4\}}.$$

  From this, we can alternatively deduce

$$\varphi_4(v) = 3 \cdot \frac{1}{2} + 0 - 3 \cdot \frac{1}{3} = \frac{1}{2}.$$

Example: A Market

- Consider the game $\langle\{1,2,3\},v\rangle$ in which

$$v(S)=\begin{cases} 1, & \text{if } S=\{1,2,3\},\{1,2\} \text{ or } \{1,3\} \\ 0, & \text{otherwise} \end{cases}.$$

- This game can be viewed as a model of a market in which there are:
  - A seller (Player 1) holding one unit of a good that she does not value;
  - Two potential buyers (Players 2 and 3) who each value the good as worth one unit of payoff.
- There are six possible orderings of the players.
  - In the four, in which Player 1 is second or third, her marginal contribution is 1 and the marginal contributions of the other two players are 0.
  - In the ordering $(1,2,3)$, Player 2's marginal contribution is 1.
  - In $(1,3,2)$, Player 3's marginal contribution is 1.

  Thus, the Shapley value of the game is $(\frac{2}{3},\frac{1}{6},\frac{1}{6})$.
- The core of the game consists of the single payoff profile $(1,0,0)$.

Example: A Majority Game

- Consider a parliament in which there are:
  - One party with $m-1$ seats;
  - m parties each with one seat.

  A majority is decisive.
- This is a generalization of the game "My Aunt and I".
- This situation can be modeled as a weighted majority game in which:
  - $N=\{1,\ldots,m+1\}$;
  - $w_1=m-1$ and $w_i=1$, for $i\neq 1$;
  - $q=m$.
- The marginal contribution of the large party is 1 in all but the $2m!$ orderings in which it is first or last.

  Hence the Shapley value of the game assigns to the large party the payoff

$$\frac{(m+1)!-2m!}{(m+1)!}=\frac{(m+1-2)m!}{(m+1)m!}=\frac{m-1}{m+1}.$$