Introduction to Game Theory

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Subsection 1

Strategic Games

Introducing Strategic Games

- A strategic game models interactive decision making in which:
  - Each decision maker chooses his plan of action once and for all;
  - These choices are made simultaneously.
- The model consists of:
  - A finite set N of players;
  - For each player i, a set A_i of actions available to player i;
  - A preference relation $\succsim_i$ of player i on the set of action profiles.
- An **outcome** is an action profile $a = (a_j)_{j \in N}$.
- The set of outcomes is $A = \bigtimes_{j \in N} A_j$.
- The requirement that the preferences of each player i be defined over A, rather than A_i, means that each player may care not only about his own action but also about the actions taken by the other players.

Definition of Strategic Games

Definition (Strategic Game)

A **strategic game** consists of:

- A finite set N of **players**;
- For each player $i \in N$, a nonempty set A_i of **actions** available to player i;
- For each player $i \in N$, a **preference relation** $\succsim_i$ on $A = \bigtimes_{j \in N} A_j$.

- If the set A_i of actions of every player i is finite, then the game is **finite**.

Generality: Advantages and Drawbacks

- The high level of abstraction of this model allows it to be applied to a wide variety of situations.
- Accordingly, a player may be:
  - An individual;
  - A collective, such as a government, board of directors etc.
- The model places no restrictions on the set of actions available to a player:
  - They may contain just a few elements;
  - They may be a huge set containing complicated plans that cover a variety of contingencies.
- Associating a preference relation $\succsim_i$ with each player limits the range of application of the model.
- The fact that the model is so abstract has the:
  - Advantage that it allows applications in a wide range of situations;
  - Drawback that the implications of the model cannot depend on any specific features of a situation.

Modeling Consequences of Actions

- In some situations the players' preferences are most naturally defined not over action profiles but over their consequences.
- When modeling an oligopoly, for example, we may take:
  - The set of players to be a set of firms;
  - The set of actions of each firm to be the set of prices.

  We work under the assumption that each firm cares only about its profit and not about the profile of prices that generates that profit.
- To accommodate such a scenario, we introduce preferences over consequences of actions.

Preferences Over Consequences of Actions

- Preferences over consequences of actions are modeled by introducing:
  - A set C of **consequences**;
  - A function
  $$g : A \to C$$
  that associates consequences with action profiles;
  - A profile $(\succsim_i^*)$ of **preference relations over** C.
- Then the preference relation $\succsim_i$ of each player i in the strategic game is defined by
$$a \succsim_i b \quad \text{if and only if} \quad g(a) \succsim_i^* g(b).$$

Consequences of Actions with Randomness

- Suppose the consequence of an action profile is affected by an exogenous random variable whose realization is not known to the players before they take their actions.
- To model this situation as a strategic game, we introduce:
  - A set C of consequences;
  - A probability space Ω;
  - A function $g : A \times \Omega \to C$ with the interpretation that $g(a, \omega)$ is the consequence when the action profile is $a \in A$ and the realization of the random variable is $\omega \in \Omega$.
- A profile a of actions induces a lottery $g(a, \bullet)$ on C.
- For each player i a preference relation $\succsim_i^*$ must be specified over the set of all such lotteries.
- Player i's preference relation in the strategic game is defined as follows:

$$a \succsim_i b \quad \text{if and only if} \quad g(a, \bullet) \succsim_i^* g(b, \bullet).$$

Payoff Functions

- Often the preference relation $\succsim_i$ of player i in a strategic game can be represented by a **payoff function** $u_i : A \to \mathbb{R}$ (also called a **utility function**), in the sense that

$$u_i(a) \geq u_i(b) \quad \text{whenever} \quad a \succsim_i b.$$

- We refer to values of such a function as **payoffs** (or **utilities**).
- If a player's preference relation is given by a payoff function, we denote the game by $\langle N, (A_i), (u_i) \rangle$ rather than $\langle N, (A_i), (\succsim_i) \rangle$.

Two Player Games in Tabular Form

- Example: A finite strategic 2-player game can be given by a table:

	L	R
T	w_1, w_2	x_1, x_2
B	y_1, y_2	z_1, z_2

  - One player's actions are identified with the rows and the other player's with the columns.
    - The set of actions of the row player is $\{T, B\}$;
    - The set of actions of the column player is $\{L, R\}$.
  - The two numbers in row r and column c are the players' payoffs when the row player chooses r and the column player chooses c.
    - The row player's payoff from the outcome (T, L) is w_1;
    - The column player's payoff from the outcome (T, L) is w_2.
- If the players' names are "1" and "2", then the convention is that the row player is player 1 and the column player is player 2.

Subsection 2

Nash Equilibrium

Introducing Nash Equilibrium

- The most commonly used solution concept in game theory is that of Nash equilibrium.
- It captures a steady state of the play of a strategic game in which each player:
  - Holds the correct expectation about the other players' behavior;
  - Acts rationally.
- The concept does not examine how a steady state is reached.

Nash Equilibrium of a Strategic Game

Definition (Nash Equilibrium)

A **Nash equilibrium** of a strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is a profile $a^* \in A$ of actions with the property that, for every player $i \in N$, we have

$$(a^*_{-i}, a^*_i) \succsim_i (a^*_{-i}, a_i), \text{ for all } a_i \in A_i.$$

- For a^* to be a Nash equilibrium, no player i has an action yielding an outcome that he prefers to that generated when he chooses a^*_i, given that every other player j chooses his equilibrium action a^*_j.
- No player can profitably deviate, given the actions of the other players.

Nash Equilibrium via Best Response Functions

- For a given $a_{-i} \in A_{-i}$, define $B_i(a_{-i})$ to be the set of player i's best actions given a_{-i}:

$$B_i(a_{-i}) = \{a_i \in A_i : (a_{-i}, a_i) \succsim_i (a_{-i}, a_i'), \text{ for all } a_i' \in A_i\}.$$

- The function B_i is called the **best-response function** of player i.
- A Nash equilibrium is a profile a^* of actions for which

$$a_i^* \in B_i(a_{-i}^*), \quad \text{for all } i \in N.$$

Finding Nash Equilibria

- We may find Nash equilibria as follows:
  - First, calculate the best response function of each player;
  - Then find a profile a^* of actions for which

$$a_i^* \in B_i(a_{-i}^*), \quad \text{for all } i \in N.$$

- If the functions B_i are singleton-valued, the second step entails solving $|N|$ equations in the $|N|$ unknowns $(a_i^*)_{i \in N}$.

Subsection 3

Examples

Bach or Stravinsky? (BoS)

- Two people wish to go out together to a concert of music by either Bach or Stravinsky.
    - They prefer to go out together;
    - However, one person prefers Bach and the other prefers Stravinsky.
- Representing the individuals' preferences by payoff functions, we have:

	Bach	Stravinsky
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

- This game is often referred to as the "Battle of the Sexes".
- BoS models a situation in which players wish to coordinate their behavior, but have conflicting interests.
- The game has two Nash equilibria: (B, B) and (S, S).
- There are two steady states: One in which both players always choose Bach and one in which they always choose Stravinsky.

A Coordination Game

- As in BoS, two people wish to go out together, but in this case they agree on the more desirable concert.

	Mozart	Mahler
Mozart	$2, 2$	$0, 0$
Mahler	$0, 0$	$1, 1$

- The game has two Nash equilibria: $(\text{Mozart}, \text{Mozart})$ and $(\text{Mahler}, \text{Mahler})$.
- In contrast to BoS, the players have a mutual interest in reaching one of these equilibria, namely $(\text{Mozart}, \text{Mozart})$.

  However, the notion of Nash equilibrium does not rule out a steady state in which the outcome is the inferior equilibrium (Mahler, Mahler).

The Prisoner's Dilemma

- Two suspects in a crime are put into separate cells.
  - If they both confess, each will be sentenced to three years in prison.
  - If only one of them confesses, he will be freed and used as a witness against the other, who will receive a sentence of four years.
  - If neither confesses, they will both be convicted of a minor offense and spend one year in prison.
- Choosing a convenient payoff representation

	Do not confess	Confess
Do not confess	$3,3$	$0,4$
Confess	$4,0$	$1,1$

- In this game, there are gains from cooperation, since the best outcome for the players is that neither confesses, but each player has an incentive to be freed.
- The game has a unique Nash equilibrium (Confess, Confess).

Hawk-Dove

- Two animals are fighting over some prey. Each can behave like a dove or like a hawk.
  - The best outcome for each animal is that in which it acts like a hawk while the other acts like a dove.
  - The worst outcome is that in which both animals act like hawks.
  - Each animal prefers to be hawkish if its opponent is dovish and dovish if its opponent is hawkish.

	Dove	Hawk
Dove	$3, 3$	$1, 4$
Hawk	$4, 1$	$0, 0$

- The game has two Nash equilibria, $(\text{Dove}, \text{Hawk})$ and $(\text{Hawk}, \text{Dove})$.
- They correspond to two different conventions about the player who yields.

Matching Pennies

- Each of two people chooses either Heads or Tails.
  - If the choices differ, person 1 pays person 2 a dollar.
  - If the choices are the same, person 2 pays person 1 a dollar.

  Each person cares only about the amount of money that he receives.

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

- Such a game, in which the interests of the players are diametrically opposed, is called **strictly competitive**.
- The game Matching Pennies has no Nash equilibrium.

Subsection 4

Existence of a Nash Equilibrium

Nash Equilibria as Fixed Points

- Not every strategic game has a Nash equilibrium.
- The conditions under which the set of Nash equilibria of a game is nonempty have been investigated extensively.
- To show that a game has a Nash equilibrium it suffices to show that there is a profile a^* of actions such that

$$a_i^* \in B_i(a_{-i}^*), \text{ for all } i \in N.$$

- Define the set-valued function $B : A \to A$ by

$$B(a) = \bigtimes_{i \in N} B_i(a_{-i}).$$

- In vector form, a Nash equilibrium is a fixed point of B:

$$a^* \in B(a^*).$$

- Fixed point theorems give conditions on B to possess fixed points.

Kakutani's Fixed Point and Quasi-Concave Preferences

Theorem (Kakutani's Fixed Point Theorem)

Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a set-valued function for which

- For all $x \in X$, the set $f(x)$ is nonempty and convex;
- The graph of f is closed, i.e., for all sequences $\{x_n\}$ and $\{y_n\}$, such that $y_n \in f(x_n)$, for all n, $x_n \to x$, and $y_n \to y$, we have $y \in f(x)$.

Then there exists $x^* \in X$, such that $x^* \in f(x^*)$.

- Define a preference relation $\succsim_i$ over A to be **quasi-concave on** A_i if, for every $a^* \in A$, the set $\{a_i \in A_i : (a^*_{-i}, a_i) \succsim_i a^*\}$ is convex.

Existence of Nash Equilibria

Proposition (Existence of Nash Equilibria)

The strategic game $\langle N, (A_i), (\succsim_i) \rangle$ has a Nash equilibrium if, for all $i \in N$:

- The set A_i of actions of player i is a nonempty compact convex subset of a Euclidian space;
- The preference relation $\succsim_i$ is:
  - Continuous;
  - Quasi-concave on A_i.

- Define $B : A \to A$ by

$$B(a) = \bigtimes_{i \in N} B_i(a_{-i}),$$

where B_i is the best response function of player i.

It suffices to show that B has a fixed point. This is done by showing that B satisfies the hypotheses of Kakutani's Fixed Point Theorem.

Existence of Nash Equilibria (Cont'd)

- For every $i \in N$, $\succsim_i$ is continuous and A_i is compact.

  Thus, the set $B_i(a_{-i})$ is nonempty.

  For every $i \in N$, $\succsim_i$ is quasi-concave on A_i.

  Therefore, $B_i(a_{-i})$ is convex.

  Finally, B has a closed graph since each $\succsim_i$ is continuous.

  So the hypotheses of Kakutani's Theorem are satisfied for B.

  Thus, B has a fixed point.

  Any such fixed point is a Nash equilibrium.

Subsection 5

Strictly Competitive Games

Strictly Competitive Strategic Games

- Suppose the names of the players are "1" and "2", i.e., $N = \{1, 2\}$.

Definition (Strictly Competitive Strategic Games)

A strategic game $\langle \{1, 2\}, (A_i), (\succsim_i) \rangle$ is **strictly competitive** if, for any $a \in A$ and $b \in A$, we have

$$a \succsim_1 b \quad \text{if and only if} \quad b \succsim_2 a.$$

- A strictly competitive game is sometimes called **zero-sum** because, if Player 1's preference relation $\succsim_1$ is represented by the payoff function u_1, then Player 2's preference relation is represented by u_2 with $u_1 + u_2 = 0$.

Maxminimization in Strictly Competitive Games

- We say that Player i **maxminimizes** if he chooses an action that is best for him on the assumption that, whatever he does, Player j will choose her action to hurt him as much as possible.
- We show that for a strictly competitive game that possesses a Nash equilibrium, a pair of actions is a Nash equilibrium if and only if the action of each player is a maxminimizer.
- This result provides a link between individual decision-making and the reasoning behind the notion of Nash equilibrium.
- We also prove that, for strictly competitive games that possess Nash equilibria, all equilibria yield the same payoffs.
- This property is rarely satisfied in non strictly competitive games.

Maxminimizers

Definition (Maxminimizers)

Let $\langle \{1,2\}, (A_i), (u_i) \rangle$ be a strictly competitive strategic game.
The action $x^* \in A_1$ is a **maxminimizer for Player 1** if

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y), \text{ for all } x \in A_1.$$

Similarly, the action $y^* \in A_2$ is a **maxminimizer for Player 2** if

$$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y), \text{ for all } y \in A_2.$$

- In words, a maxminimizer for Player i is an action that maximizes the payoff that Player i can guarantee.

Maxminimizers (Cont'd)

- A maxminimizer for Player 1 solves the problem

$$\max_x \min_y u_1(x, y).$$

- A maxminimizer for Player 2 solves the problem

$$\max_y \min_x u_2(x, y).$$

- We assume, next, without loss of generality, that

$$u_2 = -u_1.$$

Maxminimization and Minmaximization

- Maxminimization of Player 2's payoff is equivalent to the minmaximization of Player 1's payoff.

Lemma

Let $\langle \{1,2\}, (A_i), (u_i) \rangle$ be a strictly competitive strategic game. Then

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -\min_{y \in A_2} \max_{x \in A_1} u_1(x,y).$$

Further,

$y \in A_2$ solves the problem $\max_{y \in A_2} \min_{x \in A_1} u_2(x,y)$

if and only if it solves the problem $\min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$.

Maxminimization and Minmaximization (Cont'd)

- For any function f we have

$$\begin{aligned} \min_z(-f(z)) &= -\max_z f(z); \\ \operatorname{argmin}_z(-f(z)) &= \operatorname{argmax}_z f(z). \end{aligned}$$

It follows that, for every $y \in A_2$, we have

$$\begin{aligned} -\min_{x \in A_1} u_2(x,y) &= \max_{x \in A_1}(-u_2(x,y)) \\ &= \max_{x \in A_1} u_1(x,y). \end{aligned}$$

Hence

$$\begin{aligned} \max_{y \in A_2}\min_{x \in A_1} u_2(x,y) &= -\min_{y \in A_2}[-\min_{x \in A_1} u_2(x,y)] \\ &= -\min_{y \in A_2}\max_{x \in A_1} u_1(x,y). \end{aligned}$$

In addition, $y \in A_2$ is a solution of $\max_{y \in A_2}\min_{x \in A_1} u_2(x,y)$ if and only if it is a solution of the problem $\min_{y \in A_2}\max_{x \in A_1} u_1(x,y)$.

Nash Equilibria and Maxminimizers

- We connect Nash equilibria of a strictly competitive game with the set of pairs of maxminimizers.

Proposition (Nash Equilibria and Maxminimizers)

Let $G = \langle \{1,2\}, (A_i), (u_i) \rangle$ be a strictly competitive strategic game.

a. If (x^*, y^*) is a Nash equilibrium of G, then x^* is a maxminimizer for Player 1 and y^* is a maxminimizer for Player 2.

b. If (x^*, y^*) is a Nash equilibrium of G, then

$$\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y) = u_1(x^*, y^*),$$

and, thus, all Nash equilibria of G yield the same payoffs.

c. If $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$ (and, thus, in particular, if G has a Nash equilibrium), x^* is a maxminimizer for Player 1, and y^* is a maxminimizer for Player 2, then (x^*, y^*) is a Nash equilibrium of G.

Proof of the Proposition (Parts (a) and (b))

- Let (x^*, y^*) be a Nash equilibrium of G.

  Then $u_2(x^*, y^*) \geq u_2(x^*, y)$, for all $y \in A_2$.

  Since $u_2 = -u_1$, $u_1(x^*, y^*) \leq u_1(x^*, y)$, for all $y \in A_2$.

  So
  $$u_1(x^*, y^*) = \min_y u_1(x^*, y) \leq \max_x \min_y u_1(x, y).$$

  Similarly, $u_1(x^*, y^*) \geq u_1(x, y^*)$, for all $x \in A_1$.

  Hence, $u_1(x^*, y^*) \geq \min_y u_1(x, y)$, for all $x \in A_1$.

  So we have
  $$u_1(x^*, y^*) \geq \max_x \min_y u_1(x, y).$$

  Thus, $u_1(x^*, y^*) = \max_x \min_y u_1(x, y)$.

  This shows that x^* is a maxminimizer for Player 1.

Proof of the Proposition (Parts (a) and (b) Cont'd)

- A similar argument shows y^* is a maxminimizer for Player 2. Moreover,

$$u_2(x^*, y^*) = \max_y \min_x u_2(x, y).$$

So we get

$$\begin{aligned} u_1(x^*, y^*) &= -u_2(x^*, y^*) \\ &= -\max_y \min_x u_2(x, y) \\ &= \min_y \max_x (-u_2(x, y)) \\ &= \min_y \max_x u_1(x, y). \end{aligned}$$

Proof of the Proposition (Part (c))

- Let $v^* = \max_x \min_y u_1(x, y) = \min_y \max_x u_1(x, y)$.
  By the previous lemma, we have $\max_y \min_x u_2(x, y) = -v^*$.
  By hypothesis, x^* is a maxminimizer for Player 1.
  So we have $u_1(x^*, y) \geq v^*$, for all $y \in A_2$.
  Also by hypothesis, y^* is a maxminimizer for Player 2.
  So we have $u_2(x, y^*) \geq -v^*$, for all $x \in A_1$.
  Letting $y = y^*$ and $x = x^*$ in these two inequalities, we obtain

$$u_1(x^*, y^*) = v^*.$$

  Using the fact that $u_2 = -u_1$, we have for all $x \in A_1$ and $y \in A_2$,
  - $u_1(x^*, y^*) = v^* \geq -u_2(x, y^*) = u_1(x, y^*)$;
  - $u_2(x^*, y^*) = -v^* \geq -u_1(x^*, y) = u_2(x^*, y)$.

  We now conclude that (x^*, y^*) is a Nash equilibrium of G.

Consequences of the Proposition

- By Part (c), we may find the players' Nash equilibrium strategies by solving the problems:
  - $\max_x \min_y u_1(x, y)$;
  - $\max_y \min_x u_2(x, y)$.
- By Parts (a) and (c), the Nash equilibria of a strictly competitive game are interchangeable:

  If (x, y) and (x', y') are equilibria then so are (x, y') and (x', y).
- Part (b) shows that

$$\max_x \min_y u_1(x, y) = \min_y \max_x u_1(x, y),$$

for any strictly competitive game that has a Nash equilibrium.

Consequences of the Proposition (Cont'd)

- The inequality

$$\max_x \min_y u_1(x, y) \leq \min_y \max_x u_1(x, y)$$

  holds more generally.

  For any x', we have

$$u_1(x', y) \leq \max_x u_1(x, y), \quad \text{for all } y.$$

  So $\min_y u_1(x', y) \leq \min_y \max_x u_1(x, y)$.
- Thus, in any game (whether or not it is strictly competitive) the payoff that Player 1 can guarantee herself is at most the amount that Player 2 can hold her down to.
- However, the hypothesis that the game has a Nash equilibrium is essential in establishing the opposite inequality (see next slide).

The General Case

- We always have

$$\max_x \min_y u_1(x, y) \leq \min_y \max_x u_1(x, y).$$

- However,

$$\max_x \min_y u_1(x, y) = \min_y \max_x u_1(x, y)$$

requires the existence of a Nash equilibrium.

- To see this, consider Matching Pennies:

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Note that

$$\max_x \min_y u_1(x, y) = -1 < 1 = \min_y \max_x u_1(x, y).$$

Value of a Game

- If
$$\max_x \min_y u_1(x, y) = \min_y \max_x u_1(x, y),$$
then we say that this payoff, the equilibrium payoff of Player 1, is the **value** of the game.
- Suppose v^* is the value of a strictly competitive game.
Then:
    - Any equilibrium strategy of Player 1 guarantees that her payoff is at least her equilibrium payoff v^*;
    - Any equilibrium strategy of Player 2 guarantees that his payoff is at least his equilibrium payoff $-v^*$.

  So any such strategy of Player 2 guarantees that Player 1's payoff is at most her equilibrium payoff.

Value in Non Strictly Competitive Games

- In a game that is not strictly competitive a player's equilibrium strategy does not in general have the aforementioned properties.
- Consider BoS:

	Bach	Stravinsky
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

We have

$$\max_x \min_y u_1(x, y) = 0;$$

On the other hand,

$$\min_y \max_x u_1(x, y) = 1.$$