Introduction to Game Theory

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LSSU Math 500

1. Mixed, Correlated and Evolutionary Equilibrium
   - Mixed Strategy Nash Equilibrium
   - Correlated Equilibrium
   - Evolutionary Equilibrium

Subsection 1

Mixed Strategy Nash Equilibrium

Strategic Games

- Mixed strategy Nash equilibrium models a steady state of a game in which the players' choices are regulated by probabilistic rules.
- Recall that a strategic game is a triple $\langle N, (A_i), (\succsim_i) \rangle$, where the preference relation $\succsim_i$ of each player i is defined over the set $A = \times_{i \in N} A_i$ of action profiles.
- Now, we allow the players' choices to be nondeterministic.
- So we add to the primitives of the model a specification of each player's preference relation over lotteries on A.
- We assume that the preference relation of each player i can be represented by the expected value of some function $u_i : A \to \mathbb{R}$.
- We adopt a triple $\langle N, (A_i), (u_i) \rangle$, where, for each $i \in N$, $u_i : A \to \mathbb{R}$ is a function whose expected value represents player i's preferences over the set of lotteries on A.
- We refer to the model simply as a **strategic game**.

Pure and Mixed Strategies

- Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game.
- We denote by $\Delta(A_i)$ the set of probability distributions over A_i.
- We refer to a member of $\Delta(A_i)$ as a **mixed strategy** of player i.
- We assume that the players' mixed strategies are independent randomizations.
- We refer to a member of A_i as a **pure strategy**.
- For any finite set X and $\delta \in \Delta(X)$, we denote by $\delta(x)$ the probability that δ assigns to $x \in X$.
- The **support** of δ is the set of elements $x \in X$ for which $\delta(x) > 0$.

Induced Distributions on Action Profiles

- A profile $(\alpha_j)_{j\in N}$ of mixed strategies induces a probability distribution over the set A.
- If, for example, each A_j is finite, then, given independence:
  - The probability of the action profile $a=(a_j)_{j\in N}$ is
  $$\prod_{j\in N}\alpha_j(a_j);$$
  - Player i's evaluation of $(\alpha_j)_{j\in N}$ is
  $$\sum_{a\in A}\left(\prod_{j\in N}\alpha_j(a_j)\right)u_i(a).$$

Mixed Extensions

- We derive from a strategic game G another strategic game, called the mixed extension of G, in which the set of actions of each player i is the set $\Delta(A_i)$ of his mixed strategies in G.

Definition (Mixed Extension)

The **mixed extension** of the strategic game $\langle N, (A_i), (u_i)\rangle$ is the strategic game $\langle N, (\Delta(A_i)), (U_i)\rangle$ in which

- $\Delta(A_i)$ is the set of probability distributions over A_i;
- $U_i : \times_{j\in N} \Delta(A_j) \to \mathbb{R}$ assigns to each $\alpha \in \times_{j\in N} \Delta(A_j)$ the expected value under u_i of the lottery over A that is induced by α.

  I.e., if A is finite,

$$U_i(\alpha) = \sum_{a\in A} \left(\prod_{j\in N} \alpha_j(a_j) \right) u_i(a).$$

Remarks on Mixed Extensions

- Each function U_i is multilinear. I.e., for any mixed strategy profile α, any mixed strategies β_j and γ_j of Player j, and any number $\lambda \in [0,1]$, we have

$$U_i(\alpha_{-j}, \lambda\beta_j + (1-\lambda)\gamma_j) = \lambda U_i(\alpha_{-j}, \beta_j) + (1-\lambda)U_i(\alpha_{-j}, \gamma_j).$$

- Let $e(a_i)$ be the degenerate mixed strategy of Player i that attaches probability one to $a_i \in A_i$.
- When each A_i is finite, for any mixed strategy profile α,

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) U_i(\alpha_{-i}, e(a_i)).$$

Mixed Strategy Nash Equilibrium

Definition (Mixed Strategy Nash Equilibrium)

A **mixed strategy Nash equilibrium** of a strategic game is a Nash equilibrium of its mixed extension.

- We show that the set of Nash equilibria of a strategic game is a subset of its set of mixed strategy Nash equilibria.

Proposition (Nash Equilibria are Mixed Strategy Nash Equilibria)

Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game. Then $a^* = (a_i^*)$ is a Nash equilibrium of G if and only if it is a degenerate mixed strategy Nash equilibrium of G.

Mixed Strategy Nash Equilibrium (Proof)

- Suppose that $\alpha^* \in \times_{j\in N} \Delta(A_j)$ is a mixed strategy Nash equilibrium of $G = \langle N, (A_i), (u_i)\rangle$ in which each player i's mixed strategy α_i^* is degenerate in the sense that it assigns probability one to a single member - say a_i^* - of A_i. But A_i can be identified with a subset of $\Delta(A_i)$. So the action profile a^* is a Nash equilibrium of G.
- Conversely, suppose that a^* is a Nash equilibrium of G.

  By the linearity of U_i in α_i, no probability distribution over actions in A_i yields Player i a payoff higher than that generated by $e(a_i^*)$.

  Thus, the profile $(e(a_i^*))$ is a mixed strategy Nash equilibrium of G.

Mixed Strategy Nash Equilibria of Finite Games

- We saw that there are games for which there are no Nash equilibria.
- The same applies to mixed strategy Nash equilibria.
- However, every game in which each player has finitely many actions has at least one mixed strategy Nash equilibrium.

Proposition

Every finite strategic game has a mixed strategy Nash equilibrium.

- Recall, by a preceding proposition, that it suffices to show that:
  - $A_i \neq \emptyset$ is a compact and convex subset of a Euclidean space;
  - $\succsim_i$ is continuous and quasi-concave.

Mixed Strategy Nash Equilibria of Finite Games (Cont'd)

- Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game, and $m_i = |A_i|$.

  We can identify the set $\Delta(A_i)$ of player i's mixed strategies with the set of vectors $(p_1, \ldots, p_{m_i})$ for which:
  - $p_k \geq 0$, for all k;
  - $\sum_{k=1}^{m_i} p_k = 1$.

  This set is nonempty, convex and compact.

  The expected payoff is linear in the probabilities.

  So each player's payoff function in the mixed extension of G is both quasi-concave in his own strategy and continuous.

  Thus, there exists a mixed strategy Nash equilibrium.
- It can also be shown that a game in which:
  - Each action set is a convex compact subset of a Euclidian space;
  - Each payoff function is continuous,

  has a mixed strategy Nash equilibrium.

Characterization of Mixed Strategy Nash Equilibria

Lemma

Let $G = \langle N, (A_i), (u_i)\rangle$ be a finite strategic game. Then $\alpha^* \in \bigtimes_{i\in N} \Delta(A_i)$ is a mixed strategy Nash equilibrium of G if and only if, for every player $i \in N$, every pure strategy in the support of α_i^* is a best response to α_{-i}^*.

- Suppose that there is an action a_i in the support of α_i^* that is not a best response to α_{-i}^*.

  Then, by linearity of U_i in α_i, player i can increase his payoff by transferring probability from a_i to an action that is a best response.

  Hence α_i^* is not a best response to α_{-i}^*.

Characterization of Mixed Strategy Equilibria (Cont'd)

- Conversely, suppose that there is a mixed strategy α'_i that gives a higher expected payoff than does α^*_i in response to α^*_{-i}.

  Then, again by the linearity of U_i, at least one action in the support of α'_i must give a higher payoff than some action in the support of α^*_i.

  So, not all actions in the support of α^*_i are best responses to α^*_{-i}.
- It follows that every action in the support of any player's equilibrium mixed strategy yields that player the same payoff.

Analog for Infinite Action Sets

- Suppose that the set of actions of some player is not finite.
- Then α^* is a mixed strategy Nash equilibrium of G if and only if:
  - (i) For every player i, no action in A_i yields, given α^*_{-i}, a payoff to Player i that exceeds his equilibrium payoff;
  - (ii) The set of actions that yield, given α^*_{-i}, a payoff less than his equilibrium payoff has α^*_i-measure zero.
- The assumption that the players' preferences can be represented by expected payoff functions plays a key role in these characterizations of mixed strategy equilibrium.

Example

- Consider the game BoS

	Bach	Stravinsky
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

- The payoffs of Player i were interpreted as representing Player i's preferences over the set of (pure) outcomes.
- We now re-interpret the payoffs as von Neumann-Morgenstern utilities.
- As we noted previously this game has two (pure) Nash equilibria, (B, B) and (S, S), where $\mathrm{B} = \mathrm{Bach}$ and $\mathrm{S} = \mathrm{Stravinsky}$.

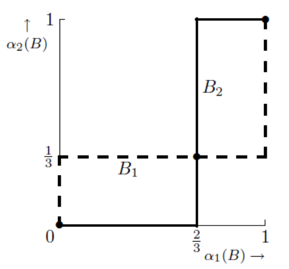
Example (Cont'd)

- Let (α_1, α_2) be a mixed strategy Nash equilibrium.
  - Suppose $\alpha_1(\text{B})$ is zero or one.
    Then we obtain the two pure Nash equilibria.
  - Suppose $0 < \alpha_1(\text{B}) < 1$.
    Given α_2, Player 1's actions B and S must yield the same payoff.
    So, we must have $2\alpha_2(\text{B}) = \alpha_2(\text{S})$.
    Thus, $\alpha_2(\text{B}) = \frac{1}{3}$.
    Since $0 < \alpha_2(B) < 1$, it follows from the same result that Player 2's actions B and S must yield the same payoff.
    So $\alpha_1(B) = 2\alpha_1(S)$.
    Thus, $\alpha_1(\text{B}) = \frac{2}{3}$.

  So the only non-degenerate mixed strategy Nash equilibrium of the game is $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$.

Example (Cont'd)

	Bach	Stravinsky
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$



- We construct the players' best response functions in the mixed extension of this game.
  - If $0 \leq \alpha_2(B) < \frac{1}{3}$, then Player 1's unique best response α_1 has $\alpha_1(B) = 0$;
  - If $\frac{1}{3} < \alpha_2(B) \leq 1$, then her unique best response has $\alpha_1(B) = 1$;
  - If $\alpha_2 = \frac{1}{3}$, then all of her mixed strategies are best responses.

  Making a similar computation for Player 2 we obtain the functions shown in the diagram.

Mixed Extensions of Strictly Competitive Strategic Games

- We defined and studied the class of strictly competitive games.
- In any strictly competitive strategic game that has a Nash equilibrium, the set of equilibria coincides with the set of pairs of maxminimizers.
- This fact can be used to find the set of mixed strategy Nash equilibria of games whose mixed extensions are strictly competitive.
- The fact that a game is strictly competitive does not imply that its mixed extension is strictly competitive:

  Example: Consider a game in which there are three possible outcomes a^1, a^2 and a^3.
  - We may have $a^1 \succ_1 a^2 \succ_1 a^3$ and $a^3 \succ_2 a^2 \succ_2 a^1$.
    So the game is strictly competitive.
  - But both players may prefer a^2 to the lottery in which a^1 and a^3 occur with equal probabilities.
    Then, the mixed extension is not strictly competitive.

Subsection 2

Correlated Equilibrium

Correlated Information

- An interpretation of a mixed strategy Nash equilibrium is as a steady state in which each player's action depends on a signal that he receives from "nature".
- Signals are seeing as private and independent.
- Suppose, next, that the signals are not private and independent.

Example

- Consider again BoS.

	Bach	Stravinsky
Bach	2, 1	0, 0
Stravinsky	0, 0	1, 2

  Assume both players observe a random variable that takes each of the two values x and y with probability $\frac{1}{2}$.

  Then there is a new equilibrium, in which both players choose Bach if the realization is x and Stravinsky if the realization is y.

  Given each player's information, his action is optimal:
  - If the realization is x, then he knows that the other player chooses Bach, so that it is optimal for him to choose Bach.
  - Symmetrically, if the realization is y.
- In this example the players observe the same random variable.
- In general, their information may be less than perfectly correlated.

Imperfectly Correlated Information

- Suppose, that there is a random variable that takes the three values x, y and z, and:
  - Player 1 knows only that the realization is either x or in $\{y, z\}$;
  - Player 2 knows only that it is either a member of $\{x, y\}$ or that it is z.
- Under these assumptions:
  - A strategy of Player 1 consists of two actions:
    - One that she uses when she knows that the realization is x;
    - One that she uses when she knows that the realization is in $\{y, z\}$.
  - A strategy of Player 2 consists of two actions:
    - One that he uses when he knows that the realization is in $\{x, y\}$;
    - One that he uses when he knows that the realization is z.
- A player's strategy is optimal if, given the strategy of the other player, for any realization of his information he can do no better by choosing an action different from that dictated by his strategy.

Imperfectly Correlated Information (Cont'd)

- We are still under the hypothesis that:
  - Player 1's information partition is $\{\{x\},\{y,z\}\}$;
  - Player 2's information partition is $\{\{x,y\},\{z\}\}$.
- Suppose that the probabilities of y and z are η and ζ.
- Moreover, assume that Player 2's strategy is:
  - Take the action a_2, if he knows that the realization is in $\{x,y\}$;
  - Take action b_2, if he knows that the realization is z.
- Suppose Player 1 is informed that either y or z has occurred.
- Then he chooses an action that is optimal, given that Player 2 chooses:
  - a_2 with probability $\frac{\eta}{\eta+\zeta}$ (the probability of y conditional on $\{y,z\}$);
  - b_2 with probability $\frac{\zeta}{\eta+\zeta}$.

Correlated Equilibrium

Definition (Correlated Equilibrium)

A **correlated equilibrium** of a strategic game $\langle N, (A_i), (u_i) \rangle$ consists of:

- A finite probability space (Ω, π) (Ω is a set of **states** and π is a probability measure on Ω);
- For each player $i \in N$, a partition $\mathcal{P}_i$ of Ω (player i's **information partition**);
- For each player $i \in N$, a function $\sigma_i : \Omega \to A_i$, such that, whenever $\omega \in P_i$ and $\omega' \in P_i$, for some $P_i \in \mathcal{P}_i$,

$$\sigma_i(\omega) = \sigma_i(\omega')$$

  (σ_i is Player i's **strategy**).

These data should satisfy some conditions shown in the next slide.

Correlated Equilibrium (Cont'd)

Definition (Correlated Equilibrium)

A **correlated equilibrium** of a strategic game $\langle N, (A_i), (u_i) \rangle$ consists of:

- (Ω, π);
- $\mathcal{P}_i$;
- $\sigma_i : \Omega \to A_i$,

such that, for every $i \in N$ and every function $\tau_i : \Omega \to A_i$, such that, whenever $\omega \in P_i$ and $\omega' \in P_i$, for some $P_i \in \mathcal{P}_i$,

$$\tau_i(\omega) = \tau_i(\omega')$$

(i.e., for every strategy of Player i), we have

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)).$$

Some Comments

- The probability space and information partition are not exogenous but are part of the equilibrium.
- The defining inequality

$$\sum_{\omega\in\Omega}\pi(\omega)u_i(\sigma_{-i}(\omega),\sigma_i(\omega))\geq\sum_{\omega\in\Omega}\pi(\omega)u_i(\sigma_{-i}(\omega),\tau_i(\omega)).$$

  is equivalent to the requirement that, for every state ω that occurs with positive probability, the action $\sigma_i(\omega)$ is optimal, given the other players' strategies and Player i's knowledge about ω.
- We show next that the set of correlated equilibria contains the set of mixed strategy Nash equilibria.

Mixed Strategy as Correlated Equilibria

Proposition

For every mixed strategy Nash equilibrium α of a finite strategic game $\langle N, (A_i), (u_i)\rangle$, there is a correlated equilibrium $\langle(\Omega, \pi), (\mathcal{P}_i), (\sigma_i)\rangle$ in which, for each player $i \in N$, the distribution on A_i induced by σ_i is α_i.

- Let $\Omega = A$ $(= \times_{j \in N} A_j)$. Define π by $\pi(a) = \prod_{j \in N} \alpha_j(a_j)$.

  For each $i \in N$ and $b_i \in A_i$, let:
  - $P_i(b_i) = \{a \in A : a_i = b_i\}$;
  - $\mathcal{P}_i$ consist of the $|A_i|$ sets $P_i(b_i)$.

  Define σ_i by $\sigma_i(a) = a_i$, for each $a \in A$.

  Then $\langle(\Omega, \pi), (\mathcal{P}_i), (\sigma_i)\rangle$ is a correlated equilibrium.

Mixed Strategy as Correlated Equilibria

- $\langle (\Omega, \pi), (\mathcal{P}_i), (\sigma_i) \rangle$ is a correlated equilibrium.

  We must show that, for every $i \in N$ and every strategy $\tau_i : \Omega \to A_i$ of Player i,

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)).$$

  - The left-hand side of the inequality is player i's payoff in the mixed strategy Nash equilibrium α;
  - The right-hand side is his payoff when he uses the mixed strategy in which he chooses the action $\tau_i(a)$ with probability $\alpha_i(a_i)$ and every other player j uses the mixed strategy α_j.

  Finally, the distribution on A_i induced by σ_i is α_i.

Example

- The three mixed strategy Nash equilibrium payoff profiles in BoS are $(2,1),(1,2)$ and $(\frac{2}{3},\frac{2}{3})$.

  We show that, in addition, one of the correlated equilibria yields the payoff profile $(\frac{3}{2},\frac{3}{2})$.

  Define the following:
  - $\Omega=\{x,y\}$;
  - $\pi(x)=\pi(y)=\frac{1}{2}$;
  - $\mathcal{P}_1=\mathcal{P}_2=\{\{x\},\{y\}\}$;
  - $\sigma_i(x)=$ Bach, and $\sigma_i(y)=$ Stravinsky, for $i=1,2$.
- One interpretation of this equilibrium is that the players observe the outcome of a public coin toss, which determines which of the two pure strategy Nash equilibria they play.

Convexity of Correlated Equilibria

Proposition

Let $G = \langle N, (A_i), (u_i) \rangle$ be a strategic game. Any convex combination of correlated equilibrium payoff profiles of G is a correlated equilibrium payoff profile of G.

- Let $u^1, \ldots, u^K$ be correlated equilibrium payoff profiles.

  Let $(\lambda^1, \ldots, \lambda^K) \in \mathbb{R}^K$, with $\lambda^k \geq 0$, for all k, and $\sum_{k=1}^K \lambda^k = 1$.

  For each value of k, let $\langle (\Omega^k, \pi^k), (\mathcal{P}_i^k), (\sigma_i^k) \rangle$ be a correlated equilibrium that generates the payoff profile u^k.

  Without loss of generality, assume that the sets Ω^k are disjoint.

  We define a correlated equilibrium with payoff profile $\sum_{k=1}^K \lambda^k u^k$.
  - $\Omega = \bigcup_k \Omega^k$;
  - For any $\omega \in \Omega$, define π by $\pi(\omega) = \lambda^k \pi^k(\omega)$, where k is s.t. $\omega \in \Omega^k$;
  - For each $i \in N$, let $\mathcal{P}_i = \bigcup_k \mathcal{P}_i^k$;
  - Define σ_i by $\sigma_i(\omega) = \sigma_i^k(\omega)$, where k is such that $\omega \in \Omega^k$.

Interpretation of the Correlated Equilibrium

- We give an interpretation of the correlated equilibrium constructed in the proof.
    - First, a public random device determines which of the K correlated equilibria is to be played;
    - Then the random variable corresponding to the k-th correlated equilibrium is realized.

Example

- Consider the game shown on the left.

	L	R
T	$6,6$	$2,7$
B	$7,2$	$0,0$

	L	R
T	y	z
B	x	$-$

The Nash equilibrium payoff profiles are:
- $(2,7)$ and $(7,2)$ (pure);
- $(4\frac{2}{3},4\frac{2}{3})$ (mixed).

The following correlated equilibrium yields a payoff profile that is outside the convex hull of these three profiles.
- $\Omega = \{x,y,z\}$;
- $\pi(x) = \pi(y) = \pi(z) = \frac{1}{3}$;
- Player 1's partition is $\{\{x\},\{y,z\}\}$ and Player 2's is $\{\{x,y\},\{z\}\}$;
- $\sigma_1(x) = \mathrm{B}$, $\sigma_1(y) = \sigma_1(z) = \mathrm{T}$ and $\sigma_2(x) = \sigma_2(y) = \mathrm{L}$, $\sigma_2(z) = \mathrm{R}$.

Example (Cont'd)

- We have

	L	R
T	$6,6$	$2,7$
B	$7,2$	$0,0$

	L	R
T	y	z
B	x	$-$

Then Player 1's behavior is optimal given Player 2's:

  - In state x, Player 1 knows that Player 2 plays L.
    Thus, it is optimal for her to play B.
  - In states y, z she assigns equal probabilities to Player 2 using L and R.
    So it is optimal for her to play T.

Symmetrically, Player 2's behavior is optimal given Player 1's.

Hence we have a correlated equilibrium.

The payoff profile is $(5,5)$.

Replacing States by Strategic Profiles

Proposition

Let $G = \langle N, (A_i), (u_i) \rangle$ be a finite strategic game. Every probability distribution over outcomes that can be obtained in a correlated equilibrium of G can be obtained in a correlated equilibrium in which the set of states is A and, for each $i \in N$, Player i's information partition consists of all sets of the form $\{a \in A : a_i = b_i\}$, for some action $b_i \in A_i$.

- Let $\langle (\Omega, \pi), (\mathcal{P}_i), (\sigma_i) \rangle$ be a correlated equilibrium of G.

  Then we define a new correlated equilibrium $\langle (\Omega', \pi'), (\mathcal{P}_i'), (\sigma_i') \rangle$, satisfying the requirements.
  - $\Omega' = A$;
  - $\pi'(a) = \pi(\{\omega \in \Omega : \sigma(\omega) = a\})$, for each $a \in A$;
  - $\mathcal{P}_i'$ consists of sets of the type $\{a \in A : a_i = b_i\}$, for some $b_i \in A_i$;
  - σ_i' is defined by $\sigma_i'(a) = a_i$.

Comment on Players' Beliefs

- In the definition of a correlated equilibrium we assume that the players share a common belief about the probabilities with which the states occur.
- If there is a random variable about which the players hold different beliefs, then additional equilibrium payoff profiles are possible.
- Suppose, e.g., that, in some contest between teams T_1 and T_2:
  - Player 1 is sure that team T_1 will win;
  - Player 2 is sure that team T_2 will win.

  Consider the BoS

	Bach	Stravinsky
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

  There is an equilibrium in which the outcome is:
  - $(\text{Bach}, \text{Bach})$ if T_1 wins;
  - $(\text{Stravinsky}, \text{Stravinsky})$ if team T_2 wins.

  This gives each player an expected payoff of 2!

Subsection 3

Evolutionary Equilibrium

Simple Evolutionary Games

- We describe the basic idea behind a variant of Nash equilibrium called evolutionary equilibrium.
- This notion is designed to model situations in which the players' actions are determined by the forces of evolution.
- We discuss only a simple case in which the members of a single population of organisms (animals, human beings, plants, etc.) interact with each other pairwise.

The Framework

- We consider a two-player symmetric strategic game

$$\langle \{1,2\}, (B,B), (u_i) \rangle.$$

  - In each match each organism takes an action from a set B.
  - The organisms do not consciously choose actions.
    They either inherit them or they are assigned by mutation.
  - We assume that there is a function u that measures each organism's ability to survive.
    If an organism takes the action a, when it faces the distribution β of actions, then its ability to survive is measured by the expectation of $u(a,b)$ under β. That is, we have

$$u_1(a,b) = u(a,b) \quad \text{and} \quad u_2(a,b) = u(b,a).$$

Idea Behind Evolutionary Equilibria

- A candidate for an evolutionary equilibrium is an action in B.
- The notion of equilibrium is designed to capture a steady state in which all organisms take this action and no mutant can survive.
- More precisely, the idea is:
  - For every action $b \in B$, evolution occasionally transforms a small fraction of the population into mutants who follow b.
  - In an equilibrium, any such mutant must obtain an expected payoff lower than that of the equilibrium action, so that it dies out.

Idea Behind Evolutionary Equilibria (Cont'd)

- Suppose that:
  - A fraction $\epsilon > 0$ of the population consists of mutants taking action b;
  - All other individuals take action b^*.
- Then the average payoff of:
  - A mutant is $(1-\epsilon)u(b,b^*)+\epsilon u(b,b)$;
  - A non-mutant is $(1-\epsilon)u(b^*,b^*)+\epsilon u(b^*,b)$.
- Therefore for b^* to be an evolutionary equilibrium we require

$$(1-\epsilon)u(b,b^*)+\epsilon u(b,b)<(1-\epsilon)u(b^*,b^*)+\epsilon u(b^*,b),$$

  for all values of ϵ sufficiently small.
- This inequality is satisfied if and only if for every $b \neq b^*$, one of the following holds:
  - $u(b,b^*)<u(b^*,b^*)$;
  - $u(b,b^*)=u(b^*,b^*)$ and $u(b,b)<u(b^*,b)$.

Evolutionarily Stable Strategies

Definition (Evolutionarily Stable Strategy)

Let $G = \langle \{1,2\}, (B,B), (u_i) \rangle$ be a symmetric strategic game, where

$$u_1(a,b) = u_2(b,a) = u(a,b),$$

for some function u.

An **evolutionarily stable strategy** (**ESS**) of G is an action $b^* \in B$ for which

- (b^*, b^*) is a Nash equilibrium of G;
- $u(b,b) < u(b^*,b)$, for every best response $b \in B$ to b^* with $b \neq b^*$.

Example

- Example (Hawk-Dove): From time to time pairs of animals in a population fight over a prey with value 1.

  Each animal can behave either like a dove (D) or like a hawk (H).

	D	H
D	$\frac{1}{2}, \frac{1}{2}$	$0, 1$
H	$1, 0$	$\frac{1}{2}(1-c), \frac{1}{2}(1-c)$

  - If both animals in a match are dovish then they split the value of the prey.
  - If they are both hawkish then the value of the prey is reduced by c and is split evenly.
  - If one of them is hawkish and the other is dovish then the hawk gets 1 and the dove 0.

Example (Cont'd)

- Let B be the set of all mixed strategies over $\{\text{D}, \text{H}\}$.
  - Suppose $c > 1$.
    Then the game has a unique symmetric mixed strategy Nash equilibrium, in which each player uses the strategy $(1 - \frac{1}{c}, \frac{1}{c})$.
    This strategy is the only ESS.
  - Suppose $c < 1$.
    Then, the game has a unique mixed strategy Nash equilibrium in which each player uses the pure strategy H.
    This strategy is the only ESS.
- By the Definition, if (b^*, b^*) is a symmetric Nash equilibrium and no strategy other than b^* is a best response to b^* ((b^*, b^*) is a **strict equilibrium**), then b^* is an ESS.

Example

- A nonstrict equilibrium strategy may not be an ESS.
- Consider the two-player symmetric game

	a	b
a	$1, 1$	$1, 1$
b	$1, 1$	$1, 1$

Clearly (a, a) is a Nash equilibrium.

However, b is a best response to a, with $b \neq a$, and

$$u(b, b) \not< u(a, b).$$

So (a, a) is not an ESS.

ESS versus Nash Equilibrium

- Here is another nonstrict equilibrium strategy that is not an ESS.

  Consider the following game, with $0 < \gamma \leq 1$.

γ, γ	$1, -1$	$-1, 1$
$-1, 1$	γ, γ	$1, -1$
$1, -1$	$-1, 1$	γ, γ

  This game has a unique symmetric mixed strategy Nash equilibrium in which each player's mixed strategy is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

  In this equilibrium the expected payoff of each player is $\frac{\gamma}{3}$.

  A mutant who uses any of the three pure strategies obtains an expected payoff of:

  - $\frac{\gamma}{3}$ when it encounters a non-mutant;
  - $\gamma > \frac{\gamma}{3}$ when it encounters another mutant.

  Hence the equilibrium mixed strategy is not an ESS.