Introduction to Game Theory

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1. Rationalizability and Iterated Elimination of Dominated Actions
   - Rationalizability
   - Iterated Elimination of Strictly Dominated Actions
   - Iterated Elimination of Weakly Dominated Actions

Subsection 1

Rationalizability

Knowledge of Other Players' Equilibrium Behavior

- In the previous sets, we discussed solution concepts for strategic games under certain hypotheses.
  - Each player knows the other players' equilibrium behavior.
  - Each player's choice is required to be optimal given this knowledge.
- Concerning knowing the other players' equilibrium behavior, we remark the following:
  - If the players participate repeatedly in the situation that the game models, then they can obtain this knowledge from the steady state behavior that they observe.
  - If the game is a one-shot event in which all players choose their actions simultaneously, then it is not clear how each player can know the other players' equilibrium actions.
- The need arises for solution concepts that do not presuppose knowledge of the other players' equilibrium behavior.

Rationality Assumptions without Correctness

- We study some solution concepts, in which:
  - The players' beliefs about each other's actions may not be correct;
  - The player's beliefs are constrained by considerations of rationality.
- The resulting solution concepts are weaker than Nash equilibrium.
- In many games they do not exclude any action from being used.
- The approach explores the logical implications of assumptions about the players' knowledge that are weaker than those adopted in previous discussions.

Beliefs and Best Responses

- Fix a strategic game $\langle N, (A_i), (u_i) \rangle$ in which the expectation of u_i represents player i's preferences over lotteries on $A = \times_{j \in N} A_j$ for each $i \in N$.
- We assume that A_i is finite (even though it is not necessary).
- A **belief** of player i (about the actions of the other players) is a probability measure on $A_{-i} = \times_{j \in N - \{i\}} A_j$.
  - A belief is not necessarily a product of independent probability measures on each of the action sets A_j, for $j \in N - \{i\}$.
  - So a player may believe that the other players' actions are correlated.
- An action $a_i \in A_i$ of player i is a **best response** to a belief if there is no other action that yields player i a higher payoff given the belief.
- Here, "rationality" means that Player i thinks that whatever action Player j chooses is a best response to player j's belief about the actions of players other than j.

Rationality Assumption Regarding Other Palyers

- Suppose Player i thinks that every other player j is rational.
- Then he must be able to rationalize his belief μ_i about the other players' actions as follows:

  Every action of any other player j to which the belief μ_i assigns positive probability must be a best response to a belief of player j.
- Supose Player i also thinks that every other player j thinks that every player $h \neq j$ (including player i) is rational.
- Then Player i must also have a view about Player j's view about Player h's beliefs.
- If Player i's reasoning has unlimited depth, then the following definition applies.

Rationalizable Actions

Definition (Rationalizable Action)

An action $a_i \in A_i$ is **rationalizable** in $\langle N, (A_i), (u_i) \rangle$ if there exist:

- A collection $((X_j^t)_{j \in N})_{t=1}^{\infty}$ of sets, with $X_j^t \subseteq A_j$, for all j and t;
- A belief μ_i^1 of player i whose support is a subset of X_{-i}^1;
- For each $j \in N$, each $t \geq 1$, and each $a_j \in X_j^t$, a belief $\mu_j^{t+1}(a_j)$ of player j whose support is a subset of X_{-j}^{t+1},

satisfying the constraints given in the following slide.

Rationalizable Actions (Cont'd)

Definition (Rationalizable Action)

The sets $X_j^t \subseteq A_j$ and the beliefs μ_i^t must satisfy:

- a_i is a best response to the belief μ_i^1 of player i;
- $X_i^1 = \emptyset$ and, for each $j \in N - \{i\}$, the set X_j^1 is the set of all $a_j' \in A_j$, such that there is some a_{-i} in the support of μ_i^1 for which $a_j = a_j'$;
- For every player $j \in N$ and every $t \geq 1$, every action $a_j \in X_j^t$ is a best response to the belief $\mu_j^{t+1}(a_j)$ of player j;
- For each $t \geq 2$ and each $j \in N$, the set X_j^t is the set of all $a_j' \in A_j$ such that there is some player $k \in N - \{j\}$, some action $a_k \in X_k^{t-1}$ and some a_{-k} in the support of $\mu_k^t(a_k)$ for which $a_j' = a_j$.

Comments on the Definition

- The second and fourth conditions in the second part of this definition are superfluous.
- They are included so that the definition corresponds more closely to the motivation given before the definition.
- We include the set X_i^1 in the collection $((X_j^t)_{j\in N})_{t=1}^{\infty}$, even though it is required to be empty, to simplify the notation.
- If $|N| \geq 3$ then X_i^1 is the only such superfluous set.
- If $|N| = 2$ there are many superfluous sets:
  - X_i^t, for any odd t;
  - For $j \neq i$, X_j^t, for any even t.

Comments on the Definition (Cont'd)

- The set X_j^1 for $j \in N - \{i\}$ is interpreted to be the set of actions of Player j that are assigned positive probability by the belief μ_i^1 of Player i about the actions of the players other than i that justifies i choosing a_i.
- For any $j \in N$ the interpretation of X_j^2 is that it is the set of all actions a_j of Player j such that there exists at least one action $a_k \in X_k^1$ of some player $k \neq j$ that is justified by the belief $\mu_k^2(a_k)$ that assigns positive probability to a_j.

Illustration of the Definition

- To illustrate what the definition entails, suppose there are three players, each of whom has two possible actions, A and B.
- Assume that:
  - The action A of Player 1 is rationalizable
  - Player 1's belief μ_1^1 used in the rationalization assigns positive probability to the choices of Players 2 and 3 being either (A, A) or (B, B).
- Then $X_2^1 = X_3^1 = \{A, B\}$.
- The beliefs $\mu_2^2(A)$ and $\mu_2^2(B)$ of Player 2 that justify his choices of A and B concern the actions of Players 1 and 3.
- The beliefs $\mu_3^2(A)$ and $\mu_3^2(B)$ of Player 3 concern Players 1 and 2.
- These four beliefs do not have to induce the same beliefs about player 1 and do not have to assign positive probability to the action A.
- The set X_1^2 consists of all the actions of player 1 that are assigned positive probability by $\mu_2^2(A)$, $\mu_3^2(A)$, $\mu_2^2(B)$, or $\mu_3^2(B)$.

An Alternative Equivalent Definition

- An equivalent definition of rationalizability is the following.

Definition (Rationalizable Action)

An action $a_i \in A_i$ is **rationalizable** in the strategic game $\langle N, (A_i), (u_i) \rangle$ if, for each $j \in N$ there is a set $Z_j \subseteq A_j$, such that

- $a_i \in Z_i$
- Every action $a_j \in Z_j$ is a best response to a belief $\mu_j(a_j)$ of player j whose support is a subset of Z_{-j}.

- Note that, if $(Z_j)_{j \in N}$ and $(Z'_j)_{j \in N}$ satisfy the definition, then so does $(Z_j \cup Z'_j)_{j \in N}$.
- It follows that the set of profiles of rationalizable actions is the largest set $\times_{j \in N} Z_j$ for which $(Z_j)_{j \in N}$ satisfies the definition.

Equivalence of the Definitions

Lemma (Equivalence of Definitions)

The two definitions of rationalizability are equivalent.

- Suppose $a_i \in A_i$ is rationalizable according to the first definition.
  Then define
  $$Z_i = \{a_i\} \cup (\cup_{t=1}^{\infty} X_i^t), \quad Z_j = (\cup_{t=1}^{\infty} X_j^t), \quad j \in N - \{i\}.$$
  Suppose $a_i \in A_i$ is rationalizable according to the second definition.
  Then define:
  - $\mu_i^1 = \mu_i(a_i)$;
  - $\mu_j^t(a_j) = \mu_j(a_j)$, for each $j \in N$ and each integer $t \geq 2$.

  Let X_j^t, $j \in N$, $t = 1, 2, \ldots$, be the sets defined in the second and fourth parts of the first definition. Then we have:
  - $X_j^t \subseteq Z_j$, for all $j \in N$, $t = 1, 2, \ldots$;
  - The sets X_j^t satisfy the conditions in the first and third parts.

Rationalizability in Nash Equilibria

- Consider a finite game.
- Any action that a player uses with positive probability in some mixed strategy Nash equilibrium is rationalizable.
- E.g., we may take Z_j to be the support of player j's mixed strategy.
- The same is true for actions used with positive probability in some correlated equilibrium.
- This will be shown in the next slide.

Rationalizability in Correlated Equilibria

Lemma

Every action used with positive probability by some player in a correlated equilibrium of a finite strategic game is rationalizable.

- Denote the game by $\langle N, (A_i), (u_i) \rangle$.

  Choose a correlated equilibrium.

  For each player $i \in N$, let Z_i be the set of actions that Player i uses with positive probability in the equilibrium.

  Then any $a_i \in Z_i$ is a best response to the distribution over A_{-i} generated by the strategies of the players other than i, conditional on Player i choosing a_i.

  The support of this distribution is a subset of Z_{-i}.

  Hence, by the second definition, a_i is rationalizable.

Example

- Consider the game:

M_1:

	L	R
U	8	0
D	0	0

M_2:

	L	R
U	4	0
D	0	4

M_3:

	L	R
U	0	0
D	0	8

M_4:

	L	R
U	3	3
D	3	3

In this game there are three players:
  - Player 1 chooses one of the two rows;
  - Player 2 chooses one of the two columns;
  - Player 3 chooses one of the four tables.

All three players obtain the same payoffs, given by the numbers in the boxes.

Example (Claim)

- Claim: The action M_2 of Player 3:
  - Is rationalizable if a player may believe that his opponent's actions are correlated;
  - Is not rationalizable if players are restricted to beliefs that are products of independent probability distributions.

Note that:
  - The action U of Player 1 is a best response to a belief that assigns probability one to (L, M_2);
  - The action D is a best response to the belief that assigns probability one to (R, M_2).

Similarly, both actions of Player 2 are best responses to beliefs that assign positive probability only to U, D and M_2.

Further, the action M_2 of Player 3 is a best response to the belief in which Players 1 and 2 play (U, L) and (D, R) with equal probabilities.

Thus, M_2 is rationalizable, with $Z_1 = \{U, D\}$, $Z_2 = \{L, R\}$, $Z_3 = \{M_2\}$.

Example (Cont'd)

- We now show that M_2 is not rationalizable if players are restricted to beliefs that are products of independent probability distributions.

  To see this, we must show that M_2 is not a best response to any pair of (independent) mixed strategies.

  Suppose that:
  - $(p, 1-p)$ is a mixed strategy of Player 1;
  - $(q, 1-q)$ is a mixed strategy of Player 2.

  In order for M_2 to be a best response we need

$$4pq + 4(1-p)(1-q) \geq \max\{8pq, 8(1-p)(1-q), 3\}.$$

  However, this is not satisfied for any values of p and q.

Subsection 2

Iterated Elimination of Strictly Dominated Actions

Idea of Iterated Elimination

- We assume that players exclude from consideration actions that are not best responses whatever the other players do.
- So we eliminate actions that a player should definitely not take.
- A player who knows that the other players are rational can assume that they too will exclude such actions from consideration.
- Next, we consider the game G' obtained from the original game G by eliminating all such actions.
- A player should not choose an action that is not a best response whatever the other players do in G'.
- Moreover, she knows that other players will not choose actions that are never best responses in G' either.
- Continuing to argue in this way suggests that the outcome of G must survive an unlimited number of such elimination rounds.

Never-best Responses and Strictly Dominated Actions

Definition (Never-best Response)

An action of Player i in a strategic game is a **never-best response** if it is not a best response to any belief of Player i.

- Clearly any action that is a never-best response is not rationalizable.
- If an action a_i of Player i is a never-best response, then, for every belief of Player i, there is some action, which may depend on the belief, that is better for Player i than a_i.
- We now show that if a_i is a never-best response in a finite game, then there is a mixed strategy that is better for Player i than a_i.

Strictly Dominated Actions and Mixed Strategies

Definition (Strictly Dominated Action)

The action $a_i \in A_i$ of Player i in the strategic game $\langle N, (A_i), (u_i) \rangle$ is **strictly dominated** if, there is a mixed strategy α_i of Player i, such that $U_i(a_{-i}, \alpha_i) > u_i(a_{-i}, a_i)$, for all $a_{-i} \in A_{-i}$, where $U_i(a_{-i}, \alpha_i)$ is the payoff of Player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}.

- We show that, in a game in which the set of actions of each player is finite, an action is a never-best response if and only if it is strictly dominated.
- Thus, in such games, the notion of strict domination has a decision-theoretic basis that does not involve mixed strategies.
- It follows that even if one rejects the idea that mixed strategies can be objects of choice, one can still argue that a player will not use an action that is strictly dominated.

Never-Best Responses vs. Strictly Dominated Actions

Lemma

An action of a player in a finite strategic game is a never-best response if and only if it is strictly dominated.

- Let the strategic game be $G = \langle N, (A_i), (u_i) \rangle$ and let $a_i^* \in A_i$.
  We consider the auxiliary strictly competitive game G' in which:
  - The set of actions of Player 1 is $A_i - \{a_i^*\}$;
  - The set of actions of Player 2 is A_{-i};
  - The preferences of Player 1 are represented by the payoff function

$$v_1(a_i, a_{-i}) = u_i(a_{-i}, a_i) - u_i(a_{-i}, a_i^*).$$

  Note that:
  - The argument (a_i, a_{-i}) of v_1 is a pair of actions in G';
  - The arguments (a_{-i}, a_i) and (a_{-i}, a_i^*) are action profiles in G.

  For any given mixed strategy profile (m_1, m_2) in G' we denote by $v_1(m_1, m_2)$ the expected payoff of Player 1.

Proof of the Equivalence

- The action a_i^* is a never-best response in G if and only if, for any mixed strategy of Player 2 in G', there is an action of Player 1 that yields a positive payoff.

  In other words, if and only if

$$\min_{m_2} \max_{a_i} v_1(a_i, m_2) > 0.$$

  This is so, by the linearity of v_1 in m_1, if and only if

$$\min_{m_2} \max_{m_1} v_1(m_1, m_2) > 0.$$

  Now, by a preceding result, the game G' has a mixed strategy Nash equilibrium. Therefore,

$$\min_{m_2} \max_{m_1} v_1(m_1, m_2) > 0 \quad \text{iff} \quad \max_{m_1} \min_{m_2} v_1(m_1, m_2) > 0.$$

Proof of the Equivalence (Cont'd)

- The latter holds if and only if, there exists a mixed strategy m_1^* of Player i in G', for which $v_1(m_1^*, m_2) > 0$, for all m_2 (that is, for all beliefs on A_{-i}).

  Since m_1^* is a probability measure on $A_i - \{a_i^*\}$, it is a mixed strategy of Player 1 in G.

  The condition $v_1(m_1^*, m_2) > 0$, for all m_2, is equivalent to

  $$U_i(a_{-i}, m_1^*) - U_i(a_{-i}, a_i^*) > 0,$$

  for all $a_{-i} \in A_{-i}$.

  Thi is equivalent to a_i^* being strictly dominated.

Surviving Elimination of Strictly Dominated Actions

Definition (Outcomes Surviving Iterated Elimination)

The set $X \subseteq A$ of outcomes of a finite strategic game $\langle N, (A_i), (u_i)\rangle$ **survives iterated elimination of strictly dominated actions** if $X = \times_{j \in N} X_j$ and there is a collection $((X_j^t)_{j \in N})_{t=0}^T$ of sets that satisfies the following conditions for each $j \in N$:

- $X_j^0 = A_j$ and $X_j^T = X_j$;
- $X_j^{t+1} \subseteq X_j^t$, for each $t = 0, \ldots, T-1$
- For each $t = 0, \ldots, T-1$ every action of player j in $X_j^t - X_j^{t+1}$ is strictly dominated in the game $\langle N, (X_i^t), (u_i^t)\rangle$, where, for each $i \in N$, u_i^t is the function u_i restricted to $\times_{j \in N} X_j^t$;
- No action in X_j^T is strictly dominated in the game $\langle N, (X_i^T), (u_i^T)\rangle$.

Example of Iterated Elimination

- Consider the game

	L	R
T	$3, 0$	$0, 1$
M	$0, 0$	$3, 1$
B	$1, 1$	$1, 0$

- The action B is dominated by the mixed strategy in which T and M are each used with probability $\frac{1}{2}$.
- After B is eliminated from the game, L is dominated by R.
- After L is eliminated, T is dominated by M.
- Thus (M, R) is the only outcome that survives iterated elimination of strictly dominated actions.

Surviving Outcomes are Profiles of Rationalizable Actions

- We now show that, in a finite game:
  - A set of outcomes that survives iterated elimination of strictly dominated actions exists;
  - Moreover, it coincides with the set of profiles of rationalizable actions.

Proposition

If $X = \times_{j \in N} X_j$ survives iterated elimination of strictly dominated actions in a finite strategic game $\langle N, (A_i), (u_i) \rangle$, then X_j is the set of Player j's rationalizable actions, for each $j \in N$.

- Suppose, first, that $a_i \in A_i$ is rationalizable.

  Let $(Z_j)_{j \in N}$ be the profile of sets that supports a_i.

  Each action in Z_j is a best response to some belief over Z_{-j}.

  Hence, it is not strictly dominated in $\langle N, (X_i^t), (u_i^t) \rangle$.

  So, for any value of t, we have $Z_j \subseteq X_j^t$. Hence, $a_i \in X_i$.

The Reverse Inclusion

- We now show that, for every $j \in N$, every member of X_j is rationalizable. By definition, no action in X_j is strictly dominated in the game in which the set of actions of each player i is X_i,

  By a preceding lemma, every action in X_j is a best response among the members of X_j to some belief on X_{-j}.

  We need to show that every action in X_j is a best response among all the members of the set A_j to some belief on X_{-j}.

  Suppose $a_j \in X_j$ is not a best response among all the members of A_j. Then, there is a value of t, such that a_j:
  - Is a best response among the members of X_j^t to a belief μ_j on X_{-j};
  - Is not a best response among the members of X_j^{t-1}.

  Thus, there is an action $b_j \in X_j^{t-1} - X_j^t$ that is a best response among the members of X_j^{t-1} to μ_j, contradicting the fact that b_j is eliminated at the tth stage of the procedure.

Iterated Elimination and Independent Beliefs

- The preceding lemmas fail if the definition of rationalizability requires the players to believe that their opponents' actions are independent.

  Example: Consider the following game.

	L	R
U	8	0
D	0	0

M_1

	L	R
U	4	0
D	0	4

M_2

	L	R
U	0	0
D	0	8

M_3

	L	R
U	3	3
D	3	3

M_4

  The action M_2 is a best response to the belief of Player 3 in which Players 1 and 2 play (U, L) and (D, R) with equal probabilities.

  Thus, M_2 is not strictly dominated.

  However, it is not a best response to any pair of (independent) mixed strategies. So it is not rationalizable if each player's belief is restricted to be a product of independent beliefs.

Subsection 3

Iterated Elimination of Weakly Dominated Actions

Weakly Dominated Actions

- We say that a player's action is weakly dominated if the player has another action that is:
  - At least as good no matter what the other players do;
  - Better for at least some vector of actions of the other players.

Definition (Weakly Dominated Action)

The action $a_i \in A_i$ of player i in the strategic game $\langle N, (A_i), (u_i)\rangle$ is **weakly dominated** if, there is a mixed strategy α_i of Player i, such that:

- $U_i(a_{-i}, \alpha_i) \geq u_i(a_{-i}, a_i)$, for all $a_{-i} \in A_{-i}$;
- $U_i(a_{-i}, \alpha_i) > u_i(a_{-i}, a_i)$, for some $a_{-i} \in A_{-i}$,

where $U_i(a_{-i}, \alpha_i)$ is the payoff of Player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}.

Iterated Elimination of Weakly Dominated Actions

- By a preceding lemma, an action that is weakly dominated but not strictly dominated is a best response to some belief.
- There is no advantage to using a weakly dominated action.
- So it seems natural to eliminate such actions in the process of simplifying a complicated game.
- The notion of weak domination leads to a procedure analogous to iterated elimination of strictly dominated actions.
- This procedure is less compelling since the set of actions that survive iterated elimination of weakly dominated actions may depend on the order in which actions are eliminated.

Example

- Consider the following game.

	L	R
T	$1,1$	$0,0$
M	$1,1$	$2,1$
B	$0,0$	$2,1$

- Consider the sequence in which:
  - We eliminate T, which is weakly dominated by M;
  - We eliminate L, which is weakly dominated by R.

  In the outcome, Player 2 chooses R and the payoff profile is $(2,1)$.
- Consider the sequence in which:
  - We eliminate B, which is weakly dominated by M;
  - We eliminate R, which is weakly dominated by L.

  In the outcome, Player 2 chooses L and the payoff profile is $(1,1)$.