College Geometry

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

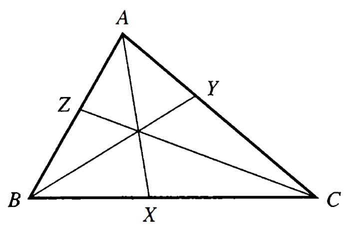
LSSU Math 325

Subsection 1

Introduction

A Geometric Figure

- Consider the angle bisectors AX, BY and CZ of the triangle $\triangle ABC$.



- Three facts that seem clear in the figure are:
  1. $BX < XC$, i.e., line segment BX is shorter than line segment XC.
  2. The point X, where the bisector of $\angle A$ meets line BC, lies on the line segment BC.
  3. The three angle bisectors are concurrent.

Meaning of Technical Words and Notation

- We assume that the nouns **point**, **line**, **angle**, **triangle** and **bisector** are familiar.
- Two lines, unless they happen to be parallel, always meet at a point.
- If three or more lines all go through a common point, then we say that the lines are **concurrent**.
- A **line segment** is that part of a line that lies between two given points on the line.
- In Assertion 1, the notation BX is used in two different ways:
  - In the inequality, BX represents the length of the line segment, which is a number;
  - Later, BX is used to name the segment itself.
  - In addition, the notation BX is often used to represent the entire line containing the points B and X, and not just the segment they determine.

It will be clear from context which of the three possible meanings is intended.

On the Types of Truth of the Three Statements

- We need to distinguish among three different types of truth represented by 1-3:
  - The fact in 1 that $BX < XC$ is an accident.
    - The inequality happens to be true in this figure, but it is not an instance of some general or universal truth.
    - Even in a particular case, this sort of information can be unreliable since it depends on the accuracy of the diagram.
  - Fact 2, that point X lies between points B and C on line BC, is not accidental.
    - Indeed, the bisector of each angle of an arbitrary triangle must always intersect the opposite side of the triangle.
    - Although this is true about angle bisectors, it can fail for other important lines associated with a triangle, e.g., the **altitude** drawn from A, which is the line through A perpendicular to line BC, may not meet the segment BC.
  - It is a fact that for every triangle, the three angle bisectors are concurrent, as in 3, but we do not consider this to be obvious and require instead a proof.

Axioms and Postulates

- Unproved assumptions are sometimes called **axioms** and **postulates**.
  - Traditionally, axioms concerned general logical reasoning;
  - On the other hand, postulates were more specifically geometric.
- For example:
  - An axiom is "things equal to the same thing are equal to each other";
  - The famous **parallel postulate** essentially asserts that "given a line and a point not on that line, there exists one and only one line through the given point parallel to the given line".

The Parallel Postulate

- The parallel postulate asserting the existence and uniqueness of a line parallel to a given line through a given point seemed less obvious than the facts asserted by the other postulates.
- Geometers attempted to prove it by trying to deduce it from the remaining axioms and postulates.
- One way to prove it, would be to assume that it is false and then try to derive some contradictory conclusions:
  - Assume, for example, that there is some line AB and some point P not on AB, and there is no line parallel to AB through P.
    If by means of this assumption one could deduce the existence of a triangle $\triangle XYZ$ for which $XY > YZ$ and also $XY < YZ$, then this contradiction would prove at least the existence part of the parallel postulate.
- When this was tried, apparent contradictions were derived and the existence of figures that seem impossible was proved.

Attempts at the Parallel Postulate

- The assumption that no line parallel to AB goes through point P yields a triangle $\triangle XYZ$ that has three right angles.

  Even though, such a conclusion seems "impossible", how can we prove this impossibility?
- It may seem that by proving the existence of a triangle with three right angles, we have the desired contradiction, but this is wrong.

  The proof that, in a triangle $\triangle XYZ$, $\angle X + \angle Y + \angle Z = 180°$ relies on the parallel postulate, which should not be assumed.
- After repeated attempts to obtain contradictions from the denial of either the existence or the uniqueness part of the parallel postulate, J. Bolyai, N. Lobachevski, and C. Gauss in the 19th century realized that no proof of a contradiction was possible.
- In fact, it was proved more generally that no proof of the parallel postulate (by contradiction or otherwise) is possible.

Non-Euclidean Geometry

- A perfectly consistent deductive geometry can be obtained by replacing Euclid's parallel postulate with either one of two alternative new postulates:
  - One denies the existence of a parallel to some line through some point;
  - The other asserts the existence of at least two such parallels.
- Each of the two types of geometry that arise in this way is said to be **non-Euclidean**, and each has its own set of proved theorems:
  - The geometry where no parallel exists is called **elliptic geometry**;
  - When more than one parallel to a line goes through a point, we have **hyperbolic geometry**.
- The deductions of each of the two types of non-Euclidean geometry contradict each other and the theorems of classical Euclidean geometry, but each appears to be internally consistent.
- It is known that if Euclidean geometry is internally consistent, then the two non-Euclidean geometries are also consistent, but no formal proof of the consistency of Euclidean geometry has been found.

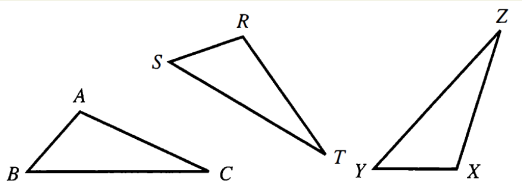
Subsection 2

Congruent Triangles

Congruence

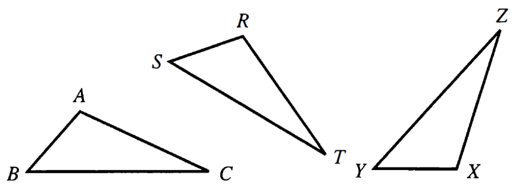
- A **rigid motion** is one of the following transformations:
  - a translation or shift;
  - a rotation in the plane;
  - a reflection in a line.
- Two figures are **congruent** if one can be subjected to a rigid motion so as to make it coincide with the other.

  Example: The three triangles are congruent:



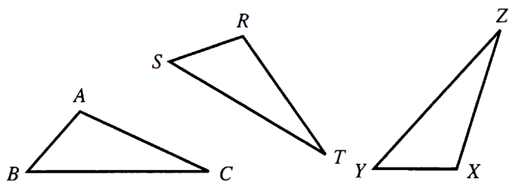
We write $\triangle ABC \cong \triangle RST$ to report that the first two triangles are congruent.

Corresponding Points



- Note that there is more to the notation $\triangle ABC \cong \triangle RST$ than may at first be apparent:
  - The only way that these two triangles can be made to coincide is for point R to coincide with point A, for S to coincide with B, and for T to coincide with C.
  - We say that A and R, B and S, and C and T are **corresponding points** of these two congruent triangles.
- The only correct ways to report the congruence is to list corresponding points in corresponding positions.
  - It is correct, therefore, to write $\triangle ABC \cong \triangle RST$ or $\triangle BAC \cong \triangle SRT$;
  - It is wrong to write $\triangle ABC \cong \triangle SRT$.

Equality of Corresponding Sides and Angles



- Since $\triangle RST \cong \triangle XYZ$:
  - The corresponding sides of these triangles have equal length;
  - The corresponding angles have equal measure (contain equal numbers of degrees or radians).
- We can thus write, for example, $RS = XY$ and $\angle SRT = \angle YXZ$.
- The notation $\angle SRT$ refers to the measure of the angle in some convenient units, such as degrees or radians.
  In other situations $\angle SRT$ may refer to the angle itself.
- This mimics the fact that RS can refer either to a line segment or to its length in some convenient unit.

Congruence and the SSS Criterion

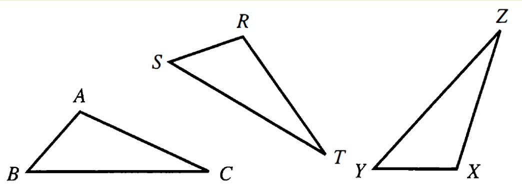
- Congruence of two triangles implies six equalities:
  - three of lengths;
  - and three of measures of angles.
- Conversely, given two triangles, if all six equalities hold, then the triangles can be made to "fit" on top of each other, so are congruent.
- However, it is not necessary to know all six equalities to conclude that two triangles are congruent.
  - If we know, for example, that the three sides of one triangle equal, respectively, the three corresponding sides of the other triangle, we can safely deduce that the triangles are congruent.

    Example: If we know that $AB = RS$, $AC = RT$, and $BC = ST$, we can conclude that $\triangle ABC \cong \triangle RST$.
  - In a proof using this criterion, we say that the triangles are "congruent by SSS".
  - SSS stands for "side-side-side" and refers to the theorem that says that if the three sides of a triangle are equal in length to the corresponding sides of another triangle, then the two triangles are congruent.

Three More Criteria for Congruence

- Other valid criteria that can be used to prove that two triangles are congruent are
  - SAS for "side-angle-side";
  - ASA for "angle-side-angle";
  - SAA for "side-angle-angle".

Example:



If we somehow know that $ST = YZ$ and that $\angle S = \angle Y$ and $\angle R = \angle X$, we can write in a proof: "We conclude by SAA that $\triangle SRT \cong \triangle YXZ$".

The Logical Status of the Criteria for Congruence

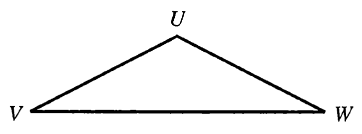
- Each of the congruence criteria SSS, SAS, ASA and SAA, i.e., the fact that it is sufficient to guarantee the congruence of two triangles is a theorem, proved by Euclid from his postulates.
- These four theorems are among the basic results that we are accepting as known to be valid and that we are willing to use without providing proofs.
- It is not hard to prove the sufficiency of some of these criteria if we are willing to accept some of the others:

  We deduce, as a first example of a proof, the sufficiency of the SSS criterion, with the understanding that we may freely use any of the other three triangle-congruence conditions.

Some Preliminary Terminology and Results

- Recall that a triangle $\triangle UVW$ is **isosceles** if two of its sides have equal lengths.

  Example: The following triangle is isosceles because $UV = UW$.



- The third side VW is called the **base** of the triangle, whether or not it actually occurs at the bottom of the diagram.
- The **base angles** of an isosceles triangle are the two angles at the ends of the base.

Theorem (Pons Asinorum)

The base angles of an isosceles triangle are equal.

Example: In the figure, we have $\angle UVW = \angle UWV$.

Deriving SSS from SAS and Pons Asinorum

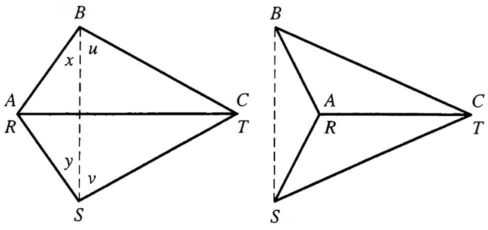
Proposition

Assume that in $\triangle ABC$ and $\triangle RST$, we know that $AB = RS$, $AC = RT$ and $BC = ST$. Prove that $\triangle ABC \cong \triangle RST$ without using the SSS congruence criterion.

- By renaming the points, if necessary, we can assume that AC is the longest side of $\triangle ABC$. It then follows that RT is the longest side of $\triangle RST$. Since we are given that $AC = RT$, we can move $\triangle RST$, flipping it over, if necessary, so that:
  - Points A and R coincide and points C and T coincide;
  - Points B and S lie on opposite sides of line AC.

  Now draw line segment BS.

Deriving SSS from SAS and Pons Asinorum (Cont'd)



- The left diagram is the only possible, since for BS to fail to meet AC, as on the right, or for BS to go through one of the points A or C would require one of BC or BA to be longer than AC. Since $AB = RS$, we see that $\triangle ABS$ is isosceles with base BS. Hence, by the pons asinorum, we deduce that $x = y$, where we are writing $x = \angle ABS$ and $y = \angle RSB$. Similarly, by a second application of the pons asinorum, we obtain $u = v$. It now follows that $x + u = y + v$, i.e., $\angle ABC = \angle RST$. Since we already know that $AB = RS$ and $BC = ST$, we can conclude by SAS that $\triangle ABC \cong \triangle RST$.

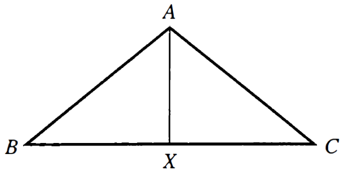
Proving the Pons Asinorum

- A **median** of a triangle is the line segment joining a vertex to the midpoint of the opposite side.

Theorem

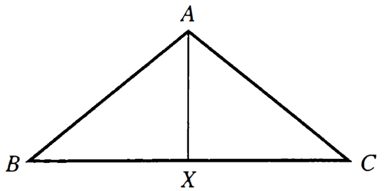
Let $\triangle ABC$ be isosceles, with base BC. Then $\angle B = \angle C$.
Also, the median from vertex A, the bisector of $\angle A$, and the altitude from vertex A are all the same line.

- In the figure, we have drawn the bisector AX of $\angle A$, and, thus, $\angle BAX = \angle CAX$. By hypothesis, we know that $AB = AC$, and of course, $AX = AX$. Thus $\triangle BAX \cong \triangle CAX$ by SAS.



It follows that $\angle B = \angle C$ since these are corresponding angles in the congruent triangles.
We also need to show that the angle bisector AX is a median and that it is an altitude too.

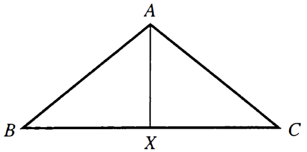
Proving the Pons Asinorum (Cont'd)



- We show that the angle bisector AX is a median and that it is an altitude too.
  - To see that it is a median, it suffices to check that X is the midpoint of segment BC. This is true because $BX = XC$, since these are corresponding sides of our congruent triangles.
  - Finally, to prove that AX is also an altitude, we must show that AX is perpendicular to BC. In other words, we need to establish that $\angle BXA = 90°$. From the congruent triangles, we know that the corresponding angles $\angle BXA$ and $\angle CXA$ are equal. Thus, $\angle BXA = \angle CXA = 90°$, since the straight angle $\angle BXC = 180°$.

Example of Invalid Circular Reasoning

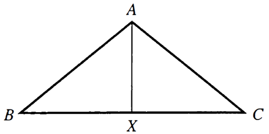
- To prove that the angle bisector, median, and altitude from vertex A are all the same, we started by drawing the bisector and showed that it was also a median and an altitude.



  Since there is only one median and one altitude from A, we know that the bisector is the median and the altitude.
- Suppose we had started by drawing the median from A instead of the bisector of $\angle A$.
  We could deduce that $\triangle BAX \cong \triangle CAX$ by SSS. We would then have $\angle BAX = \angle CAX$. We would deduce that AX is the angle bisector.
- This approach would have been less satisfactory, because it makes the pons asinorum depend on the SSS congruence criterion.
  Since the validity of the SSS criterion was shown using the pons asinorum, this would be an example of invalid circular reasoning.

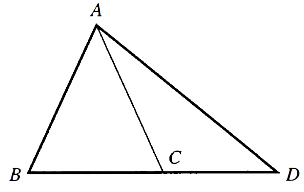
Invalidity of "SSA"

- Suppose we start by drawing altitude AX.



We would then know that $\angle BXA = \angle CXA$, since both of these are right angles. We also know that $AB = AC$ and $AX = AX$.

At this point, we might be tempted to conclude that $\triangle BAX \cong \triangle CAX$ by SSA, but we would resist that temptation, of course, because SSA is not a valid congruence criterion:



In this figure, the base BC of isosceles $\triangle ABC$ has been extended to an arbitrary point D beyond C. The two triangles $\triangle ADC$ and $\triangle ADB$ are clearly not congruent because $DB > DC$.

Yet the triangles agree in side-side-angle since $AB = AC$, $AD = AD$, and $\angle D = \angle D$.

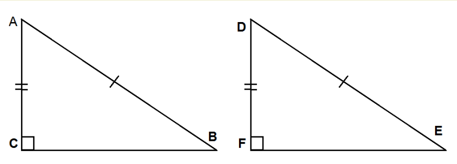
The Hypothenuse-Arm Criterion

- One case where SSA is a valid criterion is when the angle is a right angle, called the **hypotenuse-arm criterion**, abbreviated HA.
  - The longest side of a right triangle, the side opposite the right angle, is called the **hypotenuse** of the triangle.
  - The other two sides of the triangle are often called its **arms**.

Theorem

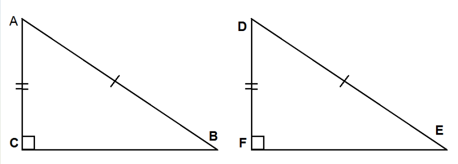
If two right triangles have equal hypotenuses and an arm of one of the triangles equals an arm of the other, then the triangles are congruent.

- We are given triangles $\triangle ABC$ and $\triangle DEF$, with right angles at C and F.

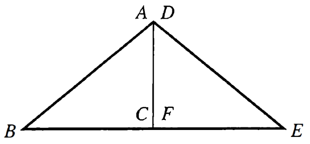


We know that $AB = DE$ and $AC = DF$, and we want to show that $\triangle ABC \cong \triangle DEF$.

The Hypothenuse-Arm Criterion (Cont'd)



- Move $\triangle DEF$, flipping it over if necessary, so that points A and D and points C and F coincide: We have $\angle BCE = \angle BCA + \angle EFD = 90^\circ + 90^\circ = 180^\circ$, and thus BCE is a line segment, which we can now call BE.



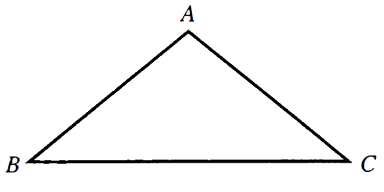
  Since $AB = DE$, we see that $\triangle ABE$ is isosceles with base BE. Thus, altitude AC is a median and hence $BC = FE$. The desired congruence now follows by SSS.

A Second Proof of Pons Asinorum

Theorem (Pons Asinorum)

The base angles of an isosceles triangle are equal.

- We are given isosceles $\triangle ABC$ with base BC,



and we want to show that $\angle B = \angle C$.

We have $AB = AC$ and $AC = AB$. Since also $\angle A = \angle A$, we can conclude that $\triangle ABC \cong \triangle ACB$, by SAS. It follows that $\angle B = \angle C$, since these are corresponding angles of the congruent triangles.

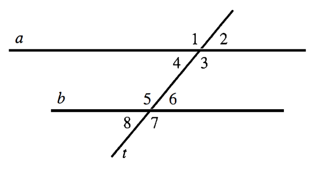
Subsection 3

Angles and Parallel Lines

Transversal and Corresponding Angles

- A **transversal** is a line that cuts across two given lines.

  Example: Line t is a transversal to lines a and b. Angles that are on the same side of the transversal and on corresponding sides of the two lines a and b are said to be **corresponding angles**.

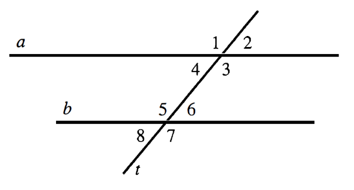
  

  $\angle 1$ and $\angle 5$ are corresponding angles, $\angle 2$ and $\angle 6$ and also corresponding angles, as are the pairs $\angle 3$ and $\angle 7$ and, of course, $\angle 4$ and $\angle 8$.

- It is a theorem that corresponding angles are equal when a transversal cuts a pair of parallel lines.
- Conversely, if any one of the equalities $\angle 1 = \angle 5$, $\angle 2 = \angle 6$, $\angle 3 = \angle 7$, or $\angle 4 = \angle 8$ is known to hold, then it is a theorem that lines a and b are parallel, and, thus, the other three equalities also hold.

Alternate Interior and Alternate Exterior Angles

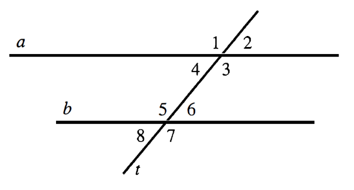
- Pairs of angles such as $\angle 4$ and $\angle 6$ or $\angle 3$ and $\angle 5$ that lie on opposite sides of the transversal and between the two given lines are called **alternate interior angles**.



- Pairs such as $\angle 1$ and $\angle 7$ or $\angle 2$ and $\angle 8$ that lie on opposite sides of the transversal and outside of the space between the two parallel lines are **alternate exterior angles**.
- Alternate interior angles are equal and alternate exterior angles are equal when a transversal cuts two parallel lines.
- It is also true that, conversely, if any one of the equalities $\angle 1 = \angle 7$, $\angle 2 = \angle 8$, $\angle 3 = \angle 5$, or $\angle 4 = \angle 6$ is known to hold, then lines a and b must be parallel, and, thus, the other three equalities also hold, as do the four equalities between corresponding angles.

Vertical, Supplementary and Complementary Angles

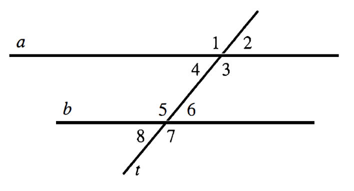
- When two lines cross, as do lines *a* and *t*, then $\angle 1$ and $\angle 3$ are said to be **vertical angles**, as are $\angle 2$ and $\angle 4$.
  Vertical angles are always equal.



- Two angles whose measures sum to 180° are said to be **supplementary**.
- If the sum is 90°, the angles are **complementary**.
- An angle of 180° is a **straight angle**.
- An angle of 90° is a **right angle**.

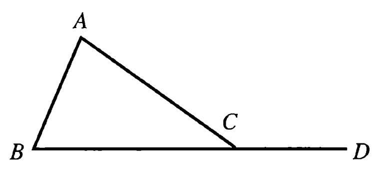
Exterior and Remote Interior Angles

- $\angle 1$ and $\angle 4$ are supplementary. If a and b are parallel, then $\angle 1 = \angle 5$. It follows that $\angle 4$ and $\angle 5$ are supplementary, as are $\angle 3$ and $\angle 6$.



- We can apply some of this to the angles of a triangle.

  Given $\angle ABC$, extend side BC to point D. Then, $\angle ACD$ is said to be an **exterior angle** of the triangle at vertex C. The two angles $\angle A$ and $\angle B$ are the **remote interior angles** with respect to this exterior angle.

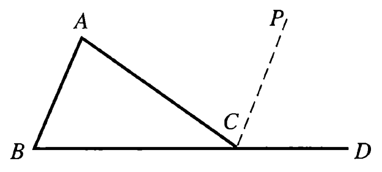


Relating Exterior and Remote Interior Angles

Theorem

An exterior angle of a triangle equals the sum of the two remote interior angles. Also, the sum of all three interior angles of a triangle is 180°.

- We must show $\angle ACD = \angle A + \angle B$. Draw a line CP through C and parallel to AB. Now $\angle A = \angle ACP$ since these are alternate interior angles for parallel lines AB and PC with respect to the transversal AC.



Also, $\angle B = \angle PCD$, since these are corresponding angles. It follows that $\angle ACD = \angle ACP + \angle PCD = \angle A + \angle B$.

Note that $\angle ACD + \angle ACB = \angle BCD = 180°$.

Substituting $\angle A + \angle B$ for $\angle ACD$ in this equation yields the conclusion that the sum of the three interior angles of $\angle ABC$ is 180°.

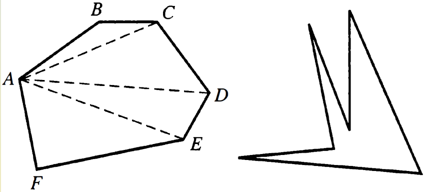
Sum of Interior Angles of an n-Gon

Proposition

The sum of the interior angles of an n-gon equals $180(n-2)^\circ$.

- Consider the case $n = 6$.

  In the left hexagon, we have drawn the three diagonals from vertex A. In general, an n-gon has exactly $n-3$ diagonals terminating at each of its n vertices.

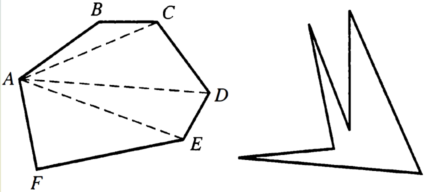


  This gives a total of $\frac{n(n-3)}{2}$ diagonals in all.
- A polygon is **convex** if all of its diagonals lie entirely in the interior.
- The interior angles of a polygon are the angles as seen from inside, and for a convex polygon these angles are all less than 180°.

Sum of Interior Angles of an n-Gon (Cont'd)

- In the right hexagon of the figure, which is not convex,

  we see that two of its six interior angles exceed 180°.
  An angle with measure larger than 180° is said to be a **reflex angle**.

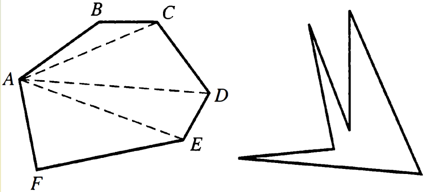


- Suppose we have a convex n-gon such as hexagon $ABCDEF$. Fix some particular vertex A and draw the $n-3$ diagonals from A. This divides the original polygon into exactly $n-2$ triangles. It should be clear that the sum of all the interior angles of all of these triangles is exactly the sum of all interior angles of the original polygon. It follows that the sum of the interior angles of a convex polygon is exactly $180(n-2)°$.

Remarks on the Non-Convex Case

- The definition of a convex polygon requires that all diagonals from all vertices should be interior.
- It is conceivable that every polygon has at least one vertex from which all diagonals are interior, but unfortunately, that is not true.

  Example: The right hexagon is a counterexample;

  at least one diagonal from every one of its vertices fails to be interior.
- It is true, but not easy to prove, that every polygon has at least one interior diagonal. It is possible to use this hard theorem to prove that for every n-gon, the sum of the interior angles is $180(n-2)^\circ$ degrees.

Another Approach to the n-Gon Proposition

- Imagine walking clockwise around a convex polygon, starting from some point other than a vertex on one of the sides.
- Each time you reach a vertex, you must turn right by a certain number of degrees.
  - If the interior angle at the kth vertex is θ_k, then it is easy to see that the right turn at that vertex is a turn through precisely $(180-\theta_k)^\circ$.
- When you return to your starting point, you will be facing in the same direction as when you started, and it should be clear that you have turned clockwise through a total of exactly 360°.
- Thus,
$$\sum_{k=1}^{n}(180-\theta_k)=360.$$
- Since the quantity 180 is added n times in this sum and each quantity θ_k is subtracted once, we see that $180n-\sum_k\theta_k=360$. Hence $\sum_k\theta_k=180n-360=180(n-2)$.

The Walk Argument Does not Require Convexity

- If when walking clockwise around the polygon you reach the kth vertex and see a reflex interior angle there, you actually turn left, and not right.
- In this case, your left turn is easily seen to be through $\theta_k - 180$ degrees, where, as before, θ_k is the interior angle at the vertex.
- If we view a left turn as being a right turn through some negative number of degrees, we see that at the kth vertex we are turning right by $180 - \theta_k$ degrees.
- This is true regardless of whether $\theta_k < 180$ as in the convex case or $\theta_k > 180$ at a reflex-angle vertex.
- It is also clearly true at a straight-angle vertex, where $\theta_k = 180$.
- Thus, this second argument works in all cases, and it shows that $180(n-2)$ is the sum of the interior angles for every polygon, convex or not.

Subsection 4

Parallelograms

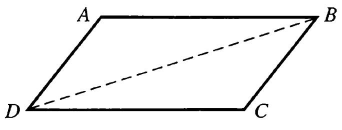
Parallelograms

- A **parallelogram** is a quadrilateral $ABCD$ for which $AB \parallel CD$ and $AD \parallel BC$, i.e., the opposite sides of the quadrilateral are parallel.
- It is also true that the opposite sides of a parallelogram are equal, but this is a consequence of the assumption that the opposite sides are parallel and not part of the definition.

Theorem

Opposite sides of a parallelogram are equal.

- We are given that $AB \parallel CD$ and $AD \parallel BC$. Our task is to show that $AB = CD$ and $AD = BC$.



Draw diagonal BD Note that $\angle ABD = \angle CDB$ since these are alternate interior angles for the parallel lines AB and CD. Similarly, $\angle DBC = \angle ADB$. Since $BD = BD$, we see that $\triangle DAB \cong \triangle BCD$ by ASA. Thus, $AB = CD$ and $AD = BC$.

Sufficient Conditions for Parallelograms

- There are two useful converses of the preceding theorem:

Theorem

In quadrilateral $ABCD$, suppose that $AB = CD$ and $AD = BC$. Then $ABCD$ is a parallelogram.

Theorem

In quadrilateral $ABCD$, suppose that $AB = CD$ and $AB \parallel CD$. Then $ABCD$ is a parallelogram.

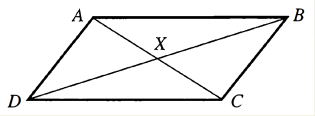
- The proofs are left as exercises.

Parallelograms and Diagonals

Theorem

A quadrilateral is a parallelogram if and only if its diagonals bisect each other.

- Let X be the point where diagonals AC and BD of quadrilateral $ABCD$ cross. Suppose first that X is the common midpoint of line segments AC and BD.



  Then $AX = XC$ and $BX = XD$ and also $\angle AXB = \angle CXD$ because these are vertical angles. It follows that $\triangle AXB \cong \triangle CXD$ by SAS. Thus $AB = CD$. Similarly, $AD = BC$. So $ABCD$ is a parallelogram.
  Conversely, assume that $ABCD$ is a parallelogram. We have $\angle BAX = \angle XCD$ and $\angle ABX = \angle CDX$ because in each case, these are pairs of alternate interior angles for the parallel lines AB and CD. Also, $AB = CD$. Thus, $\triangle ABX \cong \triangle CDX$ by ASA. We deduce that $AX = CX$ and $BX = DX$.

Rhombus and Equidistance Between Two Points

- A **rhombus** is a quadrilateral in which all four sides are equal.
- By a preceding theorem, a rhombus must be a parallelogram.
- In the case of a rhombus, the diagonals not only bisect each other, but they are also perpendicular.

  This is derived as a consequence of the following:

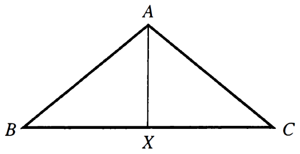
Theorem

Given a line segment BC, the locus of all points equidistant from B and C is the perpendicular bisector of the segment.

- We must show that:
  - every point on the perpendicular bisector of BC is equidistant from B and C;
  - every point that is equidistant from B and C lies on the perpendicular bisector of BC.

Equidistance Between Two Points

- Assume that AX is the perpendicular bisector of BC.



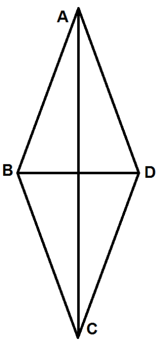
  This means that X is the midpoint of BC and AX is perpendicular to BC. In other words, we are assuming that AX is simultaneously a median and an altitude in $\triangle ABC$, and we want to deduce that $AB = AC$. This is an easy exercise.
- Assume now that A is equidistant from B and C. Draw median AX of $\triangle ABC$. Since $AB = AC$, this triangle is isosceles. Thus, median AX is also an altitude. In other words, AX is the perpendicular bisector of BC, and, of course, A lies on this line.

Diagonals of a Rhombus

Corollary

The diagonals of a rhombus $ABCD$ are perpendicular.

- Since $AB = AD$, we know by the theorem that A lies on the perpendicular bisector of diagonal BD. Similarly, C lies on the perpendicular bisector of BD. But AC is the only line that contains the two points A and C. Thus, AC is the perpendicular bisector of BD. In particular, diagonal AC is perpendicular to diagonal BD.



Rectangles and Squares

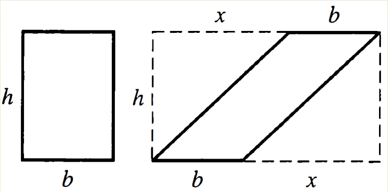
- A **rectangle** is a quadrilateral all of whose angles are right angles.
- It is easy it see that the opposite sides of a rectangle are parallel, and so a rectangle is automatically a parallelogram.
- We also know that the sum of the four angles of an arbitrary quadrilateral is 360°.
- Thus, any quadrilateral with all four angles equal, each angle must be 90°, and the figure is a rectangle.
- We remark also that adjacent vertices of a parallelogram have supplementary interior angles.
- It follows easily that if one angle of a parallelogram is a right angle, then the parallelogram must be a rectangle.
- A **square** is a quadrilateral that is both a rectangle and a rhombus.

Subsection 5

Area

Area of a Rectangle, Base and Height

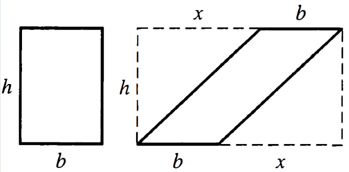
- The area of a geometric figure is often denoted K.
- We assume known the formula for the area of a rectangle $K = bh$, where b is the length of one of the sides of the rectangle and h is the length of the perpendicular sides.
- A side of length b is referred to as the **base** of the rectangle, and the **height** h is the length of the sides perpendicular to the base.
- Suppose that we have a parallelogram that is not necessarily a rectangle:



  - We designate one side of the parallelogram as the **base** and write b to denote its length.
  - But in this case, the **height** h is the perpendicular distance between the two parallel sides of length b.

Area of a Parallelogram

Claim: The parallelogram and the rectangle shown have equal areas.



Drop perpendiculars from one end of each of the sides of length b to the extensions of the opposite sides.

This results to a rectangle with base $b+x$ and height h, where x represents the amount that the base of the parallelogram had to be extended to meet the perpendicular.

The area of this rectangle is $(b+x)h$.

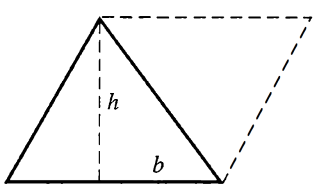
To obtain the area of the original parallelogram, we need to subtract from this the area of the two right triangles.

The two right triangles pasted together would form a rectangle with base x and height h. So the total area of the two triangles is xh.

The area of the parallelogram is therefore $(b+x)h - xh = bh$.

So the parallelogram and the rectangle have equal areas.

Areas of Triangles

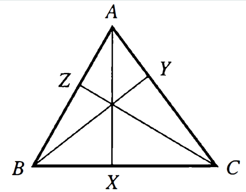


- Consider a triangle with base b and height h, where any one of the three sides can be viewed as the base and the length of the altitude drawn to that side is the corresponding height.
- To compute the area K, we have constructed a parallelogram by drawing lines parallel to our base and to one of the other sides of the triangle.
- This parallelogram has base b and height h, so its area is bh.
- By SSS, the parallelogram is divided into two congruent triangles by a diagonal.
- So the area of each of these triangles is exactly half the area of the parallelogram.
- Thus, the original triangle has area $K = \frac{1}{2}bh$.

Consequence for Isosceles Triangles

- By taking any one of the three sides as the base, we obtain three different formulas for the the area of a triangle $\triangle ABC$:

$$K_{ABC} = \frac{1}{2}AX \cdot BC = \frac{1}{2}BY \cdot AC = \frac{1}{2}CZ \cdot AB.$$



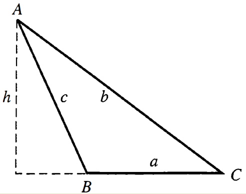
Claim: A triangle $\triangle ABC$ is isosceles with base BC if and only if the two altitudes BY and CZ are equal.

By the area formula, $BY \cdot AC = CZ \cdot AB$, since each of these quantities equals twice the area of the triangle. We can cancel the equal quantities AC and AB to obtain $BY = CZ$.

Conversely, suppose that altitudes BY and CZ are equal. Since $BY \cdot AC = CZ \cdot AB$, we get that $AB = AC$.

Area in terms of Sines and Law of Sines

- Customarily, we use the symbols a, b and c to denote the lengths of the sides of $\triangle ABC$ opposite vertices A, B and C, respectively.
- Draw the altitude of length h from A. We have $K = \frac{1}{2}ah$. We see that $\sin(C) = \frac{h}{b}$. Hence, $h = b\sin(C)$. If we substitute this into the area formula $K = \frac{1}{2}ah$, we obtain $K = \frac{1}{2}ab\sin(C)$.



  Similarly, $K = \frac{1}{2}ac\sin(B)$ and $K = \frac{1}{2}bc\sin(A)$.
- Thus, for any triangle, the following equations always hold:

$$ab\sin(C) = bc\sin(A) = ca\sin(B).$$

- If we divide by abc and take reciprocals, we get the **law of sines**:

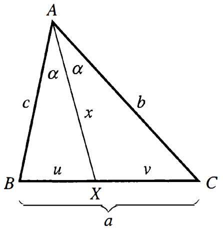
$$\frac{c}{\sin(C)} = \frac{a}{\sin(A)} = \frac{b}{\sin(B)}.$$

Angle Bisector and Opposite Segments

Theorem

Let AX be the bisector of $\angle A$ in $\triangle ABC$. Then $\frac{BX}{XC} = \frac{AB}{AC}$.
In other words, X divides BC into pieces proportional to the lengths of the nearer sides of the triangle.

- Let u and v be the lengths of BX and XC, respectively. Let h be the height of $\triangle ABC$ with respect to the base BC. Then h is also the height of each of $\triangle ABX$ and $\triangle ACX$ with respect to bases BX and XC, respectively. We have $\frac{1}{2}uh = K_{ABX} = \frac{1}{2}cx\sin(\alpha)$ and, also,



$\frac{1}{2}vh = K_{ACX} = \frac{1}{2}bx\sin(\alpha)$, where $x = AX$ and $\alpha = \frac{1}{2}\angle A$. Division of the first of these equations by the second yields $\frac{u}{v} = \frac{c}{b}$.

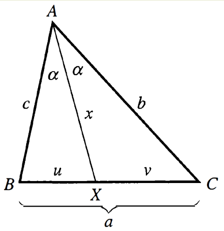
Coincidence of Median and Bisector Implies Isosceles

- As an application, we have the following proposition:

Proposition

Suppose that in $\triangle ABC$, the median from vertex A and the bisector of $\angle A$ are the same line. Then $AB = AC$.

We have $u = v$ since the angle bisector AX is assumed to be a median. Since, by the preceding theorem, $\frac{u}{v} = \frac{c}{b}$, we get that $c = b$.



Review of Formulas and Heron's Formula

- The first type of formula for the area of a triangle uses one side and an altitude:
$$K_{ABC} = \frac{1}{2} a h_a.$$
- The second type uses two sides and an angle:
$$K_{ABC} = \frac{1}{2} bc \sin(A).$$
- SSS asserts that a triangle is determined by its three sides.

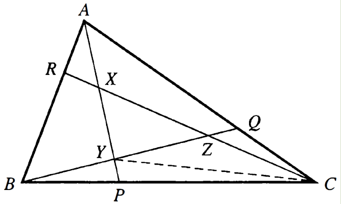
  Thus, there should be a nice way to compute the area of a triangle in terms of the lengths of its sides:
$$K_{ABC} = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \textbf{Heron of Alexandria}$$
  where $s = \frac{1}{2}(a+b+c)$ is called the **semiperimeter** of the triangle.

Subdividing Sides of a Triangle Into Thirds

Proposition

Points P, Q and R lie on the sides of $\triangle ABC$. Point P lies one third of the way from B to C, point Q lies one third of the way from C to A, and point R lies one third of the way from A to B.

Line segments AP, BQ and CR subdivide the interior of the triangle into three quadrilaterals and four triangles. The area of the only small triangle having no vertex in common with $\triangle ABC$ is exactly one seventh of the area of the original triangle.

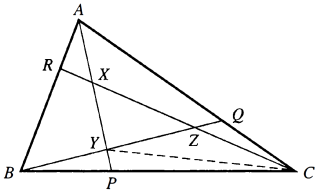


- We need to compute the area K_{XYZ}.

  Let $K_{ABC} = a$. Let, also $K_{BYP} = k$.

  We draw line segment YC and start computing areas.

Subdividing Sides Into Thirds (Cont'd)



- Since $\triangle CYP$ has the same height as $\triangle BYP$ but its base PC is twice as long, we deduce that $K_{CYP} = 2k$. Similarly, since $AQ = 2QC$, we see that $K_{ABQ} = 2K_{CBQ}$. Thus, $K_{CBQ} = \frac{1}{3}K_{ABC} = \frac{1}{3}a$.

  So $K_{CYQ} = \frac{1}{3}a - K_{BYC} = \frac{1}{3}a - 3k$. We also have $K_{AYQ} = 2K_{CYQ} = \frac{2}{3}a - 6k$. We know that $K_{ABQ} = \frac{2}{3}K_{ABC} = \frac{2}{3}a$. Hence, $K_{ABY} = \frac{2}{3}a - K_{AYQ} = \frac{2}{3}a - (\frac{2}{3}a - 6k) = 6k$. However, $K_{ABP} = \frac{1}{3}K_{ABC} = \frac{1}{3}a$. Thus, $K_{BYP} = \frac{1}{3}a - 6k$. But we know that $K_{BYP} = k$. Hence $\frac{1}{3}a - 6k = k$ and $k = \frac{1}{21}a$.

  Similar reasoning shows that $K_{ARX} = \frac{1}{21}a = K_{CQZ}$. Since $K_{ARC} = \frac{1}{3}K_{ABC} = \frac{1}{3}a$, we deduce that the area of quadrilateral $AXZQ$ is $\frac{1}{3}a - \frac{2}{21} = \frac{5}{21}a$. Finally, we recall that $K_{AYQ} = \frac{2}{3}a - \frac{6}{21}a = \frac{8}{21}a$, and it follows that $K_{XYZ} = \frac{8}{21}a - \frac{5}{21}a = \frac{1}{7}a$.

Subsection 6

Circles and Arcs

Circles

- A **circle** is the locus of all points equidistant from some given point called the **center**.
- The common distance r from the center to the points of the circle is the **radius**.
- The word **radius** is also used to denote any one of the line segments joining the center to a point of the circle.
- A **chord** is any line segment joining two points of a circle.
- A **diameter** is a chord that goes through the center.
- The length d of any diameter is given by $d = 2r$, and this is the maximum of the lengths of all chords.
- Any two circles with equal radii are **congruent**;

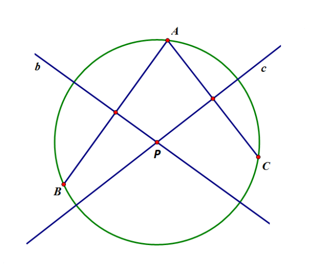
  Any point on one of two congruent circles can be made to correspond to any point on the other circle.

Three Non-collinear Points Determine a Circle

Theorem

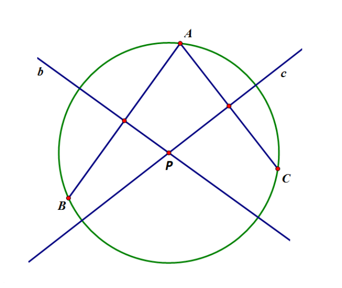
There is exactly one circle through any three given non-collinear points.

- Call the points A, B and C. Since by hypothesis, there is no line through these points, we can be sure that we are dealing with three distinct points, and we draw line segments AB and AC. Let b and c be the perpendicular bisectors of these segments. Lines b and c cannot be parallel because:



  - AB and AC are neither parallel (AB and AC have point A in common)
  - nor are they the same line (A, B and C are non-collinear).

Three Non-collinear Points Determine a Circle (Cont'd)



- Let P be the point where lines b and c meet. Since P lies on the perpendicular bisector of AB, we know that P is equidistant from A and B. In other words, $PA = PB$. Similarly, since P lies on line c, we deduce that $PA = PC$. If we let r denote the common length of the three segments PA, PB and PC, we see that the circle of radius r centered at P goes through the three given points.

Uniqueness of the Circle

- We show next that the three points cannot lie on any other circle:

  If a circle centered at some point Q, say, goes through A, B and C, then Q is certainly equidistant from A and B. Hence, it lies on the perpendicular bisector b of segment AB.

  Similarly, we see that Q lies on line c.

  Thus, $Q = P$, because P is the only point common to the two lines.

  Since the distance $PA = r$, it follows that the only circle through A, B and C is the circle of radius r centered at P.

Circumcircles and Inscribed n-Gons

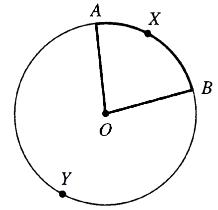
- Given $\triangle ABC$, the unique circle that goes through the three vertices is called the **circumcircle** of the triangle.

  The triangle is said to be **inscribed** in the circle.
- More generally, any polygon all of whose vertices lie on some given circle is referred to as being **inscribed** in that circle.
- The circle is **circumscribed** about the polygon.
- Every triangle is inscribed in some circle, but the same cannot be said for n-gons when $n > 3$.

Arcs

- Two points A and B on a circle divide the circle into two pieces, each of which is called an **arc**.
- We write $\widehat{AB}$ to denote one of these two arcs, usually the smaller.
- The ambiguity makes a three-point designation preferable.

  Example:

  The smaller of the two arcs determined by points A and B would be designated $\widehat{AXB}$ and the larger is $\widehat{AYB}$.
- The ambiguity in the notation $\widehat{AB}$ is related to a similar ambiguity in the notation for angles.

  Example: If we write $\angle AOB$, we generally mean the angle including point X in its interior and not the reflex angle.

Measuring Arcs

- The most common way to measure the size of an arc is in terms of the fraction of the circle it is, where the whole circle is taken to be 360° or 2π radians.
- An arc extending over a quarter of the circle, therefore, is referred to as a 90° arc, and we would write $\widehat{AB} \stackrel{\circ}{=} 90°$ or $\widehat{AB} \stackrel{\circ}{=} \frac{\pi}{2}$ radians in this case.
- This size description for an arc is meaningful only relative to the circle of which it is a part.

  If we are told that we have two 90° arcs, we cannot say that they are congruent or that they have equal length unless we know that these are two arcs of the same circle or of two circles having equal radii.
- To remind us that the number of degrees (or radians) that we assign to an arc gives only relative information, we use the symbol $\stackrel{\circ}{=}$, which we read as "equal in degrees (or radians)", and we avoid the use of $=$ in this context.

Central Angles

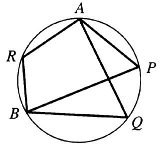
- Given an arc $\overset{\frown}{AB}$ on a circle centered at point O, we say that $\angle AOB$ is the **central angle** corresponding to the arc.
- Since a full circle is $360°$ of arc and one full rotation is $360°$ of angle, it should be clear that the number of degrees in the measure of central angle $\angle AOB$ is equal to the number of degrees in $\overset{\frown}{AB}$.

  Example: A $90°$ angle at the central point O cuts off a quarter circle, which is a $90°$ arc.
- The termnology for the phrase "cuts off" is **subtends**.
- In general, we can write $\angle AOB \overset{\circ}{=} \overset{\frown}{AB}$, i.e., a central angle is equal in degrees to the arc it subtends.

  We can also say that the arc is **measured** by the central angle.

Inscribed Angles

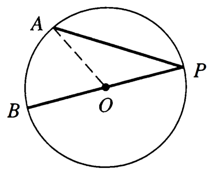
- In a given circle, the angle formed by two chords that share an endpoint is called an **inscribed angle**.
  Example: Some of the inscribed angles in the figure are $\angle APB$, $\angle AQB$ and $\angle ARB$.

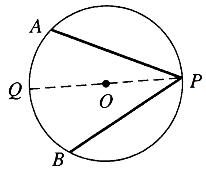


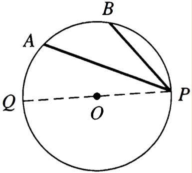
Theorem

An inscribed angle in a circle is equal in degrees to one half its subtended arc. Equivalently, the arc subtended by an inscribed angle is measured by twice the angle.

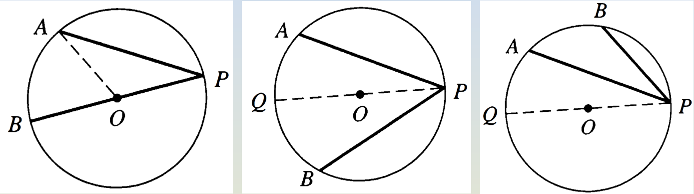
- Given $\angle APB$ inscribed in a circle centered at point O, the three cases we need to consider are:







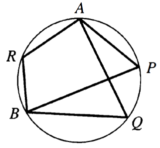
Proof of the Inscribed Angle Theorem



- We deal with each of the three cases:
  - O falls on one of the sides of $\angle APB$: Suppose O lies on PB. Draw radius AO and observe that $\triangle AOP$ is isosceles with base AP. So by the pons asinorum, $\angle A = \angle P$. Central $\angle AOB$ is an exterior angle of $\triangle AOP$. Hence, it is equal to the sum of the two remote interior angles. Thus, $\overset{\frown}{AB} \mathrel{\hat{=}} \angle AOB = \angle A + \angle P = 2\angle P$.
  - O lies in the interior of the angle: Draw diameter PQ. By the part of the theorem that we have already proved, we know that $\angle APQ \mathrel{\hat{=}} \frac{1}{2}\overset{\frown}{AQ}$ and $\angle QPB \mathrel{\hat{=}} \frac{1}{2}\overset{\frown}{QB}$. Adding these equalities gives $\angle APB \mathrel{\hat{=}} \frac{1}{2}\overset{\frown}{AQB}$.
  - O is exterior to the angle: Draw diameter PQ. We get $\angle APQ \mathrel{\hat{=}} \frac{1}{2}\overset{\frown}{AQ}$ and $\angle QPB \mathrel{\hat{=}} \frac{1}{2}\overset{\frown}{QB}$. Subtraction yields the result.

Applying the Inscribed Angle Theorem

Example: $\angle APB \doteq \frac{1}{2}\widehat{ARB}$ and $\angle AQB \doteq \frac{1}{2}\widehat{ARB}$. So we have $\angle APB = \angle AQB$.



In general, any two inscribed angles that subtend the same arc in a circle are equal.

- Now consider $\angle ARB$. This too is an inscribed angle formed by chords through A and B, but $\angle ARB$ is not necessarily equal to the other two angles because it subtends the other arc determined by A and B.

Corollary

Opposite angles of an inscribed quadrilateral are supplementary.

We have $\angle ARB \doteq \frac{1}{2}\widehat{APB}$: So we get

$\angle ARB + \angle APB = \frac{1}{2}(\widehat{APB} + \widehat{ARB}) = \frac{1}{2}360° = 180°$.

Thus, $\angle ARB$ and $\angle AQB$ are supplementary.

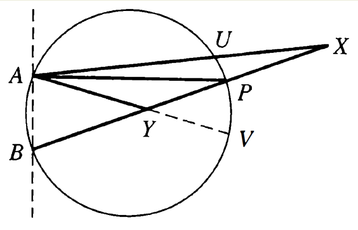
Angle Between Two Secants

- A line segment that extends a chord beyond a circle is called a **secant**.

Corollary

The angle between two secants drawn to a circle from an exterior point is equal in degrees to half the difference of the two subtended arcs.

- Consider point X, outside a given circle. $\angle APB$ is an exterior angle of $\triangle APX$. Thus, $\angle AXB = \angle APB - \angle XAP$. Notice that $\angle XAP = \angle UAP \overset{\circ}{=} \frac{1}{2}\widehat{UP}$. We see that $\angle AXB \overset{\circ}{=} \frac{1}{2}(\widehat{AB} - \widehat{UP})$.

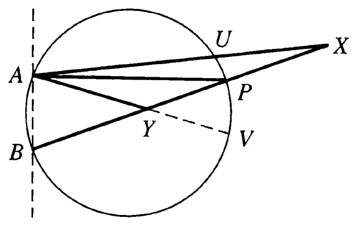


Angle Between Two Chords

Corollary

The angle between two chords that intersect in the interior of a circle is equal in degrees to half the sum of the two subtended arcs.

- Consider point Y, inside a given circle. $\angle AYB$ is an exterior angle of $\triangle AYP$. Thus, $\angle AYB - \angle APB = \angle PAY \stackrel{\circ}{=} \frac{1}{2}\overparen{PV}$. Thus, $\angle AYB \stackrel{\circ}{=} \frac{1}{2}(\overparen{AB} + \overparen{PV})$.

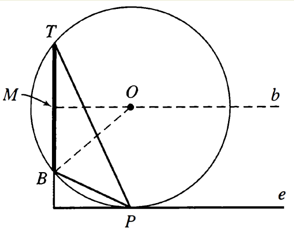


The Art Lover Problem

Proposition

A 6-foot-tall rectangular painting is hung high on a wall, with its bottom edge 7 feet above the floor. An art lover whose eyes are 5 feet above the floor wants as good a view as possible, and so she wants to maximize the angular separation from her eye to the top and the bottom of the painting. Show that she should stand 4 feet away from the wall.

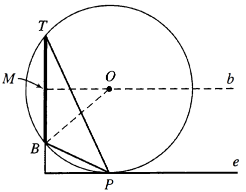
- Horizontal line e represents the possible positions of the viewer's eye, 5 feet above the floor. Line TB is the wall on which the picture is hung, and T and B represent, respectively, the top and bottom of the picture. We seek a point P on line e that maximizes $\angle BPT$.

  Point B is 2 feet above e and T is 8 feet above e.

The Art Lover Problem (Cont'd)

- The midpoint M of TB is thus at the average height $\frac{2+8}{2} = 5$ feet above e. Draw the perpendicular bisector b of TB so that b is horizontal and 5 feet above e and choose point O on b so that $OB = 5$. Draw the circle of radius 5 centered at O, tangent to e at some point P.



Claim: Point P solves the problem.

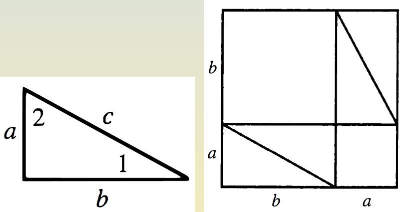
Every other point on e lies outside the circle and thus "sees" the picture TB with a smaller angle than does P. Thus, $\angle BPT$ is the maximum we seek.

We now must find how far point P is from the wall. This distance is equal to OM, and so we examine the right triangle $\triangle OMB$. We know that hypotenuse $OB = 5$ because OB is a radius of the circle. Also, $MB = 3$. By the Pythagorean theorem, we see that $OM = 4$.

The Pythagorean Theorem

Theorem (Pythagoras)

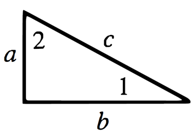
If a right triangle has arms of lengths a and b and its hypotenuse has length c, then $a^2 + b^2 = c^2$.

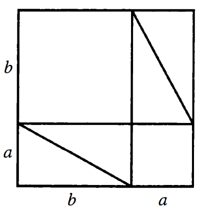
- 

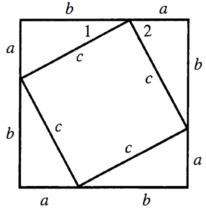
  The given triangle is on the left:. The right diagram shows a square of side $a + b$. Each of the right triangles has arms of length a and b, and so each is congruent to the original triangle by SAS.

  The two smaller squares have side lengths a and b, and so the area remaining in the big square of side $a + b$ when four copies of our given triangle are removed is exactly $a^2 + b^2$.

The Pythagorean Theorem (Cont'd)







- On the right, each of the right triangles is congruent to the given one by SAS. The quadrilateral of side c is a rhombus. The angle of the rhombus together with $\angle 1$ and $\angle 2$ make a straight angle of 180°. In the original triangle, we know that $\angle 1$, $\angle 2$, and a right angle sum to 180°. It follows that the angle of the rhombus is 90°. So the rhombus is a square. Its side length is exactly c. It follows that the area remaining when four copies of our triangle are removed from a square of side $a + b$ is c^2.

  By the preceding slide $a^2 + b^2 = c^2$.

Right Inscribed Angles

- We look at a special case of the inscribed angle theorem:

Corollary

Given $\triangle ABC$, the angle at vertex C is a right angle if and only if side AB is a diameter of the circumcircle.

- We know that the circumcircle exists. In this circle, $\widehat{AB}$ is measured by $2\angle C$. Here, $\widehat{AB}$ denotes the arc not containing C that these points determine. Chord AB is a diameter precisely when $\widehat{AB} \circeq 180°$, or equivalently, when $\angle C = 90°$.
- This is an example of an "if and only if" statement:
  - The "if part" of the statement asserts that if AB is a diameter, then $\angle C$ is a right angle.
  - The "only if part" tells us that if $\angle C$ is a right angle, then AB must be a diameter.

Acute and Obtuse Angles and Tangents

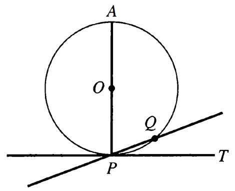
- Any line segment AB, defines a unique circle having AB as a diameter (the circle centered at the midpoint of AB and having radius $\frac{1}{2}AB$).
  - Thus, $\angle C$ is a right angle in $\triangle ABC$ if and only if point C lies on the unique circle having side AB as a diameter.
  - $\angle C < 90°$ if C lies outside of this circle.
  - $\angle C > 90°$ if C is in the interior.
- Angles smaller than $90°$ are said to be **acute**.
- Angles between $90°$ and $180°$ are **obtuse**.

Tangents to a Circle

- A line is **tangent** to a circle if it meets the circle in exactly one point.
- Through every point P on a circle, there is a unique tangent line, which is necessarily perpendicular to the radius terminating at P.

  Consider the tangent line PT. We want to compute $\angle APT$, where AP is the diameter that extends radius OP.

  Choose a point Q on the circle near P and draw the secant line PQ. If we move Q closer and closer to P, we see that $\angle APT$ is the limit of $\angle APQ \stackrel{\circ}{=} \frac{1}{2}\widehat{AQ}$. But as Q approaches P, we observe that $\widehat{AQ}$ approaches $180°$ since $\widehat{AQP}$ is a semicircle. It follows that $\angle APQ$ approaches $90°$. So AP is perpendicular to the tangent.

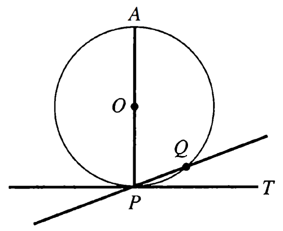


Measure of Angle Formed by Tangent and Chord

Theorem

The angle between a chord and the tangent at one of its endpoints is equal in degrees to half the subtended arc.

$\angle QPT$ between chord PQ and tangent PT is the complement of $\angle APQ$. Thus, we have: $\angle QPT = 90° - \angle APQ \stackrel{\circ}{=} 90° - \frac{1}{2}\widehat{AQ} \stackrel{\circ}{=} \frac{1}{2}(180° - \widehat{AQ}) \stackrel{\circ}{=} \frac{1}{2}\widehat{PQ}$.



Corollary

The angle between a secant and a tangent meeting at a point outside a circle is equal in degrees to half the difference of the subtended arcs.

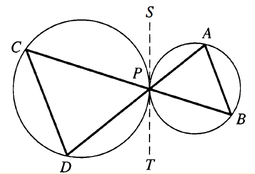
Mutually Tangent Circles

- Two circles are said to be **mutually tangent** at a point P if P lies on both circles and the same line through P is tangent to both circles .
- Mutual tangency can happen:
  - externally: circles on opposite sides of the tangent line;
  - internally, circles are on the same side of the tangent and one circle is inside the other.

Proposition

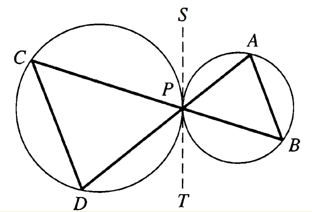
Given two externally mutually tangent circles with common point P, draw two common secants AD and BC through P. Then AB and CD are parallel.

Since BC is a transversal to the two lines AB and CD, it suffices to show that the alternate interior angles $\angle B$ and $\angle C$ are equal.



Mutually Tangent Circles (Cont'd)

- Since BC is a transversal to the two lines AB and CD, it suffices to show that the alternate interior angles $\angle B$ and $\angle C$ are equal. Draw the common tangent ST and note that $\angle DPT \stackrel{\circ}{=} \frac{1}{2}\widehat{PD}$.



Also $\angle C \stackrel{\circ}{=} \frac{1}{2}\widehat{PD}$. It follows that $\angle C = \angle DPT$. Similarly, $\angle B = \angle APS$. But $\angle DPT$ and $\angle APS$ are vertical angles. So they are equal. Hence $\angle C = \angle B$.

Subsection 7

Polygons in Circles

Regular n-Gons

- A polygon is said to be **regular** if all of its sides are equal and also all of its angles are equal.

  Example:

  - An equilateral triangle has equal sides, by definition.
    By two applications of the pons asinorum, all three angles must be equal too.
    So an equilateral triangle is a regular 3-gon.
  - An equilateral 4-gon is a rhombus.
    A rhombus need not have equal angles, and so it is not necessarily a regular polygon.
    A square, however, is a regular 4-gon.

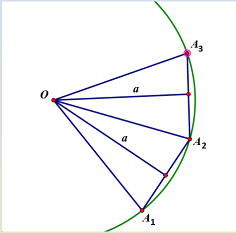
Regular Inscribed n-Gons

- For $n \geq 3$, a regular n-gon can be drawn by marking n equally spaced points around a circle and then drawing the n chords connecting consecutive points.
  - The n chords are equal in length.
  - The n angles are all equal:
    Each of the n arcs is clearly equal in degrees to $\frac{360}{n}$ degrees.
    Each angle of the polygon subtends an arc consisting of $n-2$ of these small arcs.
    So each of these angles is equal to

$$\frac{1}{2}(n-2)(\frac{360}{n}) = \frac{180(n-2)}{n} \text{ degrees.}$$

Area of a Regular n-Gon

- Draw the n radii joining the center of the circle to the n equally spaced points. This subdivides the interior of the n-gon into n isosceles triangles with equal bases of length s, the common side length of the polygon. These n triangles are all congruent by SSS.



  Thus, the lengths of the altitudes of these triangles (drawn from the center of the circle) are all equal.
- Any one of these altitudes is said to be an **apothegm** of the regular polygon, and we write a to denote their common length.
- Since the area of each of the isosceles triangles is $\frac{1}{2}sa$, the area of the entire regular n-gon is

$$\frac{1}{2}nsa = \frac{1}{2}pa,$$

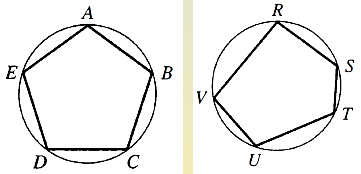
  where $p = ns$ is the perimeter of the polygon.

Maximization of the Area

Proposition

For an integer $n \geq 3$, and a circle, the regular n-gon is the n-gon that has the maximum area among all n-gons with points on the circle.

- The following figure illustrates the case $n = 5$.



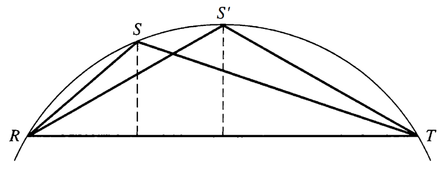
In the left diagram, $\widehat{AB}, \widehat{BC}, \widehat{CD}, \widehat{DE}$ and $\widehat{EA}$ are all equal. Thus, each is 72°. The circle in the right diagram has an equal radius, but the five points are placed so that not all of the arcs are equal.

We need to show that the area of the regular pentagon $ABCDE$ is strictly greater than that of pentagon $RSTUV$.

Proof of the Area Maximization

Claim: Given any n-gon inscribed in a circle and having arcs that are not all equal, there exists another n-gon with larger area inscribed in the same circle.

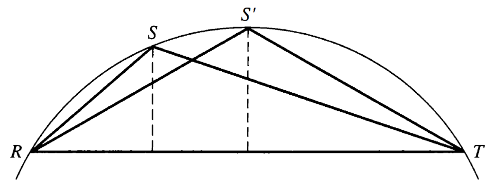
Since the arcs are not all equal, we can find two consecutive unequal arcs. Hence, we can find three consecutive vertices R, S and T of our n-gon where $\widehat{RS}$ and $\widehat{ST}$ are unequal.



We show that it is possible to move point S, leaving all of the remaining $n-1$ points fixed, so that the area is increased.

The area of the whole polygon can be viewed as the area of $\triangle RST$ plus the area of the part that lies on the other side of chord RT. We are assuming that S is not the midpoint of $\widehat{RST}$, and we label the midpoint S'.

Proof of the Area Maximization (Cont'd)



- The perpendicular distance h from S to line RT is less than the distance h' from S' to RT. As h is the height of $\triangle RST$ with respect to the base RT and h' is the height of $\triangle RS'T$ with respect to the same base, it follows that the area of $\triangle RST$ is less than that of $\triangle RS'T$. If we move point S to S', the effect is to increase the area of the triangle without affecting the rest of the polygon, and the effect on the area of the whole polygon is thus an increase. This shows that an n -gon inscribed in a circle where the arcs are unequal cannot have the maximum possible area.

Existence of Maximum Area Inscribed n-Gon

- We must also show that among all possible n-gons inscribed in a given circle, there is one for which the area is a maximum.
- We use a general principle called **compactness**.
- From calculus, if $f(x)$ is a function of a real variable x, defined for $a \leq x \leq b$, and $f(x)$ is continuous in this interval, then the function necessarily takes on a maximum value at some point c in the interval.
- Similarly, a continuous function, even of several variables, takes on a maximum value if each variable runs over a closed and bounded set.
- We think of the area of an n-gon inscribed in a circle as a function of n variables consisting of the n-points.
  - This function is continuous since small perturbation in the locations of the points results in at most a small change in the area.
  - Each point is required to lie on our circle, which as we shall see, is closed and bounded.

It follows that for some choice of n points on the circle, the area function takes on a maximum value.

Bounded Sets on the Plane

- A set of points is **bounded** if it is contained in the interior of some circle, possibly a very large circle.

  Example:
  - Bounded Sets:
    - A line segment;
    - A circle;
    - The interior of a circle.
  - Unbounded Sets:
    - A line;
    - The exterior of a circle;
    - The interior of an angle.

Points Adjacent to a Set

- Given a set $\mathscr{S}$ of points in the plane, we say that a point P is **adjacent** to $\mathscr{S}$ if every circle centered at P, no matter how small, contains at least one point of $\mathscr{S}$ in its interior.
- If P is actually a member of $\mathscr{S}$, then P is adjacent to $\mathscr{S}$.
- It is also possible for a point to be adjacent to a set without actually being in the set.

  Example: An endpoint of a line segment is not in the interior of the segment, but it is adjacent to the interior.

Closed Sets on the Plane

- A plane set is **closed** if every point adjacent to the set is actually a member of the set.
  - The interior of a circle is not a closed set because the points of the circle are adjacent and yet are not in the set.
  - The disk formed by a circle together with its interior is a closed set.
  - The circle itself is a closed set.
  - A line is a closed set.
  - A line segment including the endpoints is a closed set.
  - A line segment without its endpoints is not closed.
  - A circle with one point deleted is not closed.

Compact Sets on the Plane

- A set that is both closed and bounded is said to be **compact**.
- If we take into account the theorem that a real-valued continuous function of several variables, each of which runs over a compact set, attains a maximum and also a minimum value, then the n-gon area maximization problem is solved.

  This is because the domain of choice for each point is a circle, which is a compact set.

The Constant π

- The number $\pi = 3.1415\ldots$ is by definition the ratio of the perimeter, also called the circumference, of a circle to its diameter.
- This ratio is the same for all circles, independent of size.
- The number π is also involved in the formula $K = \pi r^2$ giving the area of a circle in terms of its radius r.

  Fix a circle of radius r and let K_n denote the area of a regular n-gon inscribed in the circle. The area K of the circle is the limit of the polygon areas K_n as $n \to \infty$. We have seen that $K_n = \frac{1}{2} p_n a_n$, where p_n and a_n are, respectively, the perimeter and apothegm of a regular n-gon inscribed in our given circle. Observe that as n gets large, p_n approaches the circumference $c = 2\pi r$ of the circle and a_n approaches the radius r. Thus, $K = \lim_{n\to\infty} K_n = \frac{1}{2}(\lim_{n\to\infty} p_n)(\lim_{n\to\infty} a_n) = \frac{1}{2}(2\pi r)(r) = \pi r^2$.
- The surface area and volume of a sphere in terms of its radius also involve the seemingly ubiquitous number π: $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$.

Algebraic Numbers

- Recall that the decimal expansions of **rational** numbers either terminate or eventually repeat, e.g., $\frac{1}{3} = 0.\overline{3}$ and $\frac{2}{7} = 0.\overline{285714}$.
- The number π is irrational, and its decimal expansion never repeats.
- The same can be said of numbers such as $\sqrt{2} = 1.4142\ldots$, but in a certain sense, π is even more unlike numbers such as $\sqrt{2}$.
- A **polynomial** is an expression of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where the constants a_i are called the **coefficients** of the polynomial $f(x)$ and we assume that $a_n \neq 0$.
- A number r is said to be a **root** of the polynomial $f(x)$ if we get 0 when we plug in r in place of x.

  Example: The number $r = \sqrt{2}$ is a root of the polynomial $f(x) = x^2 - 2$. Note that the coefficients of this polynomial are $a_2 = 1, a_1 = 0$ and $a_0 = -2$ and they are all integers.
- A number r is said to be **algebraic** if it is a root of some polynomial with integer coefficients.

Transcendental Numbers

- A number is **transcendental** if it is not algebraic, i.e., if it is not a root of any polynomial with integer coefficients.
- π is transcendental, as are "most" numbers, but it is unusual to have in hand a particular number such as π or $e = 2.7182\ldots$ that is actually known to be transcendental.
- There are many formulas that can be proved to give π exactly, even though it is not the root of any polynomial equation with integer coefficients:

$$\pi = 4\int_0^1 \frac{1}{1+x^2}dx; \quad \pi = \sqrt{6\sum_{n=1}^{\infty}\frac{1}{n^2}}; \quad \pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

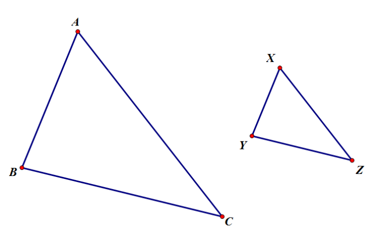
Subsection 8

Similarity

Similar Triangles

- Similarity is a weaker condition than congruence:
  - Similarity requires the "same shape".
  - Congruence requires both the "same shape" and the "same size".
- Two triangles are **similar** if the three angles of one are equal to the three angles of the other.

  Example: If we are given $\triangle ABC$ and $\triangle XYZ$ and we know that $\angle A = \angle X$, $\angle B = \angle Y$ and $\angle C = \angle Z$,



then the two triangles are similar and we write $\triangle ABC \sim \triangle XYZ$.

The AA Criterion for Similarity

Theorem

Given $\triangle ABC$ and $\triangle XYZ$, suppose $\angle A = \angle X$ and $\angle B = \angle Y$. Then $\angle C = \angle Z$, and so $\triangle ABC \sim \triangle XYZ$.

- Since the sum of the angles of a triangle is 180°, we have

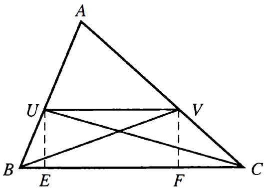
$$\angle C = 180^\circ - (\angle A + \angle B) = 180^\circ - (\angle X + \angle Y) = \angle Z.$$

- When the theorem is used to prove that two triangles are similar, we say that the triangles are similar by AA (“angle-angle").

Parallelism and Proportionality

Lemma

Let V and U be points on sides AB and AC of $\triangle ABC$. Then $UV \parallel BC$ if and only if $\frac{AU}{AB} = \frac{AV}{AC}$.

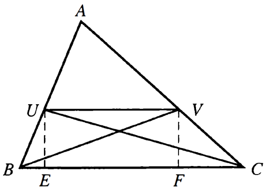


- Write $\alpha = \frac{UB}{AB}$ and $\beta = \frac{VC}{AC}$. Then $AU = AB - VB = (1-\alpha)AB$, and, similarly, $AV = (1-\beta)AC$. Thus, $\frac{AU}{AB} = 1-\alpha$ and $\frac{AV}{AC} = 1-\beta$.
  It follows that the ratios $\frac{AU}{AB}$ and $\frac{AV}{AC}$ are equal if and only if $\alpha = \beta$.
  It suffices now to show that $\alpha = \beta$ if and only if $UV \parallel BC$.

Parallelism and Proportionality

- We show $\alpha = \beta$ if and only if $UV \parallel BC$.

  Draw CU and compare the area of $\triangle BUC$ with that of $\triangle ABC$. Viewing AB and UB as bases, we see that these triangles have equal heights. Thus, $\frac{K_{BUC}}{K_{ABC}} = \frac{UB}{AB} = \alpha$. So $K_{BUC} = \alpha K_{ABC}$. Similarly, $K_{BVC} = \beta K_{ABC}$. It follows that $\alpha = \beta$ if and only if $K_{BUC} = K_{BVC}$.



  We need to show $K_{BUC} = K_{BVC}$ if and only if $UV \parallel BC$.

  Since $\triangle BUC$ and $\triangle BVC$ share base BC, their areas are equal if and only if they have equal heights UE and VF.

  So we must show that $UE = VF$ if and only if $UV \parallel EF$.

  Observe that UE and VF are parallel since each of these lines is perpendicular to BC. If $UE = VF$, it follows by a previous theorem that $UVFE$ is a parallelogram, and thus $UV \parallel EF$. Conversely, if $UV \parallel EF$, then $UVEF$ is a parallelogram by definition, and so $UE = VF$.

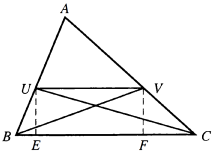
Similarity Implies Proportionality

Theorem

If $\triangle ABC \sim \triangle XYZ$, then the lengths of the corresponding sides of these two triangles are proportional.

- We show $\frac{XY}{AB} = \frac{XZ}{AC}$. By similar reasoning $\frac{XY}{AB} = \frac{YZ}{BC}$. Thus, all three ratios are equal and the sides are proportional.
  - If $XY = AB$, then $\triangle ABC \cong \triangle XYZ$ by ASA. In this case, $XZ = AC$. So $\frac{XY}{AB} = 1 = \frac{XZ}{AC}$.
  - We can suppose that XY and AB are unequal. Say XY is the shorter of these two segments.

    Choose point U on side AB of $\triangle ABC$ so that $AU = XY$. Draw UV parallel to BC, where V lies on side AC. Since $UV \parallel BC$, we have $\angle AUV = \angle B = \angle Y$ and $\angle AVU = \angle C = \angle Z$. But $AU = XY$, whence $\triangle AUV \cong \triangle XYZ$ by SAA. In particular, $AV = XZ$.

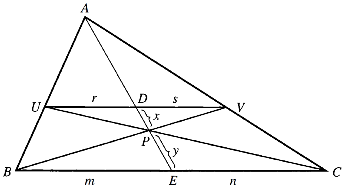
    

    Since $UV \parallel BC$, we know that $\frac{AU}{AB} = \frac{AV}{AC}$. Since $AU = XY$ and $AV = XZ$, it follows that $\frac{XY}{AB} = \frac{XZ}{AC}$.

Application of Similarity

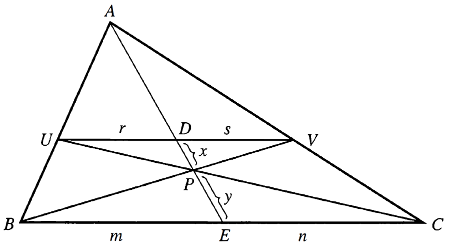
Proposition

In $\triangle ABC$, draw UV parallel to BC. The intersection of BV and UC lies on the median of $\triangle ABC$ from point A.



- Let P be the intersection point of BV and UC. Draw line AP and let D and E be the points where this line crosses UV and BC. Denote the lengths UD, DV, BE, EC, DP and PE by r, s, m, n, x and y, respectively. Our task is to prove that E is the midpoint of AC. So we need to show $m = n$. First, note that because $UD \parallel BE$, $\angle AUD = \angle ABE$ and $\angle ADU = \angle AEB$. So $\angle AUD \sim \triangle ABE$ by AA. We conclude that $\frac{r}{m} = \frac{AD}{AE}$. Similarly, $\frac{s}{n} = \frac{AD}{AE}$. This gives $\frac{r}{m} = \frac{s}{n}$. Hence, $\frac{n}{m} = \frac{s}{r}$.

Application of Similarity (Cont'd)



Consider $\triangle DUP$ and $\triangle ECP$. Because $UD \parallel EC$, we have $\angle DUP = \triangle PCE$ and $\angle UDP = \angle CEP$. Thus, $\triangle DUP \sim \triangle ECP$. Hence $\frac{r}{n} = \frac{x}{y}$.

Similar reasoning with $\triangle DVP$ and $\triangle EBP$ yields $\frac{s}{m} = \frac{x}{y}$.

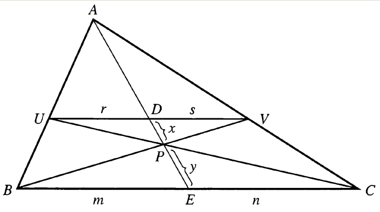
We conclude that $\frac{r}{n} = \frac{s}{m}$. Thus, $\frac{m}{n} = \frac{s}{r}$.

Since we previously had $\frac{n}{m} = \frac{s}{r}$, we see that $\frac{m}{n} = \frac{n}{m}$. Therefore, $n = m$.

Segment Joining the Midpoints of Sides

Corollary

Let U and V be the midpoints of sides AB and AC, respectively, in $\triangle ABC$. Then $UV \parallel BC$ and $UV = \frac{1}{2}BC$.



- That $UV \parallel BC$ is immediate from the lemma, since $\frac{AU}{AB} = \frac{1}{2} = \frac{AV}{AC}$. In this situation, $\triangle AUV \sim \triangle ABC$ by AA. Thus, we obtain that

$$\frac{UV}{BC} = \frac{AU}{AB} = \frac{1}{2}.$$

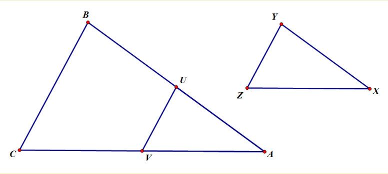
It follows that $UV = \frac{1}{2}BC$, as required.

The SSS Similarity Criterion

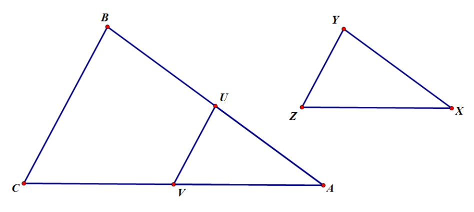
Theorem

Suppose that the sides of $\triangle ABC$ are proportional to the corresponding sides of $\triangle XYZ$. Then $\triangle ABC \sim \triangle XYZ$.

- By hypothesis, each of XY, XZ and YZ is equal, respectively, to some scalar λ times AB, AC and BC.
  - If $AB = XY$, then $\lambda = 1$ and all three sides of $\triangle ABC$ are equal to the corresponding sides of $\triangle XYZ$. Thus, by SSS, we get congruence and, a fortiori, similarity.
  - Assume that AB and XY are unequal, say XY is the shorter of these two segments. Choose point U on segment AB such that $AU = XY$ and draw UV parallel to BC with V on AC.



The SSS Similarity Criterion (Cont'd)



- By AA, $\triangle AUV \sim \triangle ABC$. Hence, the corresponding sides of these two triangles are proportional. The scale factor for this proportionality is $\frac{AU}{AB} = \frac{XY}{AB} = \lambda$. Thus $AV = \lambda AC = XZ$ and $UV = \lambda BC = YZ$. It follows that $\triangle AUV \cong \triangle XYZ$ by SSS. Hence, $\angle AUV = \angle Y$ and $\angle AVU = \angle Z$. Thus, $\angle B = \angle Y$ and $\angle C = \angle Z$. Hence $\triangle ABC \sim \triangle XYZ$ by AA.

Similarity of Arbitrary Figures on the Plane

- Two arbitrary figures are **similar** if for each point in one of them, there is a corresponding point in the other so that all distances in the first figure are proportional, with some particular scale factor λ, to the corresponding distances in the second figure.
- Caution: This is not how we defined similarity for triangles, but we proved that:
  - similar triangles satisfy this condition;
  - triangles that are similar in this proportionality sense are actually similar triangles, according to our earlier definition.

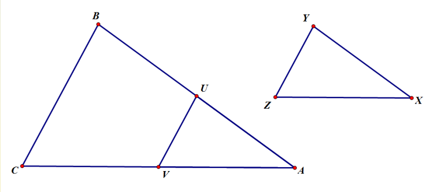
The SAS Similarity Criterion

Theorem

Given $\triangle ABC$ and $\triangle XYZ$, assume that $\angle X = \angle A$ and that $\frac{XY}{AB} = \frac{XZ}{AC}$. Then $\triangle ABC \sim \triangle XYZ$.

- If $XY = AB$, then $XZ = AC$ and the triangles are congruent by SAS. Hence, they are, a fortiori similar.

  Assume that $XY < AB$.
  Choose point U on side AB of $\triangle ABC$ so that $AU = XY$. Draw UV parallel to BC with V on side AC. Observe that $\triangle AUV \sim \triangle ABC$ by AA.

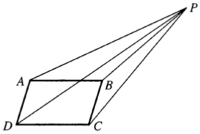
  

  Then $\frac{AV}{AC} = \frac{AU}{AB} = \frac{XY}{AB} = \frac{XZ}{AC}$. Hence $AV = XZ$. By ASA, $\triangle AUV \cong \triangle XYZ$. Thus, $\angle XYZ = \angle AUV = \angle ABC$. It now follows by AA that $\triangle ABC \sim \triangle XYZ$.

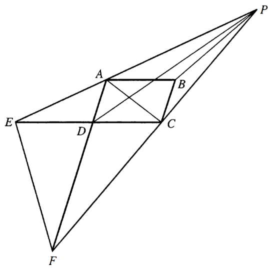
Another Application of Similarity

Proposition

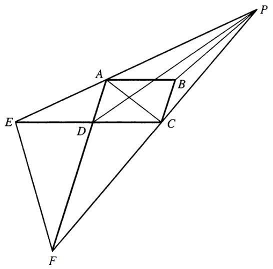
Point P lies outside of parallelogram $ABCD$ and $\angle PAB = \angle PCB$. Show that $\angle APD = \angle CPB$.



- Extend sides CD and AD of the given parallelogram so that they meet lines PA and PC at E and F, respectively, and draw segments AC and EF.
  Observe that $\angle DEA = \angle BAP$ because these are corresponding angles for the parallel lines ED and AB. Also, we have $\angle DFC = \angle BCP$ by similar reasoning.
  Since $\angle BAP = \angle BCP$ by hypothesis, we get that $\angle DEA = \angle DFC$.



Another Application of Similarity (Cont'd)



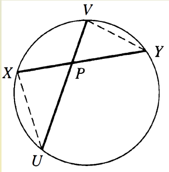
- Note $\angle ADE = \angle CDF$ since these are vertical angles. So $\triangle ADE \sim \triangle CDF$ by AA. Hence $\frac{AD}{CD} = \frac{ED}{FD}$. Using algebra, $\frac{AD}{ED} = \frac{CD}{FD}$. Since we know that $\angle ADC = \angle EDF$, it follows via SAS that $\triangle ADC \sim \triangle EDF$. So $\angle CAD = \angle FED$. Therefore, $\angle FEP = \angle FED + \angle DEA = \angle CAD + \angle BAP = \angle ACB + \angle BCP = \angle ACP$.
  Since $\angle EPF = \angle CPA$, we can now conclude that $\triangle EPF \sim \triangle CPA$ by AA. So we have $\frac{PC}{PE} = \frac{AC}{FE}$. We also have $\frac{AC}{FE} = \frac{AD}{ED}$ because we know that $\triangle ADC \sim \triangle EDF$. We now get $\frac{PC}{PE} = \frac{AC}{FE} = \frac{AD}{ED} = \frac{CB}{ED}$.
  We finally get $\triangle PCB \sim \triangle PED$: $\angle PED = \angle PCB$ since each is equal to $\angle PAB$; The similarity follows by SAS since we have $\frac{PC}{PE} = \frac{CB}{ED}$. We conclude that $\angle CPB = \angle EPD$.

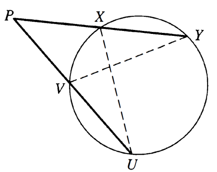
Secant Line to a Circle

Theorem

Given a circle and a point P not on the circle, choose an arbitrary line through P, meeting the circle at points X and Y. Then $PX \cdot PY$ depends only on P and is independent of the choice of the line through P.

- Draw a second line through P that also meets the circle in two points, U and V. We must show that $PX \cdot PY = PU \cdot PV$.





Draw line segments UX and VY. Observe that $\angle U = \angle Y$, since these inscribed angles subtend the same arc.

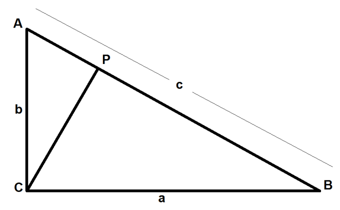
Also $\angle XPU = \angle VPY$, since these are vertical angles when P is inside the circle, and identical when P is outside the circle. In either case we have $\triangle PXU \sim \triangle PVY$ by AA. Thus, $\frac{PX}{PV} = \frac{PU}{PY}$. Equivalently, $PX \cdot PY = PU \cdot PV$.

Pythagorean Theorem through Similarity

Lemma

Suppose $\triangle ABC$ is a right triangle with hypotenuse AB and let CP be the altitude drawn to the hypotenuse. Then $\triangle ACP \sim \triangle ABC \sim \triangle CBP$.

- The first similarity follows by AA since $\angle A = \angle A$ and $\angle APC = 90° = \angle ACB$. The proof of the second similarity is similar.



- **Alternative Proof of Pythagoras' Theorem**: In the situation of the lemma, write as usual $a = BC$, $b = AC$ and $c = AB$. Since $\triangle ACP \sim \triangle ABC$, we have $\frac{AP}{AC} = \frac{AC}{AB}$. It follows that $AP = \frac{b^2}{c}$. Similarly, $BP = \frac{a^2}{c}$. Since $c = AB = AP + PB$, we conclude that $c = \frac{a^2}{c} + \frac{b^2}{c}$. Multiplication by c yields $c^2 = a^2 + b^2$.

Areas of Similar Polygons

- Suppose we have two similar polygons $\mathscr{P}$ and $\mathscr{Q}$, where the distances in $\mathscr{Q}$ are obtained from those in $\mathscr{P}$ by multiplication by some fixed scale factor λ.
    - If $\mathscr{P}$ and $\mathscr{Q}$ are squares, with side lengths p and q, respectively, then $q = \lambda p$. Hence $K_{\mathscr{Q}} = q^2 = \lambda^2 p^2 = \lambda^2 K_{\mathscr{P}}$.
    - This rule "multiply by the square of the scale factor" works for arbitrary polygons: Imagine that $\mathscr{P}$ is subdivided into a very large number of very small squares with a little left over at the edges. If $\mathscr{Q}$ is subdivided into corresponding squares, then each square in $\mathscr{Q}$ has area equal to λ^2 times the area of the corresponding square in $\mathscr{P}$. Thus the sum of the areas of the little squares that almost comprise $\mathscr{Q}$ is λ^2 times the sum of the little squares in $\mathscr{P}$. It follows that, with vanishingly small error, we have $K_{\mathscr{Q}} = \lambda^2 K_{\mathscr{P}}$.

This non-rigorous argument can be transformed into a correct proof using limits and other ideas from calculus.