College Geometry

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LSSU Math 325

Subsection 1

The Circumcircle

Circumcircle, Circumcenter and Circumradius

- We showed that there exists a unique circle through any three noncollinear points.
- Each triangle, therefore, is inscribed in exactly one circle, called its **circumcircle**.
    - The center is called the **circumcenter**;
    - The radius is called the **circumradius**.

  The usual notation is O for the circumcenter and R for the circumradius.

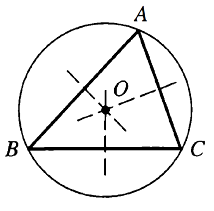
Theorem

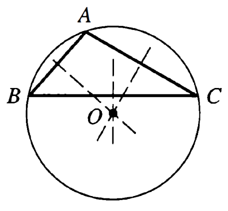
The three perpendicular bisectors of the sides of a triangle are concurrent at the circumcenter of the triangle.

- Given $\triangle ABC$, we know that its circumcenter O is equidistant from vertices A and B. So it lies on the perpendicular bisector of side AB. Similarly, O lies on the perpendicular bisectors of sides BC and AC.

Relative Position of the Circumcenter

- There are at least two possibilities concerning the circumcenter:
  - It may be an interior point of $\triangle ABC$;
  - It may be outside of the triangle.





  On the left $\angle A$ is acute while on the right it is obtuse.
- If $\angle A = 90°$, then $\widehat{BC}$ is a semicircle. So, the circumcenter O lies on the hypotenuse BC. Thus, BC is a diameter of the circumcircle.

  We conclude that, for a right triangle, the circumcenter is the midpoint of the hypotenuse and the circumradius $R = \frac{1}{2}BC$.

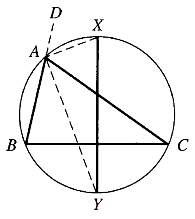
Perpendicular Bisectors of Sides and Circumference

Proposition

The perpendicular bisectors of the sides of a triangle meet the circumcircle at the interior and exterior angle bisectors.

- Consider the perpendicular bisector XY of side BC of $\triangle ABC$.

Since Y lies on the perpendicular bisector of BC, Y is equidistant from B and C. Thus, chords BY and CY are equal. It follows that $\overset{\frown}{BY} \overset{\circ}{=} \overset{\frown}{CY}$, whence $\angle BAY = \angle CAY$. So Y is the point where the bisector of $\angle A$ meets the circumcircle.



Extend AB to D and draw AX. We show that $\angle DAX = \angle CAX$. The circumcenter of $\triangle ABC$ lies on the perpendicular bisector XY of BC. So XY is a diameter of the circle. It follows that $\angle XAY = 90°$. Thus, $\angle DAX = 180° - \angle BAX = 180° - (90° + \angle BAY) = 90° - \angle BAY = 90° - \angle CAY = \angle XAY - \angle CAY = \angle CAX$.

Extended Law of Sines

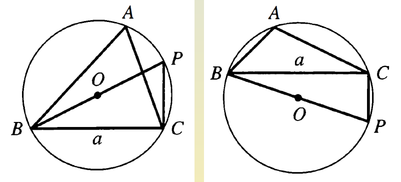
Theorem (Extended Law of Sines)

Given $\triangle ABC$ with circumradius R, write as usual a, b and c to denote the lengths of the sides opposite vertices A, B and C, respectively. Then

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = 2R.$$

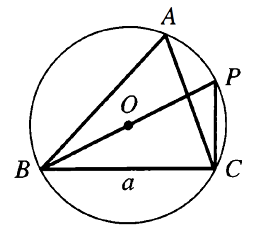
- We show $\frac{a}{\sin(A)} = 2R$.

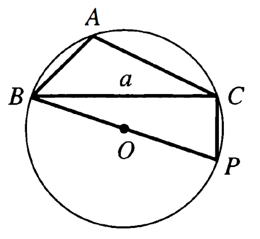
  Draw the circumcircle of $\triangle ABC$. Let BP be the diameter through B.



  There is also the possibility that $P = C$, which occurs if $\angle A = 90°$.
  - Suppose $\angle A < 90°$. BP is a diameter. So $\triangle PBC$ is a right triangle with hypotenuse BP of length $2R$. Thus, $\sin(P) = \frac{BC}{BP} = \frac{a}{2R}$. But $\angle A = \angle P$ because they subtend the same arc. Hence, $\sin(A) = \frac{a}{2R}$.

Extended Law of Sines (Cont'd)



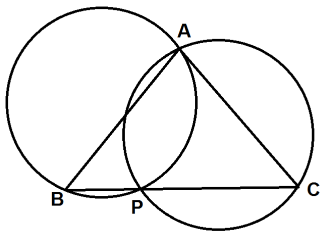


- Suppose $\angle A = 90°$. Then $\triangle ABC$ is a right triangle and the hypotenuse BC is a diameter of the circumcircle. Thus $a = BC = 2R$. But $\sin(A) = 1$. So $\frac{a}{\sin(A)} = a = 2R$.
- Suppose $\angle A > 90°$. Then $\triangle PBC$ is a right triangle. So $\sin(P) = \frac{a}{2R}$. In this case, A and P are opposite vertices of an inscribed quadrilateral. Hence $\angle A = 180° - \angle P$. It follows that $\sin(A) = \sin(P)$. Thus, $\sin(A) = \frac{a}{2R}$.

Application of the Extended Law of Sines

Proposition

Given isosceles $\triangle ABC$, choose a point P on the base BC. Then $\triangle ABP$ and $\triangle ACP$ have equal circumradii.



- Apply the extended law of sines in $\triangle ABP$: The diameter of the circumcircle of $\triangle ABP$ is $\frac{AP}{\sin(B)}$.

  Apply the extended law of sines in $\triangle ACP$: The diameter of the circumcircle of $\triangle ACP$ is $\frac{AP}{\sin(C)}$.

  Since $\triangle ABC$ is isosceles, by the pons asinorum, $\angle B = \angle C$. Thus, $\sin(B) = \sin(C)$ and the two diameters are equal.

Area, Circumradius and Sides

Corollary

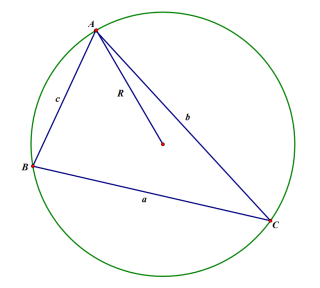
Let R and K denote the circumradius and area of $\triangle ABC$, respectively, and let a, b and c denote the side lengths. Then $4KR = abc$.

- We know $2K = ab\sin(C)$.

  By the extended law of sines,

  $$2R = \frac{c}{\sin(C)}.$$

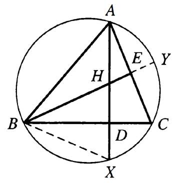
  Thus, $4KR = abc$.



Altitudes and Circumcircles

Proposition

Given acute angled $\triangle ABC$, draw altitude AD and note that point D must lie on line segment BC. Extend AD beyond D to point X on the circumcircle. Observe that $AD > DX$, and thus there is a point H on segment AD for which $HD = DX$. Then line BH is perpendicular to AC.

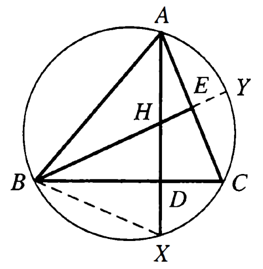


- Since $\angle B$ and $\angle C$ are acute, point D, lies between B and C. To see that $AD > DX$, we consider BC and the perpendicular bisector of AX. These are parallel lines that cut AX at point D and at the midpoint of AX, respectively. So it is enough to show that BC is below the perpendicular bisector. But the perpendicular bisector of AX goes through the center of the circle. Thus, it suffices to observe that BC is below the center. This is true because $\angle A$ is acute, and hence it subtends an arc that is less than a semicircle.

  We have now justified that H lies inside the triangle.

Altitudes and Circumcircles (Cont'd)

- Let E and Y be the points where BH meets side AC and where it meets the circle, respectively. D is the midpoint of HX. So BC is the perpendicular bisector of HX. Thus $BH = BX$. Now BD is an altitude of isosceles $\triangle HBX$. So it is also an angle bisector. We conclude $\angle YBC = \angle XBC$. It follows that $\overparen{YC} \circeq \overparen{XC}$.



Now we have

$$90^\circ = \angle ADB \circeq \frac{1}{2}(\overparen{AB} + \overparen{XC}) \circeq \frac{1}{2}(\overparen{AB} + \overparen{YC}) \circeq \angle AEB.$$

Thus, $\angle AEB$ is a right angle.

The Orthocenter

- We showed that the altitude of $\triangle ABC$ from vertex B crosses altitude AD at the specified point H.
- Similarly, the altitude from vertex C also crosses AD at this same point H.
- Thus, the three altitudes are concurrent.
- It is true for every triangle that the altitudes are concurrent at a point called the **orthocenter**, but we have so far proved this only in the case of acute angled triangles.
- A point Q is the **reflection** of a point P in a given line if the line is the perpendicular bisector of segment PQ.
- The preceding proposition also shows that the three points that result when the orthocenter of an acute angled triangle is reflected in the sides of the triangle all lie on the circumcircle.

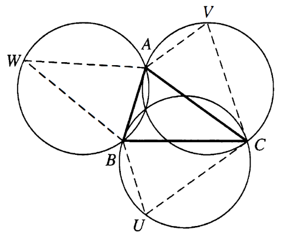
A Uniqueness Property of the Orthocenter

Proposition

The only point whose reflections in the sides of the triangle all lie on the circumcircle is the orthocenter.

- Start with acute angled $\triangle ABC$ and reflect each of the vertices A, B and C in the opposite side of the triangle to obtain points U, V and W, respectively.

  Next, draw the circumcircles of $\triangle BCU$, $\triangle CAV$ and $\triangle ABW$. Observe that these appear to go through a common point. To understand what is going on here, note that $\triangle UBC$ is the reflection of $\triangle ABC$ in line BC.

  Thus, the circumcircle of $\triangle UBC$ is just the reflection of the circumcircle of $\triangle ABC$ in this line.

A Uniqueness Property of the Orthocenter (Cont'd)

- It follows that the circumcircle of $\triangle UBC$ is the locus of all points whose reflection in line BC lies on the circumcircle of the original triangle. In particular, we know that the orthocenter H of $\triangle ABC$ lies on this circle. By similar reasoning, H lies on each of the other two circles. Hence the three circles do indeed have a point in common. We know the orthocenter H lies on all three circles. Since these circles cannot have more than one common point, H is the only point all of whose reflections in the sides of $\triangle ABC$ lie on the circumcircle of this triangle.
- The three circles through point H are clearly the circumcircles of $\triangle HBC, \triangle AHB$, and $\triangle ABH$. Each of these circles is a reflection of the circumcircle of $\triangle ABC$. It follows that all four circumcircles have equal radii.

Subsection 2

The Centroid

The Centroid

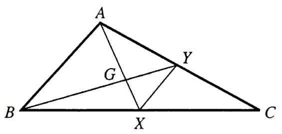
- In every triangle, the three medians are concurrent and the point of concurrence is called the **centroid** of the triangle, denoted G.

Theorem

The three medians of an arbitrary triangle are concurrent at a point that lies two thirds of the way along each median from the vertex of the triangle toward the midpoint of the opposite side.

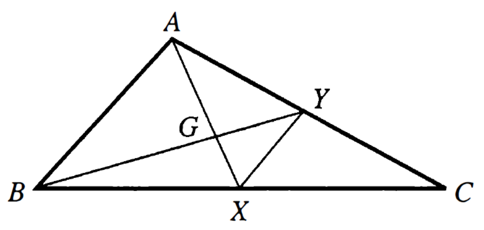
- Let G be the point where median AX crosses median BY.

  We first show that G lies two thirds of the way from A to X along AX, i.e., that $AG = 2GX$. Draw segment XY that joins the midpoints of two sides of $\triangle ABC$.



  We conclude that XY must be parallel to the third side, AB, and that $XY = \frac{1}{2}AB$. Since $XY \parallel AB$, we have equality of alternate interior angles, and thus $\angle BAG = \angle YXG$ and $\angle ABG = \angle XYG$. It follows that $\triangle BAG \sim \triangle YXG$ by AA, and, hence, $\frac{AG}{GX} = \frac{AB}{XY} = 2$.

The Centroid (Cont'd)



- Similarly, median CZ crosses AX at a point that lies two thirds of the way from A to X. Since G is the only point on AX that has this property, CZ goes through G. So the three medians are concurrent, and we know that the point of concurrence lies two thirds of the way along median AX. By similar reasoning, we deduce that the point of concurrence lies two thirds of the way along each median.

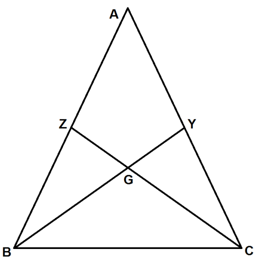
Medians and Isosceles Triangles

Proposition

Suppose that in $\triangle ABC$, medians BY and CZ have equal lengths. Then $AB = AC$.

- Medians BY and CZ intersect at the centroid G. By the Theorem, $BG = \frac{2}{3}BY = \frac{2}{3}CZ = CG$. Thus $\triangle BGC$ is isosceles. By the pons asinorum, $\angle GBC = \angle GCB$.

  We are given that $BY = CZ$, and of course, $BC = BC$. We now know that $\angle YBC = \angle ZCB$. By SAS, $\triangle BYC \cong \triangle CZB$. Thus, $YC = ZB$. We conclude that $AB = 2ZB = 2YC = AC$.



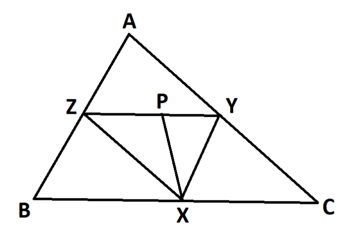
The Medial Triangle

- Assume that the mass is uniformly distributed along the sides of $\triangle ABC$ and that the interior is massless.

  In this case, we can assume that:
  - the total mass of side BC is a (the length of BC), and we pretend that it is concentrated at the midpoint X of BC;
  - a mass of b units is at the midpoint Y of AC;
  - a mass of c units is at the midpoint Z of AB.
- We need to consider only $\triangle XYZ$ with point masses a, b and c at vertices X, Y and Z.
- $\triangle XYZ$, the triangle formed by the midpoints of the sides of $\triangle ABC$, is called the **medial triangle** of $\triangle ABC$.
- We know that $XY = \frac{c}{2}$, $XZ = \frac{b}{2}$ and $YZ = \frac{a}{2}$. By SSS, $\triangle ABC \sim \triangle XYZ$.

Center of Mass of Wire Triangle is at the Incenter

- Replace the two point masses at Y and Z by a single mass at the center of mass P of side YZ. Since the masses at Y and Z are b and c, not necessarily equal, P need not be the midpoint of YZ.



Let $YP = u$ and $ZP = v$. Then $bu = cv$. Hence $\frac{u}{v} = \frac{c}{b} = \frac{XY}{XZ}$. So P divides side YZ of $\triangle XYZ$ into two pieces whose lengths are in the same ratio as the lengths of the nearer sides of the triangle. We conclude that the bisector of $\angle X$ meets YZ at P. Thus the center of mass of $\triangle XYZ$, lies on the angle bisector XP.

Center of Mass is at the Incenter (Cont'd)

- Similarly, it also lies on the other two angle bisectors of $\triangle XYZ$. So the center of mass of $\triangle ABC$ lies at the point of concurrence of the angle bisectors of $\triangle XYZ$.
- In general, the angle bisectors of an arbitrary triangle are concurrent at a point called the **incenter**.
- In general, it is not true that the incenter of the medial triangle is the centroid of the original triangle.

  In other words, the center of mass of a uniform wire triangle is not always at the same location as the center of mass of the corresponding uniform cardboard triangle.

Subsection 3

The Euler Line, Orthocenter and the Nine-Point Circle

Circumcenter and Centroid

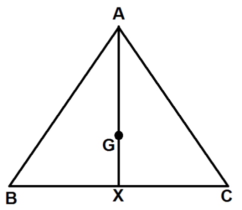
- If a given triangle is equilateral, then each median is the perpendicular bisector of the opposite side. So the circumcenter and the centroid are actually the same point.

Lemma

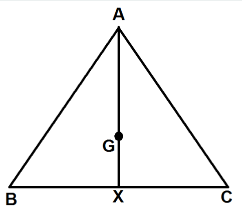
If the circumcenter and the centroid of a triangle coincide, then the triangle must be equilateral.

- Suppose that the centroid of $\triangle ABC$ is also the circumcenter.

  Let G be the centroid and let X be the midpoint of side BC. Then X, G and A are distinct points on the median from A. Since these points are collinear, A lies on line GX. But G is also the circumcenter, and so it lies on the perpendicular bisector of side BC.



Circumcenter and Centroid (Cont'd)

- 

The midpoint X of side BC also lies on the perpendicular bisector of BC. It follows that the line GX is the perpendicular bisector of BC. Since A lies on GX, we have shown that A lies on the perpendicular bisector of BC.

  Thus, A is equidistant from B and C. We now have $AB = AC$.

  Similar reasoning shows that $BA = BC$.

  Therefore, all three sides of $\triangle ABC$ are equal and the triangle is equilateral.

The Euler Line and the Orthocenter

- Given any non equilateral $\triangle ABC$:
  - The circumcenter O and the centroid G are two distinct points.
  - These two points determine a unique line that is called the **Euler line** of the triangle.
  - The three altitudes of a triangle are always concurrent at a point called the **orthocenter** of the triangle.
  - We will show that the Euler line also goes through the orthocenter.
- For equilateral triangles, there is no Euler line defined.
  - In this case, the altitudes are the medians, and we know that they are concurrent at the centroid.
  - Since there is no Euler line for an equilateral triangle, there is nothing to prove in this case.

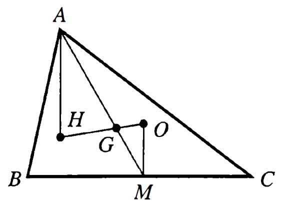
Euler Line Contains the Orthocenter

Theorem

Assume that $\triangle ABC$ is not equilateral and let G and O be its centroid and circumcenter, respectively. Let H be the point on the Euler line GO that lies on the opposite side of G from O and such that $HG = 2GO$. Then all three altitudes of $\triangle ABC$ pass through H.

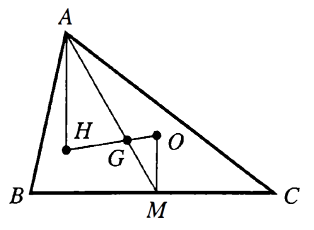
- We show that the altitude from A passes through H. If H coincides with A, there is nothing to prove. So assume that H and A are different. (H and A coincide when $\angle A = 90°$.) It suffices to show that line AH is perpendicular to BC.

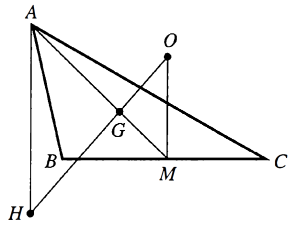
  Let M be the midpoint of side BC and observe that O and M are distinct points. Otherwise, the median GM is the Euler line. Since $AG = 2GM$, it would follow that A is H, which we are assuming is not the case.



Euler Line Contains the Orthocenter (Cont'd)

- Since both O and M lie on the perpendicular bisector of BC, line OM is perpendicular to BC.
  We will be done if we can show that OM is parallel to AH.



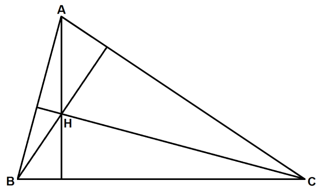


  We prove the equality of the alternate interior angles, $\angle H$ and $\angle O$.
  We know that $\frac{AG}{MG} = 2$ and that $\frac{HG}{OG} = 2$, by the construction of the point H. Since $\frac{AG}{MG} = \frac{HG}{OG}$ and $\angle AGH = \angle MGO$, we see that $\triangle AGH \sim \triangle MGO$ by SAS. It follows that $\angle H = \angle O$.

The Three Vertices and the Orthocenter

- Given $\triangle ABC$, let H be its orthocenter. If we start with a right triangle, then clearly H coincides with the right angle. In all other cases, A, B, C and H are easily seen to be four distinct points.

  Observe that the line determined by any two of these four points is perpendicular to the line determined by the other two.

  - For example, AH is perpendicular to BC because line AH is the altitude from A in $\triangle ABC$.
- We observe that each of the four points A, B, C and H is the orthocenter of the triangle formed by the other three.
  - That A is the orthocenter of $\triangle HBC$, for example, is just another way of saying that AH, AB and AC are perpendicular to BC, HC and HB, respectively.

Orthic Quadruples

- Given any four points with the property that each is the orthocenter of the triangle formed by the other three, we say that the given set of four points is an **orthic quadruple**.
- Given an arbitrary set of three points A, B and C, there almost always exists a fourth point H, such that the set $\{A, B, C, H\}$ is an orthic quadruple (take H to be the orthocenter of $\triangle ABC$).

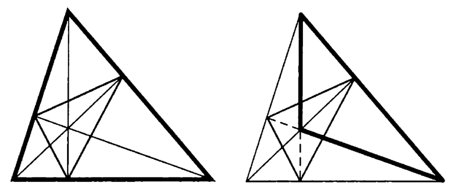
  The only exceptions are:
  - when $\triangle ABC$ is a right triangle;
  - when $\triangle ABC$ does not exist because the given three points are collinear.

Pedal or Orthic Triangle

- Given $\triangle ABC$, let D, E and F be the points where the altitudes from A, B and C meet lines BC, AC and AB, respectively.
- These points are called the **feet** of the altitudes, and they may not actually lie on the line segments BC, AC and AB.
- If $\triangle ABC$ is not a right triangle, it is not hard to see that the feet D, E and F are distinct and form a triangle.
- We refer to $\triangle DEF$ as the **pedal** or **orthic triangle** of $\triangle ABC$.

The Pedal Triangle

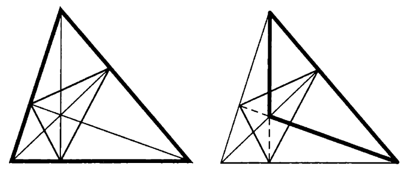
- With regards to pedal triangles two of the situations that can occur are shown below:



- The original triangle is drawn with heavy lines.
- The pedal triangle and the altitudes are drawn with solid lighter lines.
- The feet of the altitudes lie on the sides of the triangle for acute angled triangles, but two of the feet lie outside of the triangle if there is an obtuse angle.

Pedal Triangles of Triplets in Orthic Quadruples

- The triangle on the left and the one on the right happen to be two of the four triangles that can be formed using three of the four points of an orthic quadruple.



We see that in both cases, we get exactly the same pedal triangle.

Theorem

The pedal triangles of each of the four triangles determined by an orthic quadruple are all the same.

- For each choice of two of our four given points, the line determined by those two is perpendicular to the line determined by the other two. Since there are exactly three ways to pair off four objects into two sets of two, this gives three points that occur as the intersections of pairs of perpendicular lines determined by our orthic quadruple. These must be the vertices of the pedal triangle of each of the four triangles.

The Euler Points of a Triangle

- The circumcircle of the pedal triangle of $\triangle ABC$, in addition to the feet of the three altitudes, also contains the midpoints of the three sides.
- Hence it is also the circumcircle of the medial triangle of $\triangle ABC$.
- The circle has the additional property that it bisects each of the line segments AH, BH and CH, where H is the orthocenter of $\triangle ABC$.
- The midpoint X of segment AH is called the **Euler point** of $\triangle ABC$ opposite to side BC.

  Similarly, the midpoints Y and Z of BH and CH are the **Euler points** of $\triangle ABC$ opposite to sides AC and AB, respectively.
- The common circumcircle of the pedal and medial triangles contains the three Euler points and more!

The Nine-Point Circle Theorem

Theorem

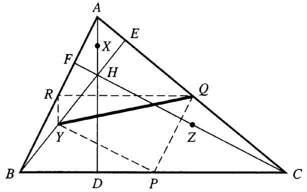
Given any triangle, all of the following points lie on a common circle:

- the three feet of the altitudes;
- the three midpoints of the sides;
- the three Euler points.

Furthermore, each of the line segments joining an Euler point to the midpoint of the opposite side is a diameter of this circle.

- This remarkable circle is called the **nine-point circle** of the triangle. However, the nine points referred to in the statement of the theorem are not always distinct.
- Points D, E and F are the feet of the altitudes of $\triangle ABC$. Points P, Q and R are the midpoints of the sides and X, Y and Z are the Euler points. We need to show that all nine of these points lie on a common circle and that XP, YQ and ZR are diameters of this circle.

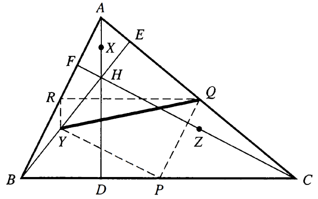
The Proof of the Nine-Point Circle Theorem



- Draw line segment YQ and consider the unique circle that has YQ as a diameter. We show that points P and R lie on this circle. To see that P lies on the circle with diameter YQ, it suffices to show that $\angle YPQ = 90°$. We will do this by proving that $YP \parallel CF$ and $PQ \parallel AB$. Since CF and AB are perpendicular, it follows that YP and PQ are perpendicular. Thus $\angle YPQ = 90°$, as required. That PQ is parallel to AB follows since P and Q are the midpoints of two sides of $\triangle ABC$ and AB is the third side. Similarly, to prove that YP is parallel to CF, we work in $\triangle BHC$. P is the midpoint of side BC. The Euler point Y is the midpoint of side BH. It follows that YP is parallel to the third side of this triangle, which is CH. Thus $YP \parallel CF$, as desired. We have shown that $\angle YPQ = 90°$. Thus, point P lies on the circle with diameter YQ.

Proof of the Nine-Point Circle Theorem (Cont'd)

- We use similar reasoning to show that R lies on this circle. It suffices to prove that $\angle YRQ = 90°$. We accomplish this by showing that $YR \parallel AD$ and $QR \parallel BC$. Since altitude AD is perpendicular to side BC, it will follow that QR is perpendicular to YR.

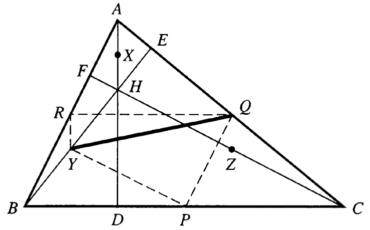


  That $QR \parallel BC$ follows by considering midpoints of sides in $\triangle ABC$. To prove that $YR \parallel AD$, we consider $\triangle ABH$.
- We have shown that points P, Q, R and Y all lie on the same circle and that YQ is a diameter of this circle.

  Since this circle contains P, Q and R, it is, of course, the circumcircle of the medial triangle of $\triangle ABC$.
- We conclude that given an arbitrary $\triangle ABC$, the circumcircle of its medial triangle has line segment QY as a diameter, where Q is the midpoint of side AC and Y is the opposite Euler point.

Concluding the Proof of the Nine-Point Circle Theorem

- It follows similarly that the line segments PX and RZ are also diameters of the medial circumcircle. In particular, the other two Euler points, X and Z, lie on this circle.



  Hence, six of the required nine points lie on the medial circumcircle and each of XP, YQ and ZR is a diameter.

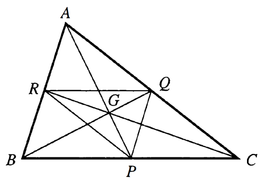
  We observe that E lies on the circle with diameter YQ since $\angle YEQ = 90°$. This shows that the altitude foot E lies on the medial circumcircle of an arbitrary $\triangle ABC$.

  It follows similarly that the medial circumcircle contains the other two altitude feet, D and F.

Radius of the Nine-Point Circle

Proposition

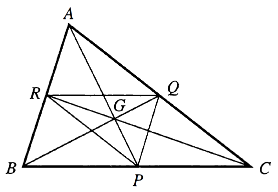
Let R be the circumradius of $\triangle ABC$. Then the distance from each Euler point of $\triangle ABC$ to the midpoint of the opposite side is R, and the radius of the nine-point circle of $\triangle ABC$ is $\frac{R}{2}$.



- Since the nine-point circle is, among other things, the circumcircle of the medial triangle of $\triangle ABC$, we can solve this problem by focusing attention on $\triangle PQR$, where P, Q and R are the midpoints of sides BC, AC and AB. We know that $QP \parallel AR$ and $RP \parallel AQ$. Thus $AQPR$ is a parallelogram, and diagonal AP bisects diagonal RQ. In other words, median AP of $\triangle ABC$ bisects side RQ of the medial triangle PQR. Hence, it contains the median from P in triangle PQR. Similarly, the other two medians of $\triangle ABC$ contain the other medians of $\triangle PQR$. It follows that the centroid G of $\triangle ABC$ is also the centroid of $\triangle PQR$.

Radius of the Nine-Point Circle: A Plane Transformation

- Consider the following two-step transformation of the plane:



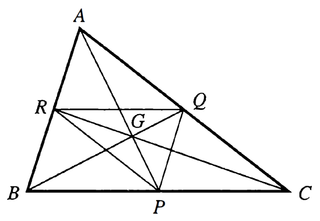
  - First, shrink the plane with a scale factor $\lambda = \frac{1}{2}$ in such a way that point G remains fixed and every other point moves toward G.
  - Next, rotate the plane $180°$ with a center of rotation at G.

  Let T denote the net effect of these two operations. Observe that $T(A) = P$, since $AG = 2GP$ and $\angle AGP = 180°$. Similarly, $T(B) = Q$ and $T(C) = R$.

  We argue informally that the transformation T carries lines to lines, triangles to triangles, and circles to circles. Furthermore, given a circle centered at some point O, the image of that circle under T is a circle centered at the point $T(O)$ and having radius half that of the original circle.

Radius of the Nine-Point Circle (Conclusion)

- Now consider the circumcircle of $\triangle ABC$, which is centered at the circumcenter O and has radius R. The transformation T carries $\triangle ABC$ to $\triangle PQR$. Hence it carries the circumcircle of $\triangle ABC$ to the circumcircle of $\triangle PQR$. The latter is the nine-point circle of $\triangle ABC$.



  Thus, the center of the nine-point circle, which we call N, is exactly the point $T(O)$. Moreover, the radius of the nine-point circle is $\frac{R}{2}$. Also, since XP, YQ, and ZR are diameters of the nine-point circle, it follows that each of these segments has length R.

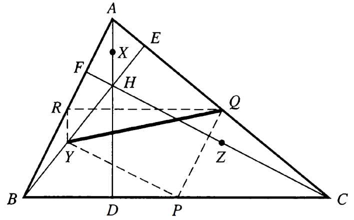
Circumradii of Triangles Formed by Orthic Quadruple

Corollary

Suppose $\triangle ABC$ is not a right triangle and let H be its orthocenter. Then $\triangle ABC$, $\triangle HBC$, $\triangle AHC$, and $\triangle ABH$ have equal circumradii.

- We know that these four triangles share a common pedal triangle.

  Since the nine-point circle of any triangle is the circumcircle of its pedal triangle, it follows that the four triangles share a common nine-point circle.

  We have just seen, however, that for an arbitrary triangle, the circumradius is exactly twice the nine-point radius. Hence, our four triangles have equal circumradii.

The Center of the Nine-Point Circle

Proposition

The center of the nine-point circle of $\triangle ABC$ is the midpoint of the segment joining the orthocenter H and the circumcenter O of the $\triangle ABC$.

- Let N be the nine-point center of $\triangle ABC$. N must be the midpoint of each of the segments PX, QY and RZ.
  - If $\triangle ABC$ is equilateral, N coincides with G and O and H.
  - We assume, therefore, that the given triangle is not equilateral, whence it has an Euler line GO. We know that $N = T(O)$. It follows from the definition of T that N lies on the line through G and O, which is the Euler line. Furthermore, N lies on the opposite side of G from O, and we have $NG = \frac{1}{2}GO$. Recall now that the orthocenter H also lies on the Euler line on the opposite side of G from O and that $HG = 2GO$. It follows that N lies on the segment GH and $HN = HG - NG = \frac{3}{2}GO$. Also, $NO = NG + GO = \frac{3}{2}GO$. We deduce that $HN = NO$. In other words, the nine-point center N is the midpoint of the segment HO.

The Center of the Nine-Point Circle

Corollary

Suppose $\triangle ABC$ is not a right triangle and let H be its orthocenter. Then the Euler lines of $\triangle ABC$, $\triangle HBC$, $\triangle AHC$ and $\triangle ABH$ are concurrent. If any of these triangles is equilateral, then the Euler lines of the remaining triangles are concurrent.

- These four triangles share a nine-point circle. The center N of this circle lies on all of the Euler lines, which are therefore concurrent.

Subsection 4

Computations

The Law of Cosines

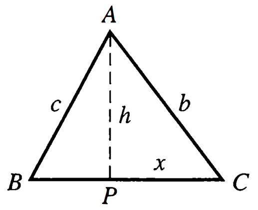
Theorem (Law of Cosines)

Given $\triangle ABC$, let a, b and c denote, as usual, the lengths of sides BC, AC and AB, respectively. Then $c^2 = a^2 + b^2 - 2ab\cos(C)$.

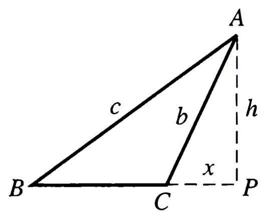
- Given a, b and c, the equation of the law of cosines can easily be solved to obtain $\cos(C) = \frac{a^2+b^2-c^2}{2ab}$.
- Similar formulas yield $\cos(A)$ and $\cos(B)$ in terms of a, b and c.
- Since a triangle can have at most one angle that fails to be acute, we can be sure that at least one of $\angle A$ or $\angle B$ is an acute angle. Assume that $\angle B < 90°$. Draw altitude AP from A to BC and write $h = AP$. Note that there are three possibilities:
  - Either $\angle C < 90°$ and P lies on segment BC or
  - $\angle C = 90°$ and point P coincides with point C, or
  - $\angle C > 90°$ and P lies on an extension of side BC.

The Law of Cosines (Cont'd)

- If $\angle C < 90^\circ$ and we write $x = PC$,

  $\triangle APC$ is a right triangle. So $\cos(C) = \frac{PC}{AC} = \frac{x}{b}$. Thus, $x = b\cos(C)$. Two applications of the Pythagorean theorem yield $b^2 = x^2 + h^2$ and $c^2 = h^2 + (a-x)^2$. This gives $c^2 = (b^2 - x^2) + (a-x)^2 = b^2 - x^2 + a^2 - 2ax + x^2 = a^2 + b^2 - 2ax$. Since $x = b\cos(C)$, we obtain the desired formula.
- Assume that $\angle C > 90^\circ$ and write $x = PC$.

  We have $\cos(C) = -\frac{x}{b}$ and $x = -b\cos(C)$. Two applications of the Pythagorean Theorem yield $b^2 = x^2 + h^2$ and $c^2 = h^2 + (a+x)^2$. We have $c^2 = (b^2 - x^2) + (a+x)^2 = b^2 - x^2 + a^2 + 2ax + x^2 = a^2 + b^2 + 2ax$.
  Since $x = -b\cos(C)$ in this case, the proof is complete.

Heron's Formula and Examples

- Define $s = \frac{1}{2}(a+b+c)$, called the **semiperimeter**.

Theorem (Heron's Formula)

The area K of $\triangle ABC$ is given by the equation

$$K = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b and c are the lengths of the sides and s is the semiperimeter.

- Note that if $\triangle ABC$ is a right triangle with arms of length 3 and 4 and hypotenuse of length 5, we know that $K = \frac{1}{2}bh = \frac{1}{2}(3)(4) = 6$.
  On the other hand, we have $s = \frac{1}{2}(3+4+5) = 6$. So Heron's formula gives $K = \sqrt{(6)(6-3)(6-4)(6-5)} = \sqrt{36} = 6$.
- If $\triangle ABC$ is an equilateral triangle, each of whose sides has length 2, we have $K = \frac{1}{2}bh = \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$.
  Heron's formula gives $K = \sqrt{(3)(3-2)(3-2)(3-2)} = \sqrt{3}$.

Proof of Heron's Formula

- We know that $K = \frac{1}{2}ab\sin(C)$. So $4K^2 = a^2b^2\sin^2(C) = a^2b^2(1-\cos^2(C))$. The law of cosines gives $\cos(C) = \frac{c^2-a^2-b^2}{2ab}$. Substituting, we obtain
$$4K^2 = a^2b^2\left(1 - \frac{(c^2-a^2-b^2)^2}{4a^2b^2}\right) = a^2b^2 - \frac{(c^2-a^2-b^2)^2}{4}.$$
Thus $16K^2 = 4a^2b^2 - (c^2-a^2-b^2)^2$. The right side of this equation factors as a difference of squares to yield $16K^2 = [2ab + (c^2-a^2-b^2)][2ab - (c^2-a^2-b^2)] = [c^2-(a-b)^2][(a+b)^2-c^2]$. Each of the factors on the right of the previous equation factors as a difference of squares, and we obtain $16K^2 = [c+(a-b)][c-(a-b)][(a+b)+c][(a+b)-c]$. Observe that $c+a-b = (a+b+c)-2b = 2(s-b)$. Similarly, the second factor in our formula for $16K^2$ equals $2(s-a)$, and the third and fourth factors are $2s$ and $2(s-c)$, respectively. It follows that $K^2 = s(s-a)(s-b)(s-c)$.

Circumradius in terms of Lengths of Sides

Proposition

The circumradius R of triangle $\triangle ABC$ is given by

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

- We know that $4KR = abc$. Thus,

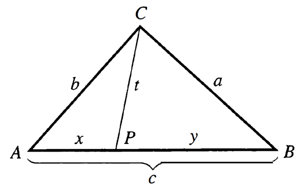
$$R = \frac{abc}{4K},$$

whence, by Heron's formula,

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

Stewart's Theorem

- Choose point P arbitrarily on side AB of $\triangle ABC$, dividing the side of length c, into pieces with lengths x and y. Let t be the length $t = CP$.



Theorem (Stewart)

$ct^2 + xyc = xa^2 + yb^2$.

- By the law of cosines, we have $t^2 = a^2 + y^2 - 2ay\cos(B)$ and $b^2 = a^2 + c^2 - 2ac\cos(B)$. We can eliminate $\cos(B)$ if we multiply the first equation by c and the second by y and then subtract. Using the fact that $c - y = x$, we obtain

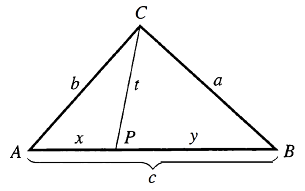
$$ct^2 - yb^2 = (c-y)a^2 + cy^2 - yc^2 = xa^2 + cy(y-c) = xa^2 - xyc.$$

Lengths of Angle Bisectors

Proposition

If CP is the angle bisector in $\triangle ABC$ and $t = CP$, then $t = \sqrt{ab\left[1 - \frac{c^2}{(a+b)^2}\right]}$.

- Recall that bisector CP divides AB into pieces proportional to the lengths of the nearer sides, i.e., $\frac{x}{y} = \frac{b}{a}$. Since $x + y = c$, a bit of algebra yields $x = \frac{bc}{a+b}$ and $y = \frac{ac}{a+b}$.



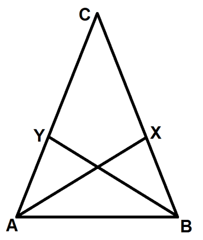
To check the algebra, observe that the sum of these two fractions is c and that the first divided by the second is equal to $\frac{b}{a}$. It follows that $xyc = \frac{abc^3}{(a+b)^2}$ and $xa^2 + yb^2 = \frac{a^2bc + b^2ac}{a+b} = abc$. We can substitute into the Stewart's Theorem equation to get $ct^2 + \frac{abc^3}{(a+b)^2} = abc$. A little more algebra now yields $t^2 = ab - \frac{abc^2}{(a+b)^2} = ab\left[1 - \frac{c^2}{(a+b)^2}\right]$.

Equality of Angle Bisectors

Proposition

Let AX and BY be angle bisectors in $\triangle ABC$ and suppose that $AX = BY$. Then $AC = BC$.

- We compute that $(AX)^2 = bc[1 - \frac{a^2}{(b+c)^2}]$ and $(BY)^2 = ac[1 - \frac{b^2}{(a+c)^2}]$. By hypothesis, we know that these two quantities are equal. Divide by c: $b[1 - \frac{a^2}{(b+c)^2}] = a[1 - \frac{b^2}{(a+c)^2}]$. Therefore, we get $b - a = \frac{ba^2}{(b+c)^2} - \frac{ab^2}{(a+c)^2}$.



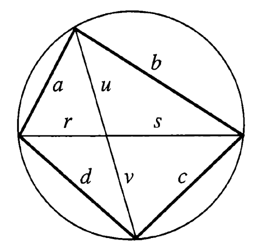
  We need to show that $a = b$. Suppose that a and b are unequal, say $b > a$. Then, the left side of the previous equation is positive. Thus, the right side must also be positive. It follows that $\frac{a}{(b+c)^2} > \frac{b}{(a+c)^2}$. But $b > a$, and so $\frac{b}{(a+c)^2} > \frac{a}{(a+c)^2} > \frac{a}{(b+c)^2}$. This contradicts the previous inequality.

Ptolemy's Theorem

Theorem (Ptolemy)

Let a, b, c and d be the lengths of consecutive sides of a quadrilateral inscribed in a circle and suppose x and y are the lengths of the diagonals. Then $ac + bd = xy$.

- Let r, s, u and v be the lengths of the four partial diagonals, where $r + s = x$ and $u + v = y$. It is easy to see from the two pairs of similar triangles that $\frac{a}{c} = \frac{u}{s} = \frac{r}{v}$ and $\frac{b}{d} = \frac{u}{r} = \frac{s}{v}$. Thus, we get $as = uc$, $br = ud$ and $uv = rs$. It follows that $sa^2 + rb^2 = uac + ubd = u(ac + bd)$ and $xu^2 + xrs = xu^2 + xuv = xu(u + v) = xuy$.



By Stewart's theorem, the quantities on the left sides of these equations are equal. Hence the right sides must also be equal. We conclude by canceling u that $ac + bd = xy$.

Subsection 5

The Incircle

The Incircle of a Triangle

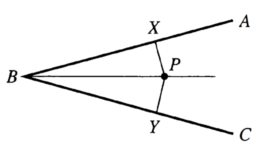
- A circle is said to be **inscribed** in a triangle if its center is interior to the triangle and all three sides of the triangle are tangent to the circle.
- Given an arbitrary triangle, we will show there must exist a unique inscribed circle, called the **incircle**.
  - Its center I is the **incenter**;
  - The length r of its radius is the **inradius**.
- Informally, to see why the incircle must exist:
  - Start with a small circle placed inside the triangle.
  - Let it grow continuously, keeping it inside the triangle by letting its center move freely as the circle grows.
  - Eventually, the circle will reach a maximum size, after which there is no room for further growth.
  - The max circle will be touching (that is, tangent to) all three sides.

Locus Property of Angle Bisectors

Lemma

The bisector of $\angle ABC$ is the locus of points P in the interior of the angle that are equidistant from the sides of the angle.

- If P is in the interior of $\triangle ABC$, drop perpendiculars PX and PY to lines AB and CB.

  P being equidistant from the sides of the angle is the same as saying that $PX = PY$. We must show that $PX = PY$ if and only if P lies on the angle bisector.

  Suppose that $PX = PY$. $\triangle PXB$ and $\triangle PYB$ are right triangles with right angles at X and Y. We have $\triangle PXB \cong \triangle PYB$ by HA. It follows that $\angle XBP = \angle YBP$.

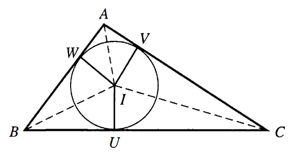
  Conversely, suppose that P lies on the angle bisector, i.e., $\angle XBP = \angle YBP$. Since $\angle BXP = 90^\circ = \angle BYP$ and $BP = BP$, $\triangle PXB \cong \triangle PYB$ by SAA. We conclude that $PX = PY$.

Angle Bisectors and Incircle

Theorem

The three angle bisectors of a triangle are concurrent at a point I, equidistant from the sides of the triangle. If we denote by r the distance from I to each of the sides, then the circle of radius r centered at I is the unique circle inscribed in the given triangle.

- Let $\triangle ABC$ be the given triangle and let I be the point where the bisectors of $\angle B$ and $\angle C$ meet. From I, drop perpendiculars IU, IV and IW from I to sides BC, AC and AB, respectively.



By the lemma, we have $IU = IW$ since I lies on the bisector of $\angle B$. Similarly, since I also lies on the bisector of $\angle C$, we see that $IU = IV$. We conclude that $IW = IV$. Thus point I must lie on the bisector of $\angle A$. Hence all three angle bisectors go through I.

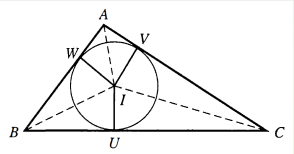
Angle Bisectors and Incircle

- We can now write $r = IU = IV = IW$, and we see that I is equidistant from sides BC, AC and AB. Point U lies on the circle of radius r centered at I. Since BC is a line through U perpendicular to radius IU, it follows that BC is tangent to this circle at U. Similarly, AC is tangent to this circle at point V, and AB is tangent at W.
- To see that this is the only circle inscribed in this triangle, suppose that we are given some inscribed circle. It suffices to show that its center is I and that its radius is r.
  - Let P be its center and X, Y and Z be the points of tangency of this circle with sides BC, AC and AB, respectively. Then radii PX and PY are perpendicular to sides BC and AC. Since $PX = PY$, we see that P must lie on the bisector of $\angle C$. Similarly, P lies on the bisector of $\angle B$, and we conclude that P is the point I.
  - It follows that each of PX and IU is a perpendicular drawn from this point to BC, and hence these are the same line segment. The radius PX of the unknown circle is thus equal to $IV = r$.

Inradius and Area

Proposition

Given a triangle with area K, semiperimeter s, and inradius r, $rs = K$. Therefore $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.



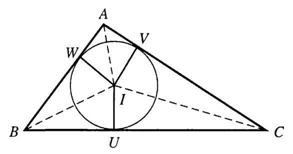
- We draw the incircle of $\triangle ABC$ and the three radii IU, IV and IW of length r joining the center I to the three points of tangency of the circle with the sides. These radii are thus perpendicular to the corresponding sides.
  We also draw segments IA, IB and IC, which bisect the angles of the triangle. Consider $\triangle BIC$. We have $K_{IBC} = \frac{1}{2}(IU)(BC) = \frac{1}{2}ra$. Similarly, $K_{IAB} = \frac{1}{2}rc$ and $K_{ICA} = \frac{1}{2}rb$. Adding these, we get
  $K = K_{IBC} + K_{IAB} + K_{ICA} = \frac{1}{2}r(a+b+c) = rs$.
  Since we know by Heron's formula, $K = \sqrt{s(s-a)(s-b)(s-c)}$, we deduce that $r = \frac{K}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

Incircle and Lengths of Side Segments

Lemma

There is exactly one way to choose points U, V and W on sides BC, AC and AB, respectively, of $\triangle ABC$ so that $AV = AW$, $BU = BW$ and $CU = CV$. The only points that satisfy these equations are the points where the sides of the triangle are tangent to the incircle. Furthermore, the distances AV, BW and CU are equal to $s-a$, $s-b$ and $s-c$, respectively.

- The lengths of the two tangents to a circle from an exterior point are equal. So $AV = AW$. To see this directly, note that $\triangle AVI \cong \triangle AWI$ by HA. Write $x = AV = AW$, $y = BU = BW$ and $z = CU = CV$.

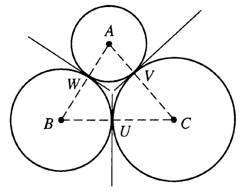


  We have $y+z = BC = a$, $x+z = b$ and $x+y = c$. The first gives $z = a-y$. Substitute into the second $x+a-y = b$. Hence $x-y = b-a$. Since $x+y = c$ from the third equation, we deduce that $x = \frac{c+b-a}{2} = s-a$. Similarly, $y = s-b$ and $z = s-c$.

Tangents to Externally Tangent Circles

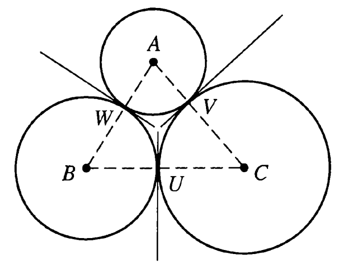
Proposition

Given three pairwise mutually externally tangent circles, show that the three common tangent lines are concurrent.



- Let U, V and W be the three points where two of the circles touch, and write A, B and C to denote the centers of the three given circles. Observe that radius BU is perpendicular to the common tangent through U and that the same is true of radius CU. So $\angle BUC = 180°$. Thus, U lies on the line segment BC. Similarly, V lies on AC and W lies on AB. We see now that U, V and W are points on the sides of $\triangle ABC$, and we have $AV = AW$ since these two segments are radii of the same circle. Similarly, $BU = BW$ and $CU = CV$. Hence, points U, V and W are the points of tangency of the incircle of $\triangle ABC$ with the sides of the triangle.

Tangents to Externally Tangent Circles (Cont'd)



- Points U, V and W are the points of tangency of the incircle of $\triangle ABC$ with the sides of the triangle. It follows that radius IU of the incircle is perpendicular to side BC at U. Since the common tangent line for the circles centered at B and C is perpendicular to BC at U, the tangent must be line IU. Similarly, the other two common tangent lines also go through the incenter I of $\triangle ABC$. Hence the three common tangents are concurrent at I.

The Law of Tangents

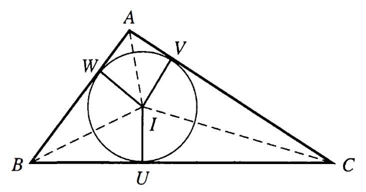
Theorem (Law of Tangents)

In $\triangle ABC$, let the quantities a, b, c and s have their usual meanings. Then

$$\tan\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

- Consider right $\triangle CUI$.

  We know that $\frac{UI}{UC}$ is the tangent of $\angle UCI$. Since CI bisects $\angle C$ of the original triangle, this gives $\tan(\frac{C}{2}) = \frac{UI}{UC} = \frac{r}{s-c}$.



  But we know that $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$. The result follows.

Coincidence of the Incenter and the Orthocenter

- In the case of an equilateral triangle, the circumcenter, the centroid, the orthocenter and the incenter coincide.
- Conversely, if any two of these points coincide, then the triangle must be equilateral:

Proposition

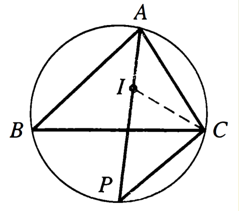
Suppose that the incenter and the circumcenter of $\triangle ABC$ are the same point. Then the triangle must be equilateral.

- Join O to vertices A and B. Note that $\triangle OAB$ is isosceles since we know that $OA = OB$. By the pons asinorum, we have $\angle OAB = \angle OBA$. O is also the incenter. So OA and OB bisect angles A and B. We conclude that $\angle A = 2\angle OAB$ and $\angle B = 2\angle OBA$. It follows that $\angle A = \angle B$. So by the converse of the pons asinorum, we see that $CA = CB$. Similarly, it follows that all of the sides are equal.

Intersection Point of Angle Bisector and Circumcircle

Lemma

Extend the bisector of one of the angles of a triangle to meet the circumcircle at point P. Then the distance from P to each of the other two vertices of the triangle is equal to the distance IP, where I is the incenter of the given triangle.



- Draw line AP bisecting $\angle A$ of $\triangle ABC$. I lies on this line, and we must show $IP = CP$. It suffices to show that $\angle ICP = \angle CIP$. Since $\angle BCP$ subtends the same arc as $\angle BAP$, we see that $\angle BCP = \angle BAP = \frac{1}{2}\angle A$. Also, IC bisects $\angle C$ of the original triangle. Thus $\angle ICB = \frac{1}{2}\angle C$.
  Hence $\angle ICP = \frac{1}{2}(\angle A + \angle C)$. To compute $\angle CIP$, we observe first that $\angle P = \angle B$ since these subtend the same arc. Thus, $\angle ICP + \angle CIP = 180° - \angle P = 180° - \angle B = \angle A + \angle C$. Since $\angle ICP = \frac{1}{2}(\angle A + \angle C)$, $\angle CIP = \frac{1}{2}(\angle A + \angle C)$.

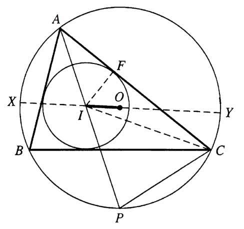
Distance Between the Circumcenter and the Incenter

Theorem (Euler)

Let $d = OI$, the distance from the circumcenter to the incenter of an arbitrary triangle. Then $d^2 = R(R-2r)$, where R and r are the circumradius and inradius of the given triangle.

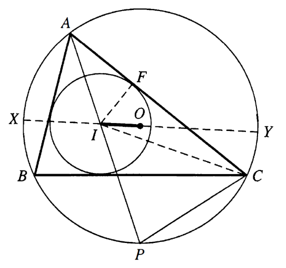
- We extend line segment OI to a diameter XY of the circumcircle.

  Since $OI = d$, $XI = R - d$ and $YI = R + d$. Thus, $XI \cdot YI = R^2 - d^2$. We know that the product of the lengths of the two pieces of each chord through I is a constant, independent of the particular chord. It follows that $R^2 - d^2 = XI \cdot YI = AI \cdot IP$. Next, we try to compute the length IP, which we know is equal to PC.



Proof of Euler's Theorem (Cont'd)

- We have $R^2 - d^2 = XI \cdot YI = AI \cdot IP$. We want to compute the length $IP = PC$:
  We use the extended law of sines in $\triangle APC$. Since $\angle PAC = \frac{1}{2}\angle A$, we have $\frac{PC}{\sin(\frac{A}{2})} = 2R$. Hence



$$R^2 - d^2 = AI \cdot IP = AI \cdot PC = AI \cdot 2R \sin(\frac{A}{2}).$$

Finally, we compute $AI \cdot \sin(\frac{A}{2})$ by working in right $\triangle AIF$, where F is the point of tangency of the incircle with side AC. Since $IF = r$, we see that $\sin(\frac{A}{2}) = \frac{r}{AI}$. Thus, $AI \cdot \sin(\frac{A}{2}) = r$.
If we substitute this into our previous formula, we get $R^2 - d^2 = 2rR$. So $d^2 = R(R - 2r)$.

Comparing the Circumradius with the Inradius

Corollary

For any triangle, $R \geq 2r$ and equality holds if and only if the triangle is equilateral.

- Since $d^2 = R(R-2r)$, we see that $R-2r$ can never be negative.

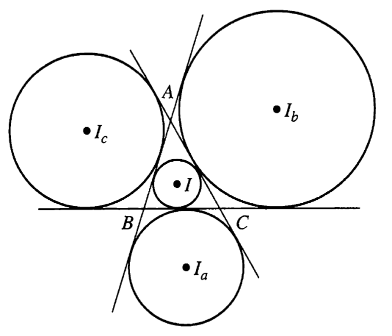
  Furthermore, $R = 2r$ if and only if $d = 0$; in other words, $R = 2r$ if and only if points I and O are identical. But I and O coincide if and only if the triangle is equilateral.

Subsection 6

Exscribed Circles

Excircles

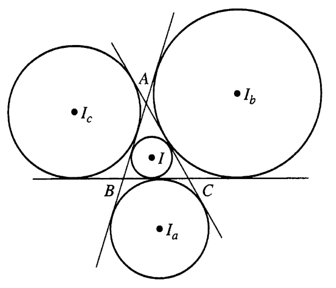
- The incircle of a triangle is sometimes referred to as a **tritangent circle** because it is tangent to all three sides of the triangle.
- If we are willing to consider circles tangent to extensions of the sides, a triangle also has three other tritangent circles.
- The three tritangent circles whose centers are exterior to the given triangle are called the **exscribed circles** or the **excircles** of the triangle.



Excenters and Exradii

- The center of each of the exscribed circles lies at the intersection of three angle bisectors.

  For example, the center of the excircle opposite vertex A, denoted I_a, lies at the point of concurrence of the bisector of $\angle A$ and the bisectors of the exterior angles at points B and C.

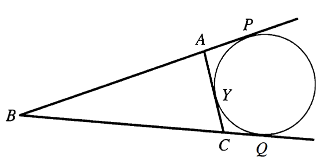


- The centers of the excircles of $\triangle ABC$ are the **excenters** of $\triangle ABC$.
- The corresponding radii are denoted r_a, r_b and r_c, and they are referred to as the **exradii** of the triangle.
- The three exradii, together with the inradius r, are collectively known as the **tritangent radii**.

Excenters and Exradii

Lemma

The length of the tangent from a vertex of a triangle to the opposite exscribed circle is equal to the semiperimeter s.



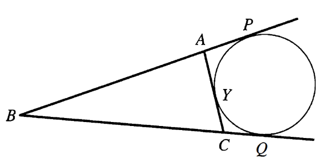
- Let Y, P and Q be the points of tangency, as shown. Since the two tangents to a circle from an exterior point are equal, we know that $BP = BQ$. We need to show that this common length is s.
  Since $AP = AY$, we have $BP = BA + AP = BA + AY$.
  Similarly, $BQ = BC + CY$.
  Adding, we get $BP + BQ = BA + AY + BC + CY = BA + BC + AC = 2s$.
  But $BP = BQ$, whence we get $BP = BQ = s$.

Relation Between the Tritangent Radii

Theorem

Given an arbitrary $\triangle ABC$, we have $\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$.

- We have $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \sqrt{\frac{s-a}{s(s-b)(s-c)}} + \sqrt{\frac{s-b}{s(s-a)(s-c)}} + \sqrt{\frac{s-c}{s(s-a)(s-b)}} = \frac{(s-a)+(s-b)+(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{s}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{s}{K} = \frac{1}{r}$.
- Note that in the figure $\triangle BPI_b$ is a right triangle with arm BP of length s. Since arm I_bP has length r_b, it follows that $\tan\left(\frac{B}{2}\right) = \frac{r_b}{s}$.



By the law of tangents, we obtain

$$r_b = s \cdot \tan\left(\frac{B}{2}\right) = s\sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{s(s-a)(s-c)}{s-b}}.$$