College Geometry

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LSSU Math 325

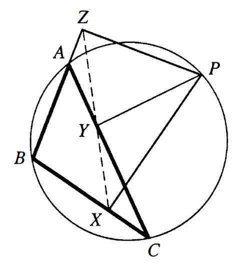
1 Circles and Lines
- Simson Lines
- The Butterfly Theorem
- Cross Ratios
- The Radical Axis

Subsection 1

Simson Lines

Simson Line of a Triangle

- Given $\triangle ABC$, choose an arbitrary point P on its circumcircle and drop perpendiculars PX, PY and PZ to sides BC, AC and AB. It is almost always necessary to extend at least one of the sides of the triangle to do this. (In the figure, side AB is extended to meet the perpendicular from P.)



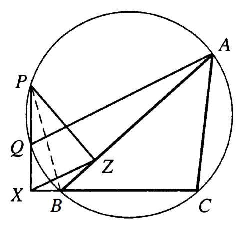
- Simson's theorem asserts that the feet X, Y and Z of the three perpendiculars from point P are always collinear.
- Appropriately, the line through X, Y, and Z is called the **Simson line** of $\triangle ABC$ with respect to point P.
- P is referred to as the **pole** for this Simson line.

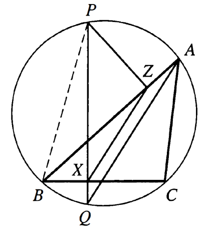
Points on Circumcircle Forming Perpendicular to a Side

Theorem

Choose a point P on the circumcircle of $\triangle ABC$ and let Q be the other point where the perpendicular to BC through P meets the circumcircle. Let X be the point where this perpendicular meets line BC and let Z be the point where the perpendicular to AB through P meets AB. If Q is different from A, then Z lies on the line parallel to QA through X.

- Point P may be such that the foot X of the perpendicular from P to BC falls on an extension of that side.
  P may also be chosen so that X actually lies on segment BC.

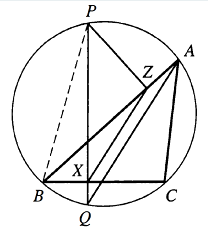




Proof of the Theorem

- If X and Z happen to be the same point, there is really nothing to prove.

  So we can assume that X and Z are distinct. The goal is to prove that $XZ \parallel QA$. If P is at the point B, then X and Z are also at B. Since we are assuming that X and Z are different, this does not happen.



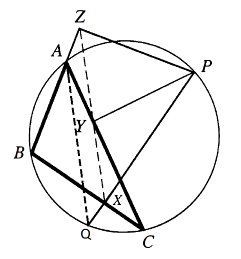
  Thus P and B are different, and we can consider the unique circle having diameter PB. Note $\angle PXB = 90^\circ = \angle PZB$. So points X and Z lie on this circle. Thus, $\angle PXZ = \angle PBZ$, since these angles subtend the same arc in that circle. We also have $\angle PBZ = \angle PQA$ because these angles subtend the same arc in the original circle. Thus, $\angle PXZ = \angle PQA$. So XZ and QA are parallel.

Simson's Theorem

Theorem (Simson's Theorem)

Let P be any point on the circumcircle of $\triangle ABC$ and let X, Y and Z be the feet of the perpendiculars dropped from P to lines BC, AC and AB, respectively. Then points X, Y and Z are collinear.

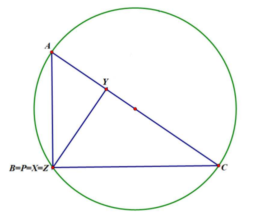
- Suppose first that P does not lie on the altitude from A in $\triangle ABC$. Let Q be as in the the preceding theorem. Since P is not on the altitude from A, the hypothesis in the preceding theorem, Q and A are distinct points is guaranteed to hold.



By the theorem, we know that Z lies on the line through X parallel to AQ. Exactly similar reasoning shows that Y also lies on the line through X parallel to AQ. Thus, X, Y and Z lie on a common line.

Simson's Theorem: The Remaining Case

- Suppose P lies on the altitude from A.
  - If P fails to lie on the altitude from B or on the altitude from C, we use similar reasoning, as before, and we get the same conclusion.
  - The only remaining case, therefore, is when P lies on all three altitudes, in which case P is the orthocenter of $\triangle ABC$. But P lies on the circumcircle of this triangle. We have seen that it is only for a right triangle that the orthocenter can lie on the circumcircle.



We can now assume that $\triangle ABC$ is a right triangle and that P is its orthocenter. We can suppose that $\angle B$ is the right angle. It follows that P is at B, and thus X and Z are also at B. Since X and Z are the same point in this case, the points X, Y and Z are certainly collinear.

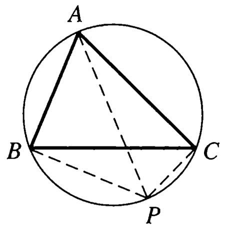
Simson's Line Parallel to Sides

Proposition

The Simson line will be parallel to one of the sides of the triangle if the pole is antidiametric to the opposite angle on the circumcircle.

- Suppose that we want a Simson line parallel to side BC. By definition, every Simson line contains at least one point on line BC. So, if a Simson line is parallel to BC, this Simson line must actually be BC.

  Is it possible to find a pole P for which the Simson line is the line BC? If such a point exists and we draw the perpendicular from P to side AC, then the foot Y of this perpendicular must lie on the Simson line BC. Since vertex C is the only point common to AC and BC, Y must be at C. So PC is perpendicular to AC.

  If it exists, therefore, the pole P must lie on the line perpendicular to AC at C. Similarly, P must lie on the perpendicular to AB at B.

Simson's Line Parallel to Sides (Cont'd)

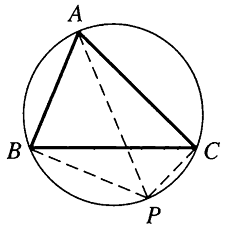
- We concluded that P is the point where the perpendiculars to AC at C and to AB at B intersect.

  The other requirement is that the pole must lie on the given circle.

  So we need to ask whether or not the intersection point P of the two perpendiculars PB and PC actually lies on the circle.

  Putting the question another way, we ask if we can find a point P on the circle such that $\angle PBA = 90° = \angle PCA$.

  The answer is to choose P so that AP is a diameter.

  We have shown, therefore, that if we take as a pole a point on the circumcircle of a triangle diametrically opposite a vertex, then the corresponding Simson line is the side opposite that vertex.

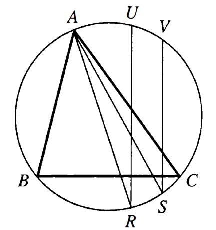
Angle Between Different Simson Lines

Theorem

Let U and V be points on the circumcircle of $\triangle ABC$. The angle between the Simson lines having U, V as poles is equal in degrees to half of $\widehat{UV}$.

- The meaning is that the smaller of the two angles is equal in degrees to half of the smaller of the two arcs.

  We extend the perpendiculars from U and V to BC to meet the circle at R and S. By the theorem, we know that the Simson line having poles U, V are parallel to AR and AS. The angle between these two Simson lines is thus equal to $\angle RAS \stackrel{\circ}{=} \frac{1}{2}\widehat{RS}$.



  Since UR and VS are parallel chords, we see that $\widehat{UV} = \widehat{RS}$.

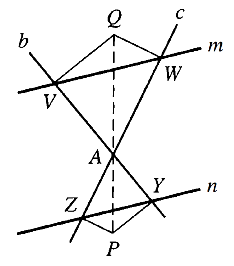
Corollary

Two Simson lines for a given triangle are perpendicular if and only if their poles are at opposite ends of a diameter.

Triple of Points Based on Parallels and Secants

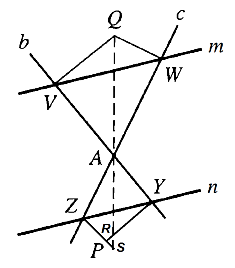
Lemma

Suppose that lines m and n are parallel, lines b and c meet at a point A, line m meets b and c at points V and W, respectively, and line n meets b and c at Y and Z, respectively. Perpendiculars to b and c are erected at V and W, and these meet at a point Q. Similarly, the perpendiculars to b and c at Y and Z meet at P. Then points P, A and Q are collinear.



- If A happens to be P or Q, there is nothing to prove. So we assume that A is neither P nor Q. Thus, neither m nor n passes through A. If point Q lies on b, m must be perpendicular to c. So n is also perpendicular to c. Thus, P must also lie on b. In this case, P, A and Q are collinear. So we suppose that Q does not lie on b, and similarly, that Q is not on c and that P is not on b or c. In particular, Q is different from V and W, and P is different from Y and Z.

Triple of Points Based on Parallels and Secants (Cont'd)



Observe that V is the foot of the perpendicular from Q to b. Since A lies on b and A is different from V, AQ is not perpendicular to b. Thus AQ is not parallel to YP. Similarly, AQ is not parallel to ZP. The goal is to show that P lies on AQ.

Since YP is not parallel to AQ, we consider the point R where YP meets AQ. Similarly, let S be the point where ZP meets AQ. We show that R and S are the same point.

Since P is the only point common to YP and ZP, it will follow that P, R and S are all the same point and that P lies on AQ.

Now $YR \parallel VQ$, since both are perpendicular to b. Using similar triangles, $\frac{AR}{AQ} = \frac{AY}{AV}$. Similarly, $\frac{AS}{AQ} = \frac{AZ}{AW}$. But since $YZ \parallel VW$, we see that $\frac{AY}{AV} = \frac{AZ}{AW}$. Hence, $\frac{AR}{AQ} = \frac{AY}{AV} = \frac{AZ}{AW} = \frac{AS}{AQ}$. So $AR = AS$, and R and S are the same point.

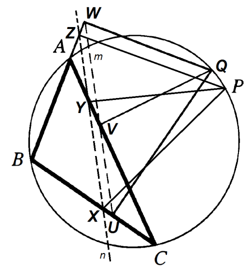
A Converse to Simson's Theorem

Theorem

Given $\triangle ABC$, suppose that the feet of the perpendiculars from some point Q to the three sides of the triangle are collinear. Then Q must lie on the circumcircle of $\triangle ABC$.

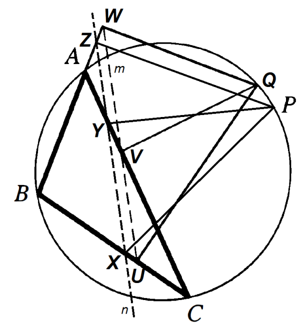
- Let U, V and W be the feet of the perpendiculars from Q to BC, AC and AB, respectively, and suppose that these three points all lie on some line m. We have seen that a Simson line can be found parallel to any given line.

  So we can choose a pole P for which the corresponding Simson line n is parallel to m. By definition, n runs through the points X, Y and Z, which are the feet of the perpendiculars from P to lines BC, AC and AB, respectively. Thus, if we define b to be the line AC and c to be the line AB, we are in the situation of the lemma.



A Converse to Simson's Theorem (Cont'd)

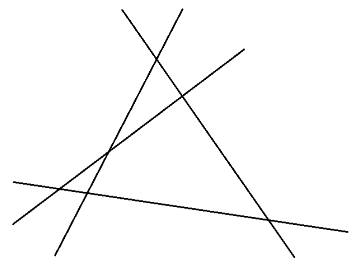
- We conclude that points P, A and Q are collinear.
  Similar reasoning shows that points P, B and Q are also collinear and P, C and Q are collinear too.
  If P and Q are not the same point, it follows that line PQ runs through all three vertices A, B and C of $\triangle ABC$, a contradiction.



  Since P lies on the circumcircle of $\triangle ABC$, it follows that Q lies on the circumcircle.

Four Lines in General Position

- Four lines are in **general position** if no two of the lines are parallel and no three of them are concurrent.



- Four lines in general position determine four triangles by taking the lines three at a time.
- Moreover, there are six points of intersection of the lines.

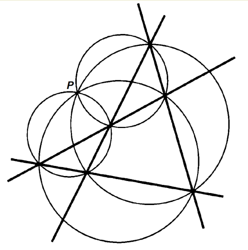
Circumcircles Formed by Lines in General Position

Corollary

The circumcircles of the four triangles determined by any four lines in general position always go through a common point.

- Draw two of the circumcircles and observe that one of their points of intersection, which we call P, lies on none of the given lines. Drop perpendiculars from P to each of the four lines, thereby determining four feet, one on each line.

  By Simson's theorem applied in one of the circles, three of the four feet are collinear. By a second application of Simson's theorem, in the other circle, another three of the feet are collinear. It follows that all four of the feet of the perpendiculars are collinear. By the theorem, the point P must lie on all four circumcircles.



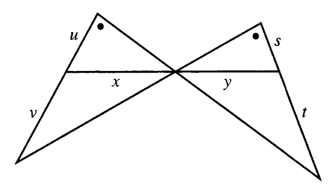
Subsection 2

The Butterfly Theorem

The Butterfly Lemma

Lemma

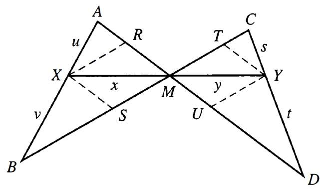
In the figure segments of length x, y, u, v, s and t are marked. If the angles marked with dots are equal, then $\frac{x^2}{y^2} = \frac{uv}{st}$.



- Label the points in the original diagram.

  Draw XR and YU parallel to BC and XS and YT parallel to AD. Since $\triangle XRM \sim \triangle YUM$ and $\triangle XSM \sim \triangle YTM$ by AA, we conclude that $\frac{RM}{UM} = \frac{x}{y} = \frac{SM}{TM}$.

  This yields $\frac{x^2}{y^2} = \frac{RM \cdot SM}{TM \cdot UM}$.



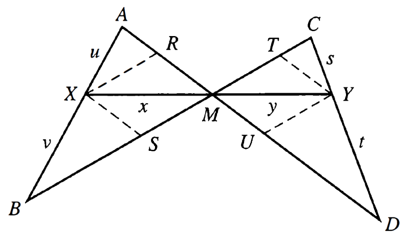
The Butterfly Lemma (Cont'd)

- We have $\frac{x^2}{y^2} = \frac{RM \cdot SM}{TM \cdot UM}$.

  Now $\angle A = \angle C$ by hypothesis. But $\angle AMB = \angle CMD$, So we must also have $\angle B = \angle D$. Also,

  $$\angle AXR = \angle B = \angle D = \angle CYT.$$

  By AA, $\triangle AXR \sim \triangle CYT$.



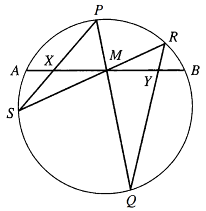
  We conclude that $\frac{u}{s} = \frac{AX}{CY} = \frac{XR}{YT} = \frac{SM}{UM}$. Similarly, $\frac{v}{t} = \frac{RM}{TM}$.
  Therefore, $\frac{x^2}{y^2} = \frac{v}{t} \frac{u}{s}$.

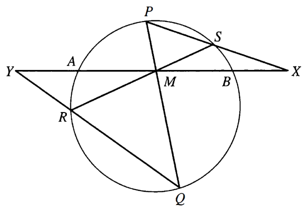
The Butterfly Theorem

Theorem

Suppose that chords PQ and RS of a given circle meet at the midpoint M of chord AB. If X and Y are the points where PS and QR meet AB, respectively, then $XM = YM$.

- Two possible configurations for the theorem are:



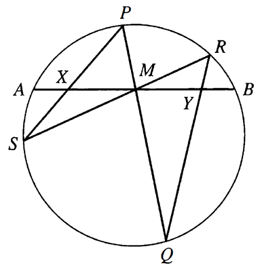


The only difference is that on the left, points P and R lie on the same side of line AB, while on the right, we have interchanged the labels R and S so that P and R lie on opposite sides of AB. In the latter situation, chord AB had to be extended to meet lines PS and QR.

The Butterfly Theorem

- We prove the case appearing on the left.

  Since $\angle P = \angle R$, we apply the lemma to conclude that



$$\left(\frac{XM}{YM}\right)^2 = \frac{PX \cdot XS}{RY \cdot YQ}.$$

Now $PX \cdot XS = AX \cdot XB$.
Similarly, $RY \cdot YQ = BY \cdot YA$.
Write $x = XM$, $y = YM$ and $AM = m = BM$. We have

$$\frac{x^2}{y^2} = \frac{AX \cdot XB}{BY \cdot YA} = \frac{(m-x)(m+x)}{(m-y)(m+y)} = \frac{m^2 - x^2}{m^2 - y^2}.$$

Since $m \neq 0$, elementary algebra now yields that $x^2 = y^2$. We conclude that $x = y$.

Subsection 3

Cross Ratios

Distance and Ratios

- Two distinct points A and B determine the number AB, the length of the line segment they determine or the distance between them.

  The number AB is not unambiguously determined by the two points since it depends on the unit of measurement.

- Three distinct collinear points A, B and C determine a number that is independent of the unit of measurement:

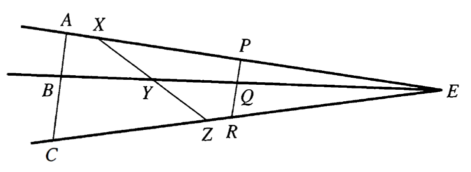
  Define the associated quantity $r(A, B, C)$ to be the ratio

$$r(A, B, C) = \frac{AB}{BC}.$$

  The ratio $r(A, B, C)$ is independent of the unit of measurement, but it suffers from another deficiency that is shared by the distance function.

Sets of Points in Perspective

- Consider three sets of collinear points $\{A, B, C\}$, $\{X, Y, Z\}$ and $\{P, Q, R\}$ that look identical to an eye located at point E. We say that these three sets are in **perspective** from E.



- If $AC \parallel PR$, using similar triangles, we can show that $r(A, B, C) = r(P, Q, R)$.
- If $AC \nparallel XZ$, as in the diagram, $r(A, B, C)$ may not be equal to $R(X, Y, Z)$.

  E.g., we have taken Y to be the midpoint of segment XZ, and hence $r(X, Y, Z) = 1$, but B is not the midpoint of AC, and so $r(A, B, C) \neq 1$.

The Cross Ratio

- If we start with four collinear points A, B, C and D, it is possible to define a unitless quantity, denoted $\mathrm{cr}(A, B, C, D)$, that is invariant under perspective:

  If $\{A, B, C, D\}$ and $\{W, X, Y, Z\}$ are sets of distinct collinear points in perspective from a point E, then $\mathrm{cr}(A, B, C, D) = \mathrm{cr}(W, X, Y, Z)$.

- The quantity $\mathrm{cr}(A, B, C, D)$ is called the **cross ratio** of the four distinct collinear points A, B, C and D, and it is defined by the formula

$$\mathrm{cr}(A, B, C, D) = \frac{AC \cdot BD}{AD \cdot BC}.$$

In Cross Ratio Order of Points Matters

- Suppose the points A, B, C and D are equally spaced and they are arrayed in that order along a line.

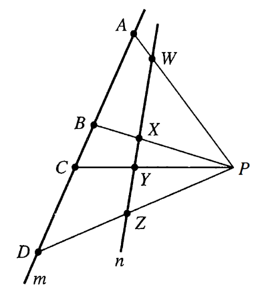
  Then we have:

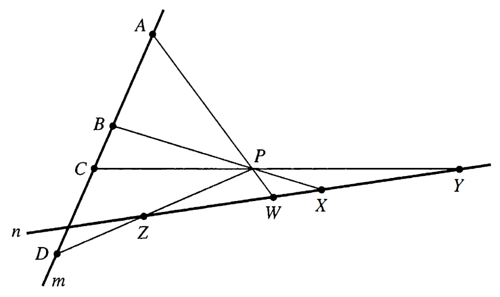
$$\begin{array}{rcl} \text{cr}(A,B,C,D) & = & \frac{AC\cdot BD}{AD\cdot BC} = \frac{4}{3}; \\ \text{cr}(B,A,C,D) & = & \frac{BC\cdot AD}{BD\cdot AC} = \frac{3}{4}. \end{array}$$

- Since the cross ratio of four points depends on the order in which the points occur, the notion of perspective must keep track of the order in which the points occur.

Set of Points in Perspective from a Point

- Suppose m and n are lines and P is a point not on either of them. Then points A, B, C and D of line m are **in perspective from** P with points W, X, Y and Z of line n, respectively, if W, X, Y and Z lie on lines PA, PB, PC and PD, respectively.

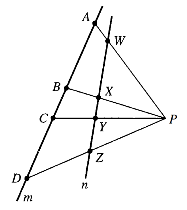




A Cross-Ratio Formula

Lemma

Let A, B, C and D be distinct collinear points and suppose P is any point not on the line through them. Then $\text{cr}(A,B,C,D) = \frac{\sin(\angle APC)\sin(\angle BPD)}{\sin(\angle APD)\sin(\angle BPC)}$.



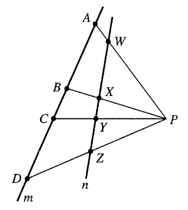
- Let h be the perpendicular distance from P to the line containing A, B, C and D. Then we can compute the area K_{APC} of $\triangle APC$ in two ways: $\frac{1}{2}h \cdot AC = K_{APC} = \frac{1}{2}(PA \cdot PC)\sin(\angle APC)$. There are similar formulas for $K_{BPD}, K_{APD}, K_{BPC}$: $\frac{1}{2}h \cdot AC = \frac{1}{2}(PA \cdot PC)\sin(\angle APC)$, $\frac{1}{2}h \cdot BD = \frac{1}{2}(PB \cdot PD)\sin(\angle BPD)$, $\frac{1}{2}h \cdot AD = \frac{1}{2}(PA \cdot PD)\sin(\angle APD)$, $\frac{1}{2}h \cdot BC = \frac{1}{2}(PB \cdot PC)\sin(\angle BPC)$. If we divide the product of the first two of these equations by the product of the second two, the lengths PA, PB, PC, PD and h all cancel and we get $\text{cr}(A,B,C,D) = \frac{AC \cdot BD}{AD \cdot BC} = \frac{\sin(\angle APC)\sin(\angle BPD)}{\sin(\angle APD)\sin(\angle BPC)}$.

Invariance of the Cross-Ratio

Theorem

Suppose that A, B, C and D are distinct collinear points in perspective from some point P, with distinct collinear points W, X, Y and Z, respectively. Then $\mathrm{cr}(A,B,C,D) = \mathrm{cr}(W,X,Y,Z)$.

- Observe that in the diagram, we have $\angle APC = \angle WPY$, $\angle BPC = \angle XPY$, $\angle APD = \angle WPZ$, $\angle BPD = \angle XPZ$. Not all four of these relations hold in the other diagram. The last two are replaced by $\angle APD + \angle WPZ = 180°$ and $\angle BPD + \angle XPZ = 180°$.

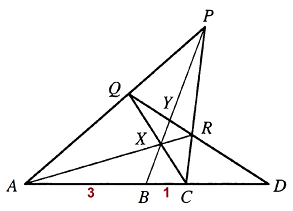


In general, each of $\angle APC, \angle BPD, \angle APD$ and $\angle BPC$ is either equal to or supplementary to $\angle WPY, \angle XPZ, \angle WPZ$ and $\angle XPY$, respectively. Thus, we always have $\sin(\angle APC) = \sin(\angle WPY)$, $\sin(\angle BPD) = \sin(\angle XPZ)$, $\sin(\angle APD) = \sin(\angle WPZ)$, $\sin(\angle BPC) = \sin(\angle XPY)$. The theorem now follows from the lemma.

Three Collinear Points and a Point not on their Line

- Start with collinear points A, B and C with distances $AB = 3$ and $BC = 1$.

  Choose point P not on line AC.
  Lines PA, PB and PC were drawn.
  Point Q, different from A and P was chosen arbitrarily on AP.
  Draw CQ meeting PB at a point labeled X.
  Then draw AX meeting PC at R.
  Finally, point D is determined as the point where QR meets the original line AC.



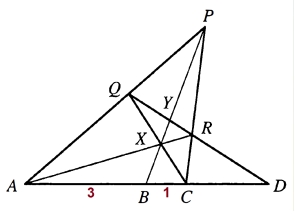
  This information is sufficient to compute the cross ratio $\mathrm{cr}(A, B, C, D)$, even without specifying the distances AB and BC.

The Cross Ratio of A, B, C and D

Lemma

In the configuration shown below $\text{cr}(A,B,C,D) = 2$.

- Since collinear Q, Y, R and D are in perspective from P with collinear A, B, C and D, $\text{cr}(Q,Y,R,D) = \text{cr}(A,B,C,D)$. In perspective from X, we have Q, Y, R and D are in perspective from C, B, A and D, respectively, whence $\text{cr}(Q,Y,R,D) = \text{cr}(C,B,A,D)$.



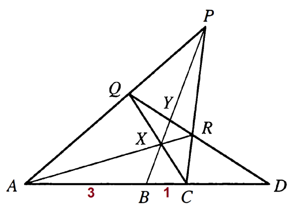
It follows that $\frac{AC \cdot BD}{AD \cdot BC} = \text{cr}(A,B,C,D) = \text{cr}(C,B,A,D) = \frac{CA \cdot BD}{CD \cdot BA}$. Since the numerators of are equal, the denominators must be equal too. Write $x = AB, y = BC$ and $z = CD$. We have $(x+y+z)y = AD \cdot BC = CD \cdot BA = zx$. Thus, $y(y+z) = xz - xy$. We can recompute the numerator $AC \cdot BD = (x+y)(y+z) = x(y+z) + y(y+z) = x(y+z) + x(z-y) = 2xz$. Since the denominator is equal to xz, we have $\text{cr}(A,B,C,D) = 2$.

Computing CD, given AB and BC

Proposition

Given $AB = 3$ and $BC = 1$ in the preceding configuration, we have $CD = 2$.

- We have $AB = 3$ and $BC = 1$. Set $CD = z$. By the lemma, we have



$$2 = \mathrm{cr}(A, B, C, D) = \frac{4(1+z)}{4+z}.$$

Solving this, we get $z = 2$.

- More generally, if we write $AB = x$, $BC = y$ and $CD = z$, we have:

$$2 = \frac{(x+y)(y+z)}{(x+y+z)y} \Rightarrow 2y(x+y) + 2yz = (x+y)y + (x+y)z$$

$$\Rightarrow (x-y)z = y(x+y) \Rightarrow z = \frac{y(x+y)}{x-y}.$$

Cocircular Points and their Cross Ratio

- We say that points located on a circle are **cocircular**.
- If three or more distinct points are cocircular, then there is a unique circle containing all of them.
- Suppose A, B, C and D are four distinct cocircular points.

  We define the **cross ratio** of these points to be the quantity

  $$\mathrm{cr}(A,B,C,D) = \frac{\sin\left(\frac{1}{2}\widehat{AC}\right)\sin\left(\frac{1}{2}\widehat{BD}\right)}{\sin\left(\frac{1}{2}\widehat{AD}\right)\sin\left(\frac{1}{2}\widehat{BC}\right)},$$

  where the arcs, of course, are on the common circle through the four given points.
- Concerning potential ambiguities:
  - We can measure the arcs in degrees or radians.
  - We can use any of the two arcs determined by X and Y.

Overloading of the Cross Ratio Notation

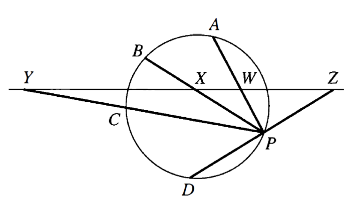
- We used the same notation $\mathrm{cr}(A, B, C, D)$ for the cross ratio of four collinear points and for the cross ratio of four cocircular points.
- We are safe, however, because four distinct points can never be both collinear and cocircular, and hence at most one of the two definitions of $\mathrm{cr}(A, B, C, D)$ applies.
- The same name "cross ratio" and the same notation "$\mathrm{cr}(,,,)$" for the concepts applying to four collinear points and to four cocircular points suggest an intimate connection between these two concepts.

Equality of Cross Ratios

Theorem

Let A, B, C and D be four distinct cocircular points and suppose P is a point on the same circle, different from all of them. Given a line not through P, let W, X, Y and Z be the four necessarily distinct and collinear points where PA, PB, PC and PD, respectively, meet the given line. Then $\mathrm{cr}(A, B, C, D) = \mathrm{cr}(W, X, Y, Z)$.

- By the preceding lemma, $\mathrm{cr}(W, X, Y, Z) = \frac{\sin(\angle WPY)\sin(\angle XPZ)}{\sin(\angle WPZ)\sin(\angle XPY)}$. By definition, $\mathrm{cr}(A, B, C, D) = \frac{\sin(\frac{1}{2}\widehat{AC})\sin(\frac{1}{2}\widehat{BD})}{\sin(\frac{1}{2}\widehat{AD})\sin(\frac{1}{2}\widehat{BC})}$. It suffices to show equality of corresponding sines.

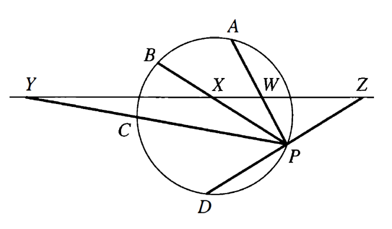


  We have $\angle WPY = \angle APC \overset{\circ}{=} \frac{1}{2}\widehat{ABC}$. Thus $\sin(\angle WPY) = \sin(\frac{1}{2}\widehat{AC})$ ($\widehat{AC}$ unspecified). Similarly, $\angle XPY = \angle BPC \overset{\circ}{=} \frac{1}{2}\widehat{BC}$, where we refer here to the arc between B and C that excludes point P.

Equality of Cross Ratios (Cont'd)

- Thus, $\sin(\angle XPY) = \sin(\frac{1}{2}\widehat{BC})$ (with $\widehat{BC}$ unspecified).

  In the configuration of the figure we see that $\angle XPZ$ is supplementary to $\angle BPD \stackrel{\circ}{=} \frac{1}{2}\widehat{BCD}$, and, thus, $\angle XPZ \stackrel{\circ}{=} 180° - \frac{1}{2}\widehat{BPD}$, and we have $\sin(\angle XPZ) = \sin(\frac{1}{2}\widehat{BD})$. Similarly, $\angle WPZ \stackrel{\circ}{=} 180° - \frac{1}{2}\widehat{APD}$, and, thus, $\sin(\angle WPZ) = \sin(\frac{1}{2}\widehat{AD})$.

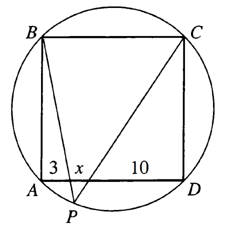
  In general, each angle is always equal in degrees to half of one of the two possible arcs corresponding to it.
  - We use the arc that contains P when P lies between exactly one of the two pairs of corresponding points;
  - We use the arc excluding P, otherwise.

  The four sines of angles are always equal to the four sines of arcs.
- In this situation, we refer to P as the **projection point**.

An Inscribed Square Problem

Proposition

A point on the circumcircle of a square is joined to the two most distant vertices, thereby cutting the nearest side of the square into three pieces. If the two extreme pieces have lengths 3 and 10, then the middle piece has length 2.



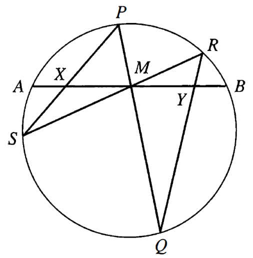
- Denote the intersection points of PB and PC with AD by R and S, respectively, and write $RS = x$. By the theorem with projection point P,

$$2 = \mathrm{cr}(A,B,C,D) = \mathrm{cr}(A,R,S,D) = \frac{AS \cdot RD}{RS \cdot AD} = \frac{(x+3)(x+10)}{x(x+13)}.$$

So we get $x^2 + 13x - 30 = 0$. This quadratic equation has roots $x = 2, -15$. We conclude $x = 2$.

Butterfly Theorem via Cross Ratios

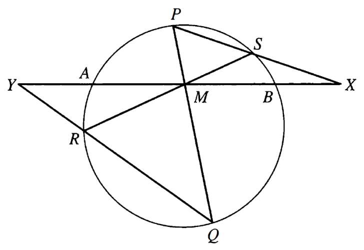
- Recall that M is the midpoint of AB and must show $MX = MY$.
  Using P as the projection point, we get $\text{cr}(A, X, M, B) = \text{cr}(A, S, Q, B)$.
  With R as the projection point, we get $\text{cr}(A, M, Y, B) = \text{cr}(A, S, Q, B)$.
  So $\text{cr}(A, X, M, B) = \text{cr}(A, M, Y, B)$.



  Write $x = MX$, $y = MY$, and $AM = m = MB$. We see that
  $\text{cr}(A, X, M, B) = \frac{AM \cdot XB}{AB \cdot XM} = \frac{m(x+m)}{2mx} = \frac{1}{2} + \frac{m}{2x}$ and
  $\text{cr}(A, M, Y, B) = \frac{AY \cdot MB}{AB \cdot MY} = \frac{(y+m)m}{2my} = \frac{1}{2} + \frac{m}{2y}$.
  Since the two are equal, $x = y$.

Butterfly Theorem via Cross Ratios (Second Case)

- Using P as the projection point, we get $\text{cr}(A,X,M,B)=\text{cr}(A,S,Q,B)$.
  With R as the projection point, we get $\text{cr}(A,M,Y,B)=\text{cr}(A,S,Q,B)$.
  So $\text{cr}(A,X,M,B)=\text{cr}(A,M,Y,B)$.



  A similar calculation, as before, yields that
  $\text{cr}(A,X,M,B)=\frac{m(x-m)}{2mx}=\frac{1}{2}-\frac{m}{2x}$ and
  $\text{cr}(A,M,Y,B)=\frac{(y-m)m}{2my}=\frac{1}{2}-\frac{m}{2y}$.
  We deduce that $x=y$, in this case too.

Subsection 4

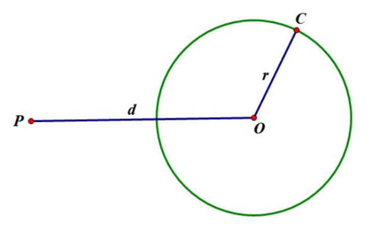
The Radical Axis

Power of a Point with respect to a Circle

- Given a point P and a circle of radius r centered at some point O, we say that the **power** of P with respect to the given circle is the quantity

$$p = d^2 - r^2,$$

where $d = PO$ is the distance from the point to the center of the circle.



  - The points exterior to the circle clearly have positive power.
  - The interior points have negative power.
  - The points of the circle itself have power $p = 0$.

Secant Drawn to a Circle and Power of a Point

Lemma

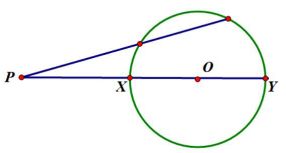
Fix a circle and a point P and let p be the power of P with respect to the given circle.

a. If P lies outside the circle and a line through P cuts the circle at X and Y, then $PX \cdot PY = p$.

b. If P is inside the circle on chord XY, then $PX \cdot PY = -p$.

c. If P lies on the line tangent to the circle at point T, then $(PT)^2 = p$.

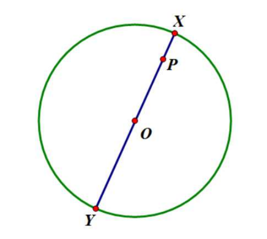
a. Assume that P lies outside the circle. The quantity $PX \cdot PY$ is the same for all lines through P that meet the circle in two points.



We can, thus, assume that the line XY through P actually goes through the center of the circle, which we denote O. If X is the nearest to P, we have $PX \cdot PY = (d-r)(d+r) = d^2 - r^2 = p$.

Secant Drawn to a Circle and Power of a Point

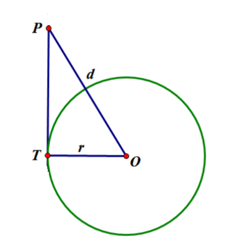
b. Suppose now that P lies inside the circle. The quantity $PX \cdot PY$ is a constant, independent of the particular chord XY through P. We can assume that chord XY is a diameter, and we can further assume that P lies on the segment OX.



We see that $PX \cdot PY = (r-d)(r+d) = r^2 - d^2 = -p.$

c. Finally, if PT is tangent to the circle at T, then the $\triangle OTP$ is a right triangle with side $OT = r$ and hypotenuse $PO = d$. By the Pythagorean theorem,

$$(PT)^2 = d^2 - r^2 = p.$$



Points with Equal Powers

Theorem

Fix two circles, centered at distinct points A and B. Then there exist points whose powers with respect to the two given circles are equal. The locus of all such points is a line perpendicular to AB.

- Suppose that points A and B lie on the x-axis so that A is the point $(a,0)$ and B is $(b,0)$, where $a \neq b$. If P is an arbitrary point with coordinates (x,y), then $(PA)^2 = y^2 + (x-a)^2$ and $(PB)^2 = y^2 + (x-b)^2$.

  Write r and s to denote the radii of the given circles centered at A and B, respectively. The powers of P with respect to the two circles are equal if and only if $y^2 + (x-a)^2 - r^2 = y^2 + (x-b)^2 - s^2$. This reduces to the linear equation $a^2 - 2ax - r^2 = b^2 - 2bx - s^2$. Since $b-a$ is nonzero, this is equivalent to $x = \frac{r^2 - s^2 + b^2 a^2}{2(b-a)}$. The right side being some constant, this is the equation of a line perpendicular to the x-axis.

The Radical Axis

- Given two circles with different centers, their **radical axis** is the line consisting of all points having equal powers with respect to the two circles.

Corollary

If two circles intersect at two points A and B, then their radical axis is their common secant AB. If two circles are tangent at a point T, then their radical axis is their common tangent at T.

- A point common to two circles has power 0 with respect to each of them, and thus its two powers are equal and the point lies on the radical axis. If A and B are two different points common to two circles, then A and B both lie on the radical axis, which we know is a line. It follows that the radical axis is the line AB.

The Radical Axis (Cont'd)

- In the case where two circles are tangent at T, then since T is on both circles, it lies on the radical axis. To see that the radical axis is tangent to each circle at T, it suffices to show that T is the only point where this line meets either circle. This is clear, however, because if a point P of the radical axis lies on one of the circles, its power with respect to that circle is 0. Hence, with respect to the other circle too, is 0. It follows that P lies on both circles, and hence P is the unique point common to the two circles, namely, T.

Concurrency of Three Radical Axes

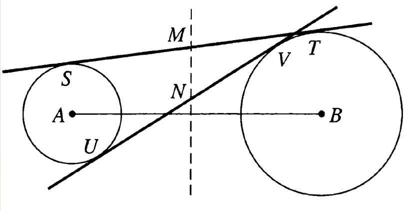
Corollary

Given three circles with noncollinear centers, the three radical axes of the circles taken in pairs are distinct concurrent lines.

- Since the radical axis of a pair of circles is perpendicular to the line of centers of the circles, it follows from the noncollinearity of the three centers that the three radical axes are distinct and nonparallel. Every two of them, therefore, have a point of intersection. For any point P, write p_1, p_2, and p_3 to denote the powers of P with respect to the three given circles. For points on one radical axis, we have $p_1 = p_2$, and on another, we have $p_2 = p_3$. At the point P, where these two radical axes meet, we have $p_1 = p_2 = p_3$. Thus, $p_1 = p_3$, and P also lies on the third radical axis.
- The unique point common to the three radical axes is called the **radical center** of the three circles.

Radical Axis of Two Non-Intersecting Circles

- If the circles are external to each other, we can draw a line tangent to both, and we let S and T be the two points of tangency.

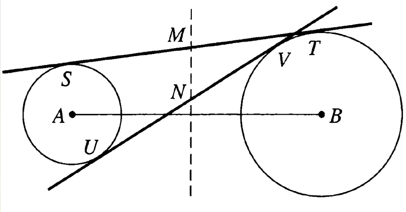
  

  If M is any point on line ST, then distances MS^2 and MT^2 are the powers of point M with respect to the two circles.

  It follows that if we take M to be the midpoint of segment ST, the two powers are equal. The midpoint M thus lies on the radical axis. But we know that the radical axis is perpendicular to the line of centers. So it suffices to draw the perpendicular to this line through M to complete the construction.

Radical Axis of Two Non-Intersecting Circles (Method 2)

- Alternatively, we can draw one of the other three lines tangent to both circles, and we let U and V be its two points of tangency.

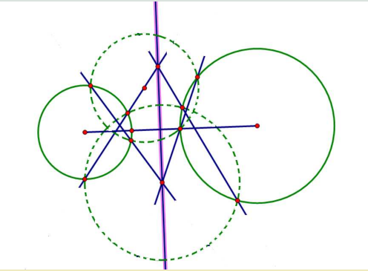
  The midpoint N of segment UV must also lie on the radical axis, which can thus be constructed by drawing line MN.

  We can avoid the possibility that M and N are the same point if we choose the second tangent appropriately.

Radical Axis of Two Non-Intersecting Circles (Method 3)

- Draw an auxiliary circle meeting each of the two given circles in two points and draw the line through each of these pairs of points.

These two lines are the radical axes of the auxiliary circle with each of the two original circles. We know that the point P where these lines meet must lie on the radical axis that we seek.



Now choose a second auxiliary circle and perform a similar construction to obtain a point Q. Since both P and Q are known to lie on the radical axis of the two given circles, we can complete our construction by drawing line PQ.

A Characterization of the Radical Axis

Corollary

Given points A and B on one circle and C and D on another, let P be the intersection of lines AB and CD. Then P lies on the radical axis of the two given circles if and only if the four points A, B, C and D are cocircular.

- If the four points all lie on some circle, then lines AB and CD are the radical axes of this circle with each of the given circles. Their intersection P lies on the radical axis of the two given circles.

  Conversely, suppose P lies on the radical axis of the two given circles and name these circles $\mathbf{X}$ and $\mathbf{Y}$, where A and B lie on $\mathbf{X}$ and C and D lie on $\mathbf{Y}$. Consider the circle $\mathbf{Z}$ through A, B and C. AB is the radical axis of $\mathbf{X}$ and $\mathbf{Z}$. Since P lies on this radical axis and also on the radical axis of the original two circles $\mathbf{X}$ and $\mathbf{Y}$, it follows that P also lies on the radical axis of $\mathbf{Y}$ and $\mathbf{Z}$. But C also lies on this radical axis, and we conclude that the radical axis of circles $\mathbf{Y}$ and $\mathbf{Z}$ must be the line PC. This line goes through D, however, and thus D has equal powers with respect to circles $\mathbf{Y}$ and $\mathbf{Z}$. Since D lies on $\mathbf{Y}$, we conclude that it also lies on $\mathbf{Z}$.