College Geometry

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Subsection 1

Ceva's Theorem

Point Giving a Fixed Ratio of Distances

Lemma

Given distinct points A and B and a positive number μ, there is exactly one point X on the line segment AB, such that $\frac{AX}{XB} = \mu$. Also, there is at most one other point on the line AB for which this equation holds.

- View X as a variable point and let $f(X)$ be the function whose value at X is the quantity $\frac{AX}{XB}$. $f(X)$ is a nonnegative real number, and it is defined everywhere except when $X = B$. As X moves from A toward B along segment AB, we see that AX increases and XB decreases. Thus, $f(X)$ is monotonically increasing from 0 when X is at A, and it approaches infinity as X approaches B. There is, thus, exactly one point X between A and B, where $f(X) = \mu$.

External Point Giving a Fixed Ratio of Distances

- If X is on line AB outside of segment AB, there are just two possibilities:
  - B is between X and A: Then, $AX = XB + BA$ and $f(X) = \frac{AX}{XB} = 1 + \frac{BA}{XB} > 1$.
  - A is between X and B: Then, $AX = XB - BA$ and $f(X) = \frac{AX}{XB} = 1 - \frac{BA}{XB} < 1$.

  For any given value $f(X) = \mu$, therefore, at most one of these two situations can occur depending on whether $\mu > 1$ or $\mu < 1$.
  - If B is between X and A, the function $f(X) = 1 + \frac{BA}{XB}$ is monotonically decreasing as X moves farther from B.
  - Otherwise, $f(X) = 1 - \frac{BA}{XB}$ is monotonically increasing as X gets farther from B.

  In either case, we see that there can be at most one point X, such that $f(X) = \mu$.

A Property of Ratios

- If two ratios are equal, say, $\dfrac{a}{b}=\dfrac{c}{d}$, then we automatically get two more ratios equal to these two, namely,

$$\frac{a+c}{b+d} \quad \text{and} \quad \frac{a-c}{b-d},$$

  assuming, of course, that $b+d$ is nonzero for the first of these and that $b-d$ is nonzero for the second.

  To see why this works, write $\lambda=\frac{a}{b}=\frac{c}{d}$. Then $a=\lambda b$ and $c=\lambda d$. So $a+c=\lambda(b+d)$ and $a-c=\lambda(b-d)$. Thus,

$$\frac{a+c}{b+d}=\lambda=\frac{a-c}{b-d}.$$

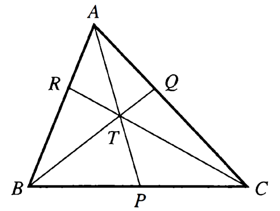
- These properties are called **addition** and **subtraction principles** for ratios.

Ceva's Theorem, Cevians and the Cevian Product

Theorem (Ceva)

Let AP, BQ and CR be three lines joining the vertices of $\triangle ABC$ to points P, Q and R on the opposite sides. Then these three lines are concurrent if and only if

$$\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = 1.$$



- A line going through exactly one vertex of a triangle is called a **Cevian** of the triangle.
- Ceva's Theorem asserts that, if we wish to prove that three Cevians are concurrent, we must compute the product of the three fractions in Ceva's theorem and show that the resulting quantity is equal to 1.
- In general, we will refer to this quantity as the **Cevian product** associated with the three given Cevians.

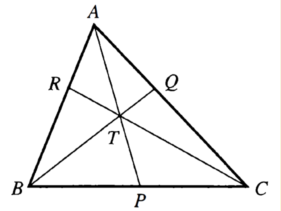
Proof of Ceva's Theorem: Necessity

- Assume first that the three Cevians are concurrent at some point T. View BP and PC as the bases of $\triangle ABP$ and $\triangle APC$, respectively, and observe that these triangles have equal heights. It follows that $\frac{BP}{PC}$ is the ratio of the areas of these two triangles. Segments BP and PC can also be viewed as the bases of $\triangle TBP$ and $\triangle TPC$, and these two triangles also have equal heights. Thus, we get $\frac{K_{APB}}{K_{APC}} = \frac{BP}{PC} = \frac{K_{TBP}}{K_{TPC}}$. By the subtraction principle for ratios, we deduce that $\frac{BP}{PC} = \frac{K_{ABP}-K_{TBP}}{K_{APC}-K_{PC}} = \frac{K_{ABT}}{K_{ABT}}$.

  Exactly similar reasoning yields

$$\frac{AR}{RB} = \frac{K_{CAT}}{K_{BCT}} \quad \text{and} \quad \frac{CQ}{QA} = \frac{K_{BCT}}{K_{ABT}}.$$

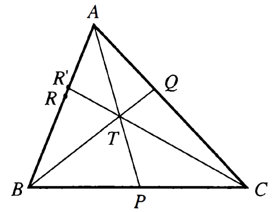
  We can now compute the Cevian product $\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = \frac{K_{CAT}}{K_{BCT}}\frac{K_{ABT}}{K_{CAT}}\frac{K_{BCT}}{K_{ABT}} = 1$.



Proof of Ceva's Theorem: Sufficiency

- To prove the converse, we assume that the Cevian product is trivial.

  We prove that AP, BQ and CR are concurrent by defining T to be the intersection of AP and BQ and showing that line CR must also pass through T.



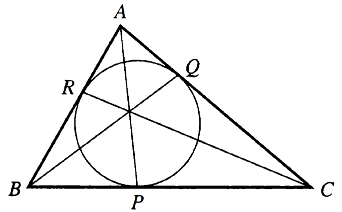
  It suffices to show that line CT goes through R. So we let R' be the point where line CT actually does meet side AB.

  Then CR' is a Cevian that is concurrent with AP and BQ. By the first part of the proof, the corresponding Cevian product is trivial. Hence, $\frac{AR'}{R'B}\frac{BP}{PC}\frac{CQ}{QA} = 1 = \frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA}$, where the second equality is by assumption. Cancelation yields that $\frac{AR'}{R'B} = \frac{AR}{RB} = \mu$. But there can only be one point X on line segment AB for which $\frac{AX}{XB} = \mu$. Thus, R and R' must actually be the same point. So CT goes through R.

The Gergonne Point of a Triangle

Proposition

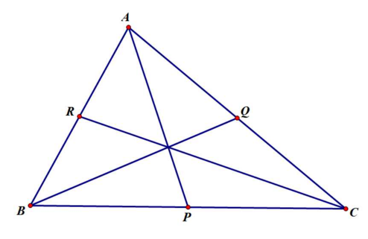
Let P, Q and R be the points of tangency of the incircle of $\triangle ABC$ with sides BC, CA and AB, respectively. Then lines AP, BQ and CR are concurrent.



- The point of concurrency of the three lines is sometimes called the **Gergonne point** of the triangle.
- The Gergonne point need not be the center of the circle. Thus, in general, the three lines are not the angle bisectors.
- We know that $AR = QA$, $BP = RB$ and $CQ = PC$. We calculate the Cevian product: $\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = \frac{AR}{BP}\frac{BP}{CQ}\frac{CQ}{AR} = 1$. It follows by Ceva's theorem that the three Cevians are concurrent.

Application: The Centroid

- Suppose that our three Cevians are the medians of $\triangle ABC$.

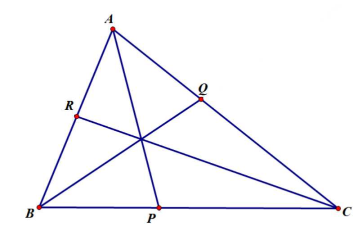


So P, Q and R are the midpoints of BC, CA and AB, respectively. Thus $AR = RB$, $BP = PC$ and $CQ = QA$. The medians are concurrent. So the Cevian product is 1. Indeed we have

$$\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = 1 \cdot 1 \cdot 1 = 1.$$

Application: The Incenter

- Consider the case where the Cevians AP, BQ and CR are the angle bisectors.



Again, the Cevian product must be 1 since the angle bisectors are always concurrent. Recall that an angle bisector of a triangle divides the opposite side into pieces whose lengths are proportional to the nearer sides of the triangle. We have $\frac{AR}{RB} = \frac{b}{a}$, $\frac{BP}{PC} = \frac{c}{b}$ and $\frac{CQ}{QA} = \frac{a}{c}$. The Cevian product $\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = \frac{b}{a}\frac{c}{b}\frac{a}{c} = 1$.

The Ratio on the Cevian

Theorem

Let AP, BQ and CR be Cevians in $\triangle ABC$, where P, Q and R lie on sides BC, CA and AB, respectively. If these Cevians are concurrent at a point T, then

$$\frac{AT}{AP} = \frac{AR \cdot CQ + QA \cdot RB}{AR \cdot CQ + QA \cdot RB + RB \cdot CQ},$$

and similar formulas hold for $\frac{BT}{BQ}$ and $\frac{CT}{CR}$.

- If masses $m_A = 1$ and $m_B = \frac{AR}{RB}$ and $m_C = \frac{QA}{CQ}$ are placed at the vertices of $\triangle ABC$, then the point T is the center of mass of the system (because of the Cevian product). We calculate:

$$\begin{aligned} \frac{AT}{AP} &= \frac{1}{\frac{AT+TP}{AT}} = \frac{1}{1+\frac{TP}{AT}} = \frac{1}{1+\frac{m_A}{m_B+m_C}} = \frac{m_B+m_C}{m_A+m_B+m_C} \\ &= \frac{\frac{AR}{RB}+\frac{QA}{CQ}}{1+\frac{AR}{RB}+\frac{QA}{CQ}} = \frac{AR \cdot CQ+QA \cdot RB}{AR \cdot CQ+QA \cdot RB+RB \cdot CQ}. \end{aligned}$$

Subsection 2

Interior and Exterior Cevians

Interior and Exterior Cevians

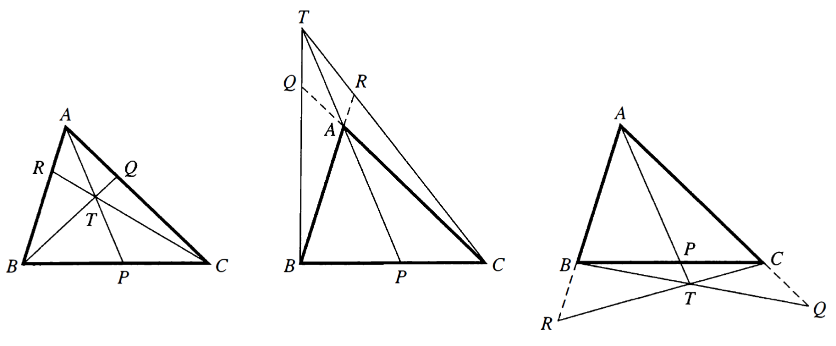
- Cevians AP, BQ and CR that "begin" at vertices of $\triangle ABC$, cut across the interior of the triangle and "terminate" at points lying on the opposite sides of the triangle are called **interior Cevians**.
- Ceva's theorem remains valid even if we expand the definition and allow **exterior Cevians**:

  These are lines that join a vertex of a triangle to a point on an extension of the opposite side, and which thus do not cut across the interior of the triangle.

  Example: AP is an exterior Cevian if point P lies on line BC, but it does not lie on the line segment BC;

  In particular, P is not one of the points B or C.

Configurations Involving Exterior Cevians



- Note that:
  - On the left, all Cevians are interior;
  - In the other two diagrams, AP is interior while both BQ and CR are exterior.
- If three Cevians are concurrent, then the number of interior Cevians among them is necessarily either one or three.

General Form of Ceva's Theorem

Theorem (Ceva's Theorem)

Let AP, BQ and CR be Cevians of $\triangle ABC$, where P, Q and R lie on lines BC, CA and AB, respectively. Then these Cevians are concurrent if and only if an odd number of them are interior and $\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = 1$.

- (**Sketch**) If the three Cevians are concurrent, we observe from an appropriate diagram that an odd number of them must be interior. In particular, at least one is interior, and so we can assume that AP is interior. We can assume that we are in the situation of one of the three preceding diagrams. In all cases, we have $\frac{AR}{RB} = \frac{K_{ACR}}{K_{RCB}} = \frac{K_{ATR}}{K_{RTB}}$, $\frac{BP}{PC} = \frac{K_{BAP}}{K_{PAC}} = \frac{K_{BTP}}{K_{PTC}}$ and $\frac{CR}{RA} = \frac{K_{CBQ}}{K_{QBA}} = \frac{K_{CTQ}}{K_{QTA}}$. We apply the addition and subtraction principles for ratios, but the appropriate principle in each case depends on which of the three diagrams is under consideration. We get $\frac{AR}{RB} = \frac{K_{CAT}}{K_{BCT}}$, $\frac{BP}{PC} = \frac{K_{ABT}}{K_{CAT}}$ and $\frac{CR}{RA} = \frac{K_{BCT}}{K_{ABT}}$. Thus, $\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = 1$.

Ceva's Theorem (Parallel Cevians and Converse)

- To prove the converse, assume that the number of interior Cevians is odd and that the Cevian product is trivial.

  Suppose two Cevians (say, AP and BQ) meet at some point T. We show that line CT goes through R. So we let R' be the point where CT meets line AB, and we work to show that R and R' are the same point. Now CR' is a Cevian that is concurrent with AP and BQ. The corresponding Cevian product is trivial. Reasoning as before, we deduce that $\frac{AR'}{R'B} = \frac{AR}{RB}$. The set $\{AP, BQ, CR\}$ contains an odd number of interior Cevians by hypothesis. The set $\{AP, BQ, CR'\}$ contains an odd number since these three Cevians are known to be concurrent by construction. Thus, either both of the Cevians CR and CR' are interior or else neither is. It follows either that both of the points R and R' lie on the line segment AB or else neither of them does. By a preceding lemma R and R' must be the same point.

Using Ceva's Theorem to Prove Concurrency of Altitudes

Proposition

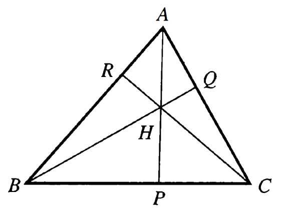
Ceva's theorem can be used to show that the altitudes of a triangle are concurrent.

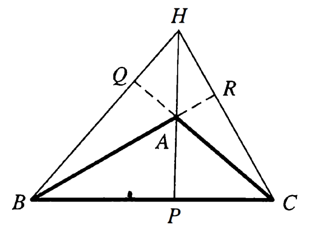
- There are exactly three possibilities for $\triangle ABC$. Either all angles are acute or one of the angles (say, $\angle A$) is obtuse or the triangle has a right angle.
  - In the case of a right triangle, each altitude clearly goes through the right angle, and so there is nothing to prove.
  - We can thus assume that one of the other two cases arises. Thus, the altitudes AP, BQ and CR are Cevians that we want to show are concurrent.

    In either case, the number of altitudes that are interior is odd. By Ceva's Theorem, it suffices to show that the product $\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA}$ is trivial.

The Acute Angle and Obtuse Angle Cases

- In the first case, we have $AR = b\cos(A)$, $BP = c\cos(B)$, $CQ = a\cos(C)$, $RB = a\cos(B)$, $PC = b\cos(C)$, $QA = c\cos(A)$. Each of a, b and c occurs once in the numerator and once in the denominator of the Cevian product. Similarly, each of $\cos(A), \cos(B)$ and $\cos(C)$ occurs once in the numerator and once in the denominator. Everything cancels and the Cevian product is trivial.





In the second case, exactly the same equations hold for the lengths of the six line segments provided that we interpret $\angle A$ as referring to the exterior angle of the triangle at A so that $\cos(A)$ will be positive. It follows that in this case too the Cevian product is trivial.

Subsection 3

Ceva's Theorem and Angles

Using Angles to Compute the Cevian Product

- Suppose that AP is a Cevian in $\triangle ABC$.
  - AP determines the two distances BP and PC, used in the Cevian product.
  - AP also determines two angles $\angle BAP$ and $\angle PAC$, neither of which is zero.
- Given three Cevians there are six angles determined.
- It is impossible to determine the six distances appearing in the Cevian product from knowledge of these six angles.
- Surprisingly, however, it is possible to compute the value of the Cevian product.
- The six angles can thus be used to determine whether or not the three Cevians are concurrent.

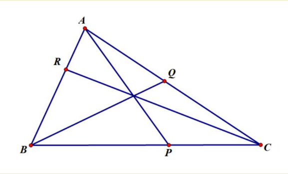
The Cevian Product in Terms of Angles

Theorem

Suppose that AP, BQ and CR are Cevians in $\triangle ABC$. Then the corresponding Cevian product is equal to $\frac{\sin(\angle ACR)}{\sin(\angle RCB)}\frac{\sin(\angle BAP)}{\sin(\angle PAC)}\frac{\sin(\angle CBQ)}{\sin(\angle QBA)}$ In particular, the three Cevians are concurrent if and only if an odd number of them are interior and this angular Cevian product is equal to 1.

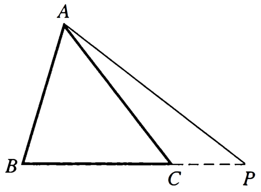
- The three factors of the Cevian product are $\frac{AR}{RB}$, $\frac{BP}{PC}$ and $\frac{CQ}{QA}$. We will use the law of sines to express each of these in terms of angles. We work first with the ratio $\frac{BP}{PC}$, and we begin with the case where the Cevian AP is interior.

  In $\triangle ABP$, we have $\frac{BP}{\sin(\angle BAP)} = \frac{AP}{\sin(\angle B)}$. In $\triangle ACP$, we have $\frac{PC}{\sin(\angle PAC)} = \frac{AP}{\sin(\angle C)}$. If we solve these equations for BP and PC and then divide and cancel AP, we obtain $\frac{BP}{PC} = \frac{\sin(\angle BAP)}{\sin(\angle PAC)}\frac{\sin(\angle C)}{\sin(\angle B)}$.



The Cevian Product in Terms of Angles (Cont'd)

- In the case where AP is an exterior Cevian, it turns out that we get exactly the same formula.



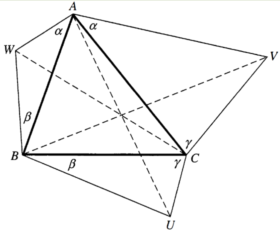
In $\triangle ABP$, $\frac{BP}{PC} = \frac{\sin(\angle BAP)}{\sin(\angle PAC)} \frac{\sin(\angle C)}{\sin(\angle B)}$. In $\triangle PAC$, the angle opposite side AP is not the original $\angle C = \angle ACB$, but instead, the corresponding exterior angle $\angle ACP = 180° - \angle C$. But the sines are equal, so we get the same formula $\frac{PC}{\sin(\angle PAC)} = \frac{AP}{\sin(\angle C)}$.

So in all cases we have $\frac{BP}{PC} = \frac{\sin(\angle BAP)}{\sin(\angle PAC)} \frac{\sin(\angle C)}{\sin(\angle B)}$. Similarly, $\frac{CQ}{QA} = \frac{\sin(\angle CBQ)}{\sin(\angle QBA)} \frac{\sin(\angle A)}{\sin(\angle C)}$ and $\frac{AR}{RB} = \frac{\sin(\angle ACR)}{\sin(\angle RCB)} \frac{\sin(\angle B)}{\sin(\angle A)}$. When we multiply the three ratios, we get $\frac{AR}{RB} \frac{BP}{PC} \frac{CQ}{QA} = \frac{\sin(\angle ACR)}{\sin(\angle RCB)} \frac{\sin(\angle BAP)}{\sin(\angle PAC)} \frac{\sin(\angle CBQ)}{\sin(\angle QBA)}$.

An Application

Theorem

Given an arbitrary $\triangle ABC$, build three outward-pointing triangles BCU, CAY and ABW, each sharing a side with the original triangle. Assume that $\angle BAW = \angle CAV$, $\angle CBU = \angle ABW$ and $\angle ACV = \angle BCU$. Assume further that lines AU, BV and CW cut across the interior of $\triangle ABC$. Then lines AU, BV and CW are concurrent.

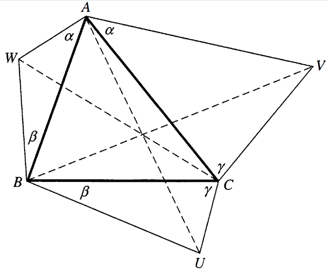


- Lines AU, BV and CW are interior Cevians. So, by the theorem, it suffices to show that $\frac{\sin(\angle ACW)}{\sin(\angle WCB)} \frac{\sin(\angle BAU)}{\sin(\angle UAC)} \frac{\sin(\angle CBV)}{\sin(\angle VBA)} = 1$. We write $\alpha = \angle BAW = \angle CAV$ and β, γ for the other two pairs.
  By the law of sines in $\triangle ACW$ to deduce that $\frac{AW}{\sin(\angle ACW)} = \frac{CW}{\sin(\angle WAC)}$. Since $\angle WAC = \angle A + \alpha$, we have $\sin(\angle ACW) = \frac{AW \sin(\angle A+\alpha)}{CW}$. Similarly, $\sin(\angle WCB) = \frac{BW \sin(\angle B+\beta)}{CW}$. Therefore, $\frac{\sin(\angle ACW)}{\sin(\angle WCB)} = \frac{AW \sin(\angle A+\alpha)}{BW \sin(\angle B+\beta)}$.

An Application (Cont'd)

- We got $\frac{\sin(\angle ACW)}{\sin(\angle WCB)} = \frac{AW\sin(\angle A+\alpha)}{BW\sin(\angle B+\beta)}$.

  The ratio $\frac{AW}{BW}$ can be computed using the law of sines in $\triangle ABW$. We have $\frac{AW}{\sin(\beta)} = \frac{BW}{\sin(\alpha)}$. Thus, $\frac{AW}{BW} = \frac{\sin(\beta)}{\sin(\alpha)}$. Substitution of this into our previous formula yields $\frac{\sin(\angle ACW)}{\sin(\angle WCB)} = \frac{\sin(\beta)}{\sin(\alpha)}\frac{\sin(\angle A+\alpha)}{\sin(\angle B+\beta)}$.

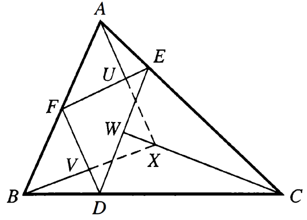


  Similar reasoning yields $\frac{\sin(\angle BAU)}{\sin(\angle UAC)} = \frac{\sin(\gamma)}{\sin(\beta)}\frac{\sin(\angle B+\beta)}{\sin(\angle C+\gamma)}$ and $\frac{\sin(\angle CBV)}{\sin(\angle VBA)} = \frac{\sin(\alpha)}{\sin(\gamma)}\frac{\sin(\angle C+\gamma)}{\sin(\angle A+\alpha)}$. So when we multiply these three ratios of sines to compute the angular Cevian product, everything cancels and the result is 1.

Perpendiculars to the Sides of the Pedal Triangle

Proposition

The pedal triangle of acute $\triangle ABC$ is $\triangle DEF$. Perpendiculars AU, BV and CW are dropped from the vertices of the original triangle to the sides of the pedal triangle. Then the lines AU, BV and CW are concurrent. The point of concurrence is the circumcenter of $\triangle ABC$.

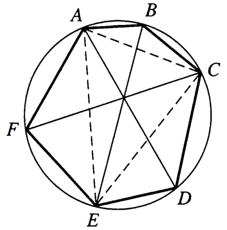


- By Ceva's theorem, we need $\frac{\sin(\angle ACW)}{\sin(\angle WCB)}\frac{\sin(\angle BAU)}{\sin(\angle UAC)}\frac{\sin(\angle CBV)}{\sin(\angle VBA)} = 1$. Since $\triangle EWC$ is a right triangle, we see that $\sin(\angle ACW) = \cos(\angle WEC)$. Similarly, $\sin(\angle UAC) = \cos(\angle UEA)$. Since $\triangle DEF$ is the pedal of $\triangle ABC$, we have $\angle UEA = \angle WEC$. Hence $\sin(\angle ACW) = \sin(\angle UAC)$. Similarly, all other factors cancel, and the lines meet at some point X. We have $\angle CAX = 90° - \angle AEU = 90° - \angle CEW = \angle ACX$. Thus $\triangle AXC$ is isosceles and $AX = CX$. Similarly, $BX = CX$. Thus, X must be the circumcenter.

Diagonals of an Inscribed Hexagon

Theorem

Let $ABCDEF$ be a hexagon inscribed in a circle. Then the diagonals AD, BE and CF are concurrent if and only if $\frac{AB}{BC}\frac{CD}{DE}\frac{EF}{FA} = 1$.



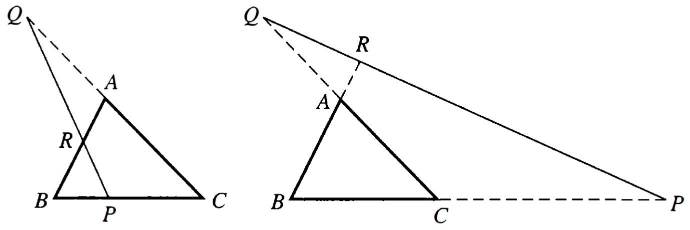
- Draw lines AC, CE and EA.
  View the three diagonals as Cevians of $\triangle ACE$. These diagonals are concurrent if and only if $\frac{\sin(\angle AEB)}{\sin(\angle BEC)}\frac{\sin(\angle CAD)}{\sin(\angle DAE)}\frac{\sin(\angle ECF)}{\sin(\angle FCA)} = 1$. We show that this angular Cevian product is equal to the hexagonal Cevian product in the statement.
  By the extended law of sines in $\triangle ABE$, $\frac{AB}{\sin(\angle AEB)} = 2R$, where R is the radius of the given circle, and hence is the circumradius of $\triangle ABE$. Thus, $\sin(\angle AEB) = \frac{AB}{2R}$. Similarly, considering $\triangle BEC$, we get $\sin(\angle BEC) = \frac{BC}{2R}$. Thus, $\frac{\sin(\angle AEB)}{\sin(\angle BEC)} = \frac{BC}{BC}$. Similarly, the other two ratios of sines are equal to the other two ratios of side lengths.

Subsection 4

Menelaus' Theorem

Collinear Points on Lines Forming a Triangle

- Given $\triangle ABC$, three arbitrary points P, Q and R on lines BC, CA and AB, respectively, might be collinear.
- This cannot happen if each of P, Q and R lies on an actual side of the triangle rather than on an extension of the side.
- For P, Q and R to be collinear, it is necessary that either exactly two of them or none of them lie on sides of the triangle.



- The fact that the number of members of the set $\{P, Q, R\}$ that lie on sides of the triangle must be even for these points to be collinear is the exact opposite of the situation in Ceva's theorem.

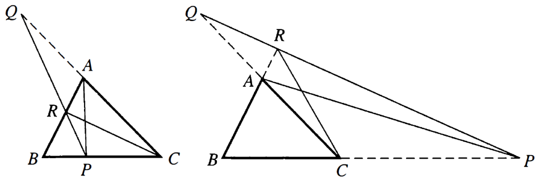
Menelaus' Theorem

Theorem (Menelaus)

Given $\triangle ABC$, let points P, Q and R lie on lines BC, CA and AB, respectively, and assume that none of these points is a vertex of the triangle. Then P, Q and R are collinear if and only if an even number of them lie on segments BC, CA and AB and $\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = 1$.

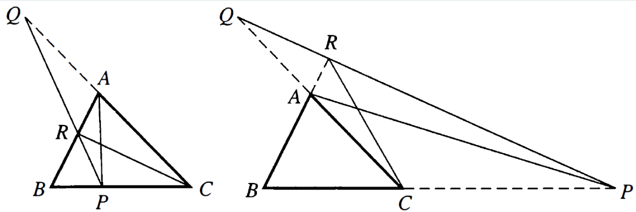
- First, assume that P, Q and R are collinear. The number of members of the set $\{P, Q, R\}$ that are on sides of the triangle must be even. We need to show that the Cevian product is trivial.

  Draw AP and CR. BP and PC can be viewed as the bases of $\triangle BPR$ and $\triangle CPR$ having equal heights. So $\frac{BP}{PC} = \frac{K_{BPR}}{K_{CPR}}$.



Menelaus' Theorem (Cont'd)

- We obtained $\frac{BP}{PC} = \frac{K_{BPR}}{K_{CPR}}$.

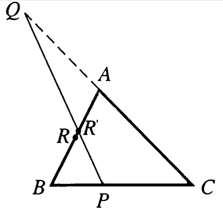


Similarly, using $\triangle APR$ and $\triangle BPR$, with bases AR and RB, we get $\frac{AR}{RB} = \frac{K_{APR}}{K_{BPR}}$. We compute the ratio $\frac{CQ}{QA}$ twice, using $\triangle CQP$ and $\triangle AQP$, and also $\triangle CQR$ and $\triangle AQR$: $\frac{CQ}{CA} = \frac{K_{CQP}}{K_{AQP}} = \frac{K_{CQR}}{K_{AQR}}$. Applying the subtraction principle, we get $\frac{CQ}{QA} = \frac{K_{CQP} - K_{CQR}}{K_{AQP} - K_{AQR}} = \frac{K_{CPR}}{K_{APR}}$. Everything now cancels when we compute $\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA}$. Thus this Cevian product is equal to 1.

Menelaus' Theorem: The Converse

- We now assume that the Cevian product is trivial and that an even number of P, Q and R lie on sides of the triangle. The only way it can happen that $PQ \parallel AB$, $QR \parallel BC$ and $RP \parallel CA$ is for all three of P, Q and R to lie on sides of the triangle, which is not the case. We can assume, therefore, that PQ is not parallel to AB, and we let R' be the point where PQ meets AB.

  We show R and R' are actually the same point. R' is neither A nor B. Since an even numbers of points in each of the sets $\{P, Q, R\}$ and $\{P, Q, R'\}$ lie on sides of the triangles, R' lies between A and B if and only if R lies between A and B.



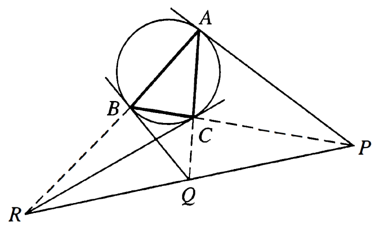
  By hypothesis, $\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = 1$. Since P, Q and R' are collinear, $\frac{AR'}{R'B}\frac{BP}{PC}\frac{CQ}{QA} = 1$. It follows that $\frac{AR}{RB} = \frac{AR'}{R'B}$. Thus R and R' must be the same point.

The Lemoine Axis of a Triangle

Proposition

Consider the tangent lines to the circumcircle of $\triangle ABC$ at the vertices of the triangle. If the tangent at A, B and C meet lines BC, CA and AB at points P, Q and R, respectively, then P, Q and R are collinear.

- Since points P, Q, and R lie outside of the circle, none of them lies on a side and Menelaus' theorem applies.

  We compute the three ratios $\frac{AR}{RB}, \frac{BP}{PC}$, and $\frac{CQ}{QA}$ and show that their product is equal to 1. We have $\angle BAQ \overset{\circ}{=} \frac{1}{2}\widehat{BC} \overset{\circ}{=} \angle CBQ$. Also $\angle BQA = \angle CQB$. Hence, $\triangle BAQ \sim \triangle CBQ$ by AA.

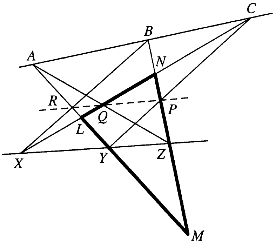
  It follows that $\frac{CQ}{BQ} = \frac{BQ}{AQ} = \frac{CB}{BA} = \frac{a}{c}$. Thus, $CQ = \frac{a}{c}BQ$ and $AQ = \frac{c}{a}BQ$. This yields $\frac{CQ}{QA} = \frac{\frac{a}{c}}{\frac{c}{a}} = \frac{a^2}{c^2}$. Similarly, $\frac{AR}{RB} = \frac{b^2}{a^2}$ and $\frac{BP}{PC} = \frac{c^2}{b^2}$. It follows that the Cevian product is equal to 1, and the points are collinear.

Pappus' Theorem

Theorem (Pappus)

Suppose that points A, B and C lie on some line ℓ and that points X, Y and Z lie on line m, where the six points are distinct and the two lines are also distinct. Assume that lines BZ and CY meet at P, lines AZ and CX meet at Q and lines AY and BX meet at R. Then points P, Q and R are collinear.

- Define point L, M, N as the intersections of XC, AY, AY, BZ and BZ, XC, respectively.

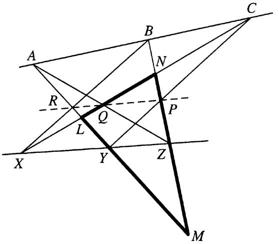
  

  We are assuming that none of the pairs of lines defining points L, M and N is parallel. Note that points P, Q and R lie on lines MN, NL and LM, respectively. We show that these three points are collinear by applying Menelaus' theorem to $\triangle LMN$. We compute the Cevian product $\frac{LR}{RM}\frac{MP}{PN}\frac{NQ}{QL}$ and show that it is equal to 1.

Pappus' Theorem (Cont'd)

- We should also check, of course, that the number of members of the set $\{P, Q, R\}$ that lie on actual sides of $\triangle LMN$ is even, but we shall omit the verification of this.

  Observe that points A, B and C are collinear and lie on lines LM, MN and NL, respectively. By Menelaus' Theorem, $\frac{LA}{AM}\frac{MB}{BN}\frac{NC}{CL} = 1$. From the fact that X, Y and Z are collinear, we get $\frac{LY}{YM}\frac{MZ}{ZN}\frac{NX}{XL} = 1$. R, B and X are collinear, and thus $\frac{LR}{RM}\frac{MB}{BN}\frac{NX}{XL} = 1$.



  From the collinearity of A, Q and Z we get $\frac{LA}{AM}\frac{MZ}{ZN}\frac{NQ}{QL} = 1$. From the collinearity of C, P and Y, we get $\frac{LY}{YM}\frac{MP}{PN}\frac{NC}{CL} = 1$. Multiplying the last three equations and the first two, we get

$$\frac{LR}{RM}\frac{MB}{BN}\frac{NX}{XL}\frac{LA}{AM}\frac{MZ}{ZN}\frac{NQ}{QL}\frac{LY}{YM}\frac{MP}{PN}\frac{NC}{CL} = 1 = \frac{LA}{AM}\frac{MB}{BN}\frac{NC}{CL}\frac{LY}{YM}\frac{MZ}{ZN}\frac{NX}{XL}.$$

  Six fractions on each side cancel, yielding the desired equation.