College Geometry

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

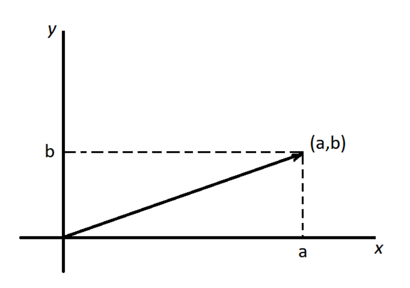
LSSU Math 325

Subsection 1

Vectors

Terminology and Notation

- Vectors are generally written as $\vec{v}$ or $\vec{A}$ or $\overrightarrow{AB}$, with a little arrow over the symbol or symbols.
- A **plane vector** $\vec{v}$ is simply an ordered pair of real numbers, which are called its **coordinates**. We write $\vec{v} = (a, b)$, where the coordinates a and b are real numbers.



Addition, Subtraction and Scalar Multiplication

- Vectors can be added or subtracted by adding or subtracting the corresponding coordinates:

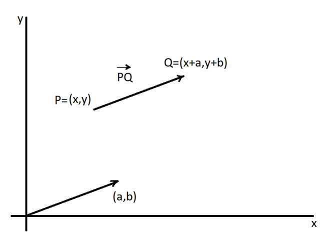
  If $\vec{v} = (a, b)$ and $\vec{w} = (c, d)$, we have:
  - $\vec{v} + \vec{w} = (a + c, b + d)$;
  - $\vec{v} - \vec{w} = (a - c, b - d)$.
- Also, we can multiply vectors by scalars simply by multiplying each coordinate by that scalar (a **scalar** is an ordinary real number):

  If z is scalar and $\vec{v} = (a, b)$ is a vector, we write $z\vec{v} = (za, zb)$.
- Many of the usual rules of arithmetic also hold for vectors, e.g., the commutative and associative laws are valid for vector addition, and two distributive laws hold for addition and scalar multiplication.
- Also, the vector $\vec{0} = (0, 0)$, which is called the **zero vector**, behaves very much like the number 0 in ordinary arithmetic:

  If $\vec{v}$ is any vector and z is any scalar, then $\vec{v} + \vec{0} = \vec{v}$ and $z\vec{0} = \vec{0}$.

Vectors as Geometric Objects

- Given a vector $\vec{v} = (a, b)$, let P be any point in the plane and suppose that its coordinates are (x, y).
  If we let Q be the point whose coordinates are $(x + a, y + b)$, then we can think of the vector $\vec{v}$ as instructions about how to get from point P to point Q:



  - Go a units right and b units up (if a is negative, we actually move left, and if b is negative, we move down).
- If we draw an arrow from P to Q with tail at P and head at Q, this arrow is a "picture" of the vector $\vec{v}$, and we write $\overrightarrow{PQ} = \vec{v}$.

From Arrows to Vectors

- Given two points P and Q and an arrow with tail at P and head at Q, we can reconstruct the vector $\vec{v} = \overrightarrow{PQ}$ by subtracting the corresponding coordinates of $P = (x_1, y_1)$ and $Q = (x_2, y_2)$.

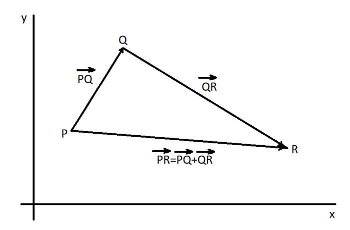
$$\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1).$$

- Note that we need the *arrow from P to Q* and not just the line segment PQ, because we need to know *which point is the head and which is the tail* so that we can subtract the tail coordinates from the head coordinates, and not vice versa:

$$\overrightarrow{QP} = -\overrightarrow{PQ}.$$

Geometric Interpretation of Vector Addition

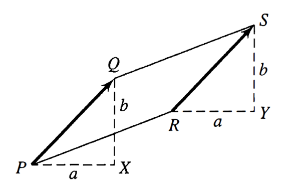
- Given vectors $\vec{v}$ and $\vec{w}$, we represent:
  - $\vec{v}$ as an arrow from P to Q, where P is arbitrary;
  - $\vec{w}$ as an arrow with starting point Q, and we write $\vec{w} = \overrightarrow{QR}$.



  It is easy to see that the arrow from P to R represents $\vec{v} + \vec{w}$, i.e., we have the vector equation $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$.
- In terms of “instructions", the instructions for going from P to R are:
  - first to go from P to Q;
  - then to go from Q to R.

Arrows Representing the Same Vector

- Given points P, Q, R and S, suppose it happens that $\overrightarrow{PQ} = \overrightarrow{RS}$.
  Claim: Line segments PQ and RS must be equal and parallel.
  Consider the figure, where $\triangle PQX$ and $\triangle RSY$ are right triangles (with horizontal and vertical arms) with the given equal vectors as hypotenuses. If we write $\overrightarrow{PQ} = (a, b) = \overrightarrow{RS}$, we see that $PX = a = RY$ and $XQ = b = YS$. Thus, by SAS, $\triangle PXQ \cong \triangle RYS$.

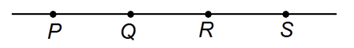
  It follows that the lengths PQ and RS are equal (by the Pythagorean theorem, PR and QS are equal to $\sqrt{a^2 + b^2}$).
  Moreover, $\overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS}$. Subtracting the equal vectors $\overrightarrow{PQ} = \overrightarrow{RS}$, we deduce that $\overrightarrow{QS} = \overrightarrow{PR}$. It follows that $QS = PR$. So the quadrilateral $PQSR$ is a parallelogram. Hence, $PQ \parallel RS$.
- Conversely, two arrows that are equal, parallel, and point in the same rather than in opposite directions correspond to equal vectors.

Geometric Interpretation of Scalar Multiplication

- The geometric significance of multiplication of a vector $\vec{v}$ by a positive scalar z is that an arrow representing $z\vec{v}$ points in the same direction as an arrow representing $\vec{v}$, but it is of length z times the length of $\vec{v}$.
- If the scalar z is negative, the direction of the vector is reversed, but otherwise, we get the same shrinking or stretching effect as with a positive scalar.

  Example: An arrow representing $-3\vec{v}$ has three times the length of an arrow representing $\vec{v}$, but it points in the opposite direction.

  If P, Q, R and S are four points lying in that order along a line and equally spaced so that $PQ = QR = RS$, then $\overrightarrow{PQ} = \overrightarrow{QR} = \overrightarrow{RS}$ and $\overrightarrow{PR} = \overrightarrow{QS}$.

  Some of the other equations that we can write in this situation are $-\overrightarrow{SP} = \overrightarrow{PS} = 3\overrightarrow{PQ}$ and $\overrightarrow{PR} = -\frac{2}{3}\overrightarrow{SP}$.

Subsection 2

Vectors and Geometry

Vectors and Points in the Plane

- Suppose that a fixed point O, called the **origin**, has been selected in the plane.
- A vector of the form $\overrightarrow{OA}$, with tail at point O, will simply be written as $\vec{A}$.
- Since for points A and B in the plane, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$, using the notational shortcut just described, we have $\vec{A} + \overrightarrow{AB} = \vec{B}$.
- Hence $\overrightarrow{AB} = \vec{B} - \vec{A}$, i.e., any vector named by two points can be described as a difference of two "single-point" vectors.
- One way to prove that two points P and Q are actually identical is to show that $\overrightarrow{PQ} = \vec{0}$. Since $\overrightarrow{PQ} = \vec{Q} - \vec{P}$, this is the zero vector precisely when $\vec{Q} = \vec{P}$.

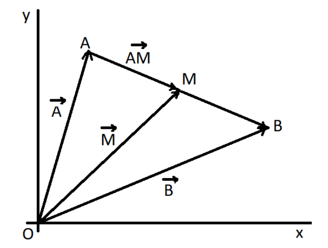
  In other words, to show that P and Q are the same point, it suffices to show that the vectors $\vec{P}$ and $\vec{Q}$ corresponding to these points are equal.

The Midpoint of a Line Segment

Proposition

The vector $\vec{M}$ corresponding to the midpoint M of line segment AB is exactly the average of the vectors $\vec{A}$ and $\vec{B}$, corresponding to the endpoints of the segment.

- To get to M from A, we need to travel exactly half of the way from A to B. This can be expressed in vector language by writing $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$. Using the notation just introduced, we rewrite $\vec{M} - \vec{A} = \frac{1}{2}(\vec{B} - \vec{A})$.



Thus, $\vec{M} = \vec{A} + \frac{1}{2}(\vec{B} - \vec{A})$. This, finally, yields $\vec{M} = \frac{1}{2}(\vec{A} + \vec{B})$.

The Centroid Revisited

Theorem

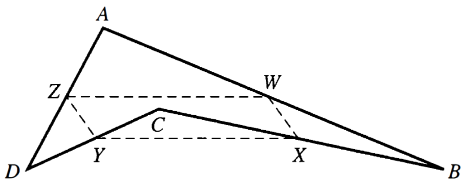
The medians of $\triangle ABC$ are concurrent at a point G that lies two thirds of the way along each median (moving from a vertex to the midpoint of the opposite side). Furthermore, $\vec{G} = \frac{1}{3}(\vec{A} + \vec{B} + \vec{C})$.

- We compute the vector $\vec{G}$ corresponding to the point G that lies two thirds of the way along median AM, where M is the midpoint of BC. By the proposition, $\vec{M} = \frac{1}{2}(\vec{B} + \vec{C})$. Hence, $\vec{G} - \vec{A} = \overrightarrow{AG} = \frac{2}{3}\overrightarrow{AM} = \frac{2}{3}(\vec{M} - \vec{A}) = \frac{2}{3}(\frac{1}{2}(\vec{B} + \vec{C}) - \vec{A})$. Therefore, $\vec{G} = \frac{1}{3}(\vec{A} + \vec{B} + \vec{C})$.
  Similarly, the vector corresponding to the point two thirds of the way along each of the other two medians must also be the average of the three vectors corresponding to the vertices.
  The vectors corresponding to the points two thirds of the way along the three medians are therefore equal, and it follows that these three points are identical.

Midpoints of the Sides of a Quadrilateral

Proposition

Let $ABCD$ be any quadrilateral and let W, X, Y and Z be the midpoints of AB, BC, CD and DA. Then $WXYZ$ is a parallelogram.



- We show that $\overrightarrow{WX} = \overrightarrow{ZY}$. This will imply that WX is both parallel and equal to ZY. The given data are the four points A, B, C and D, and so we express $\overrightarrow{WX}$ and $\overrightarrow{ZY}$ in terms of $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$. We have $\vec{W} = \frac{1}{2}(\vec{A} + \vec{B})$, $\vec{X} = \frac{1}{2}(\vec{B} + \vec{C})$, $\vec{Z} = \frac{1}{2}(\vec{A} + \vec{D})$ and $\vec{Y} = \frac{1}{2}(\vec{C} + \vec{D})$.
  - $\overrightarrow{WX} = \vec{X} - \vec{W} = \frac{1}{2}(\vec{B} + \vec{C}) - \frac{1}{2}(\vec{A} + \vec{B}) = \frac{1}{2}(\vec{C} - \vec{A})$;
  - $\overrightarrow{ZY} = \vec{Y} - \vec{Z} = \frac{1}{2}(\vec{C} + \vec{D}) - \frac{1}{2}(\vec{A} + \vec{D}) = \frac{1}{2}(\vec{C} - \vec{A})$.

A Point Dividing a Line Segment in a Given Ratio

- We determine the vector corresponding to the point obtained by moving a specified fraction γ of the way along a given line segment AB.

Lemma

Let γ be a real number with $0 < \gamma < 1$ and suppose that X is the point lying γ of the way from A to B along segment AB. Then $\vec{X} = (1-\gamma)\vec{A} + \gamma\vec{B}$.

Example: If $\gamma = \frac{1}{2}$, then X is the point that lies half of the way from A to B, and so X is the midpoint of segment AB. In this case, the lemma asserts that $\vec{X} = \frac{1}{2}\vec{A} + \frac{1}{2}\vec{B}$, as expected from preceding work.

As γ approaches 0, point X approaches point A, and so $\vec{X}$ should approach $\vec{A}$, and this is consistent with the formula given.

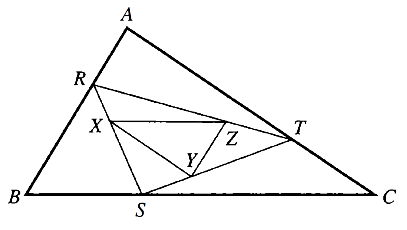
A similar reasoning applies as γ approaches 1, so that X approaches B.

- In general $\overrightarrow{AX} = \gamma\overrightarrow{AB}$. So $\vec{X} - \vec{A} = \gamma(\vec{B} - \vec{A})$. Now compute $\vec{X} = \vec{A} + \gamma(\vec{B} - \vec{A}) = (1-\gamma)\vec{A} + \gamma\vec{B}$.

An Additional Application in Similarity

Proposition

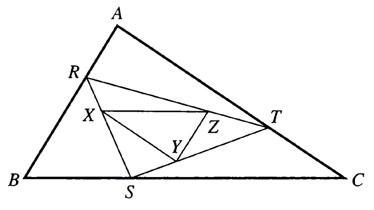
Given $\triangle ABC$, we construct $\triangle RST$ by taking points R, S and T on the sides of the original triangle, as follows: Point R lies one third of the way from A to B along AB, point S lies one third of the way from B to C along BC, and point T lies one third of the way from C to A along CA. Now repeat this process starting with $\triangle RST$ and obtain $\triangle XYZ$. Then $\triangle XYZ \sim \triangle CAB$ and the corresponding sides of these two triangles are parallel.



- The strategy is to express the vectors along the sides of $\triangle XYZ$ in terms of A, B and C. Since R is one third of the way from A to B, $\vec{R} = \frac{2}{3}\vec{A} + \frac{1}{3}\vec{B}$. Similarly, $\vec{S} = \frac{2}{3}\vec{B} + \frac{1}{3}\vec{C}$. Since X lies one third of the way from R to S, $\vec{X} = \frac{2}{3}\vec{R} + \frac{1}{3}\vec{S} = \frac{2}{3}(\frac{2}{3}\vec{A} + \frac{1}{3}\vec{B}) + \frac{1}{3}(\frac{2}{3}\vec{B} + \frac{1}{3}\vec{C})$.

An Additional Application in Similarity (Cont'd)

- We found $\vec{X} = \frac{2}{3}(\frac{2}{3}\vec{A} + \frac{1}{3}\vec{B}) + \frac{1}{3}(\frac{2}{3}\vec{B} + \frac{1}{3}\vec{C})$. Hence $\vec{X} = \frac{4}{9}\vec{A} + \frac{4}{9}\vec{B} + \frac{1}{9}\vec{C}$. Analogously, we get $\vec{Y} = \frac{4}{9}\vec{B} + \frac{4}{9}\vec{C} + \frac{1}{9}\vec{A}$. Now calculate $\overrightarrow{XY} = \vec{Y} - \vec{X} = (\frac{4}{9}\vec{B} + \frac{4}{9}\vec{C} + \frac{1}{9}\vec{A}) - (\frac{4}{9}\vec{A} + \frac{4}{9}\vec{B} + \frac{1}{9}\vec{C}) = \frac{1}{3}\vec{C} - \frac{1}{3}\vec{A} = \frac{1}{3}\overrightarrow{AC}$.



  Since the vector $\overrightarrow{XY}$ is one third of the vector $\overrightarrow{AC}$, we know that the corresponding arrows are parallel and that the former has one third the length of the latter. Thus, $XY \parallel CA$ and $XY = \frac{1}{3}CA$.
  Similarly, each side of $\triangle XYZ$ is parallel to the corresponding side of $\triangle CAB$, and each side of $\triangle XYZ$ has length equal to one third of the length of the corresponding side of $\triangle CAB$.
  Thus, $\triangle XYZ \sim \triangle CAB$ by SSS.

Subsection 3

Dot Products

The Dot Product

- If $\vec{v} = (a, b)$ and $\vec{w} = (c, d)$, then the **dot product** $\vec{v} \cdot \vec{w}$ is defined to be the scalar
$$\vec{v} \cdot \vec{w} = ac + bd.$$
- It is easy to check that the commutative and distributive laws hold for dot products: If $\vec{u}, \vec{v}$ and $\vec{w}$ are any three vectors, we have the following:
  - $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$;
  - $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

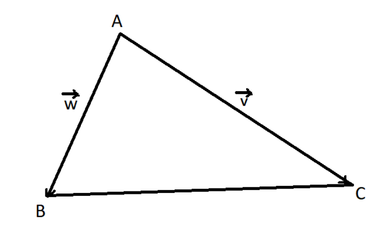
  In the last equation, the plus sign on the left represents vector addition, but the plus sign on the right represents ordinary scalar addition.

The Dot Product of a Vector by Itself

- Consider the dot product of a vector with itself:
  If $\vec{v} = (a, b)$, we see that $\vec{v} \cdot \vec{v} = a^2 + b^2$.
  This is the square of the length of an arrow representing $\vec{v}$.
- Using the absolute value notation $|\vec{v}|$ to represent the length of a vector $\vec{v}$, we write
$$\vec{v} \cdot \vec{v} = |\vec{v}|^2.$$
- If P and Q are points and we write PQ to denote the length of the line segment they determine, we can write $|\overrightarrow{PQ}| = PQ$.

The Dot Product of Two Vectors

- Now consider $\triangle ABC$, with a, b and c the lengths of sides BC, AC and AB, respectively. Write $\vec{v} = \overrightarrow{AC}$ and $\vec{w} = \overrightarrow{AB}$. Then $\vec{v} \cdot \vec{v} = |\vec{v}|^2 = b^2$. Similarly, $\vec{w} \cdot \vec{w} = c^2$.



  By the Law of cosines
  $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \overrightarrow{BC} \cdot \overrightarrow{BC} = a^2 = b^2 + c^2 - 2ac \cos A$.
  By distributivity, $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{w} - \vec{w} \cdot \vec{v} + \vec{w} \cdot \vec{w} = |\vec{v}|^2 + |\vec{w}|^2 - 2(\vec{v} \cdot \vec{w}) = b^2 + c^2 - 2(\vec{v} \cdot \vec{w})$.
  We conclude that
  $$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos A.$$

The Dot Product of Perpendicular Vectors

Corollary

Nonzero vectors are perpendicular if and only if their dot product is zero.

- Suppose θ is the angle between $\vec{v}$ and $\vec{w}$.

  If $\vec{v}$ and $\vec{w}$ are perpendicular, then $\theta = 90°$. Since $\cos(90°) = 0$, we see that $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta = 0$.

  Conversely, if $\vec{v}$ and $\vec{w}$ are nonzero, then $|\vec{v}| \neq 0 \neq |\vec{w}|$. Thus, if $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta = 0$, the only possibility is that $\cos\theta = 0$. So $\theta = 90°$ and $\vec{v}$ and $\vec{w}$ are perpendicular.

The Orthocenter Revisited

Proposition

The altitudes of $\triangle ABC$ are concurrent.

- Let H be the intersection of the altitudes from A and from B. We show that H also lies on the altitude from C. If H is the point C, there is nothing to prove. If H is different from C, we show that CH is perpendicular to AB. Since the vectors $\overrightarrow{CH}$ and $\overrightarrow{AB}$ are nonzero, it suffices to show that $\overrightarrow{CH} \cdot \overrightarrow{AB} = 0$, i.e., that $(\vec{H} - \vec{C}) \cdot (\vec{B} - \vec{A}) = 0$. By the distributive law, we must show $\vec{H} \cdot (\vec{B} - \vec{A}) = \vec{C} \cdot (\vec{B} - \vec{A})$.

  Since H lies on the altitude from A, $\overrightarrow{AH} \cdot \overrightarrow{CB} = 0$, i.e., $(\vec{H} - \vec{A}) \cdot (\vec{B} - \vec{C}) = 0$. So $\vec{H} \cdot (\vec{B} - \vec{C}) = \vec{A} \cdot (\vec{B} - \vec{C})$. Since H also lies on the altitude from B, similar reasoning yields $\vec{H} \cdot (\vec{C} - \vec{A}) = \vec{B} \cdot (\vec{C} - \vec{A})$. No we have
  $\vec{H} \cdot (\vec{B} - \vec{A}) = \vec{H} \cdot (\vec{B} - \vec{C}) + \vec{H} \cdot (\vec{C} - \vec{A}) = \vec{A} \cdot (\vec{B} - \vec{C}) + \vec{B} \cdot (\vec{C} - \vec{A}) =$
  $\vec{A} \cdot \vec{B} - \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C} - \vec{B} \cdot \vec{A} = \vec{B} \cdot \vec{C} - \vec{A} \cdot \vec{C} = \vec{C} \cdot (\vec{B} - \vec{A})$.

Circumcenter, Orthocenter and Centroid

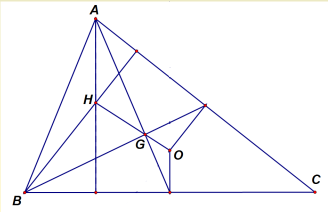
Proposition

The circumcenter of $\triangle ABC$ is collinear with the orthocenter and the centroid.

- The circumcenter O, the centroid G, and the orthocenter H actually lie on the Euler line, and the point O lies on the opposite side of G from H, and $HG = 2GO$.
- We choose the origin as follows:
  - If H and G are the same point, we let O be this point too.
  - Otherwise, we choose O on line HG, on the opposite side of G from H, and half as far from G as H is.

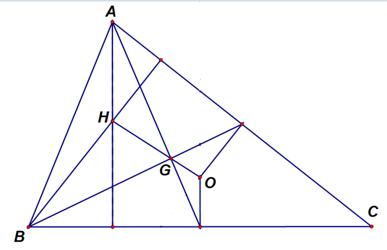
Since O is collinear with H and G, it suffices to show that O actually is the circumcenter.

We need to show, therefore, that the three distances OA, OB and OC are all equal.



Circumcenter, Orthocenter and Centroid

- By construction, $\overrightarrow{OH} = 3\overrightarrow{OG}$. Thus, we get $\vec{H} = 3\vec{G} = \vec{A} + \vec{B} + \vec{C}$, i.e., $\vec{H} - \vec{A} = \vec{B} + \vec{C}$. Since AH is perpendicular to BC, this yields $0 = \overrightarrow{AH} \cdot \overrightarrow{CB} = (\vec{H} - \vec{A}) \cdot (\vec{B} - \vec{C}) = (\vec{B} + \vec{C}) \cdot (\vec{B} - \vec{C}) = \vec{B} \cdot \vec{B} - \vec{C} \cdot \vec{C}$. Thus, $|\vec{B}|^2 = \vec{B} \cdot \vec{B} = \vec{C} \cdot \vec{C} = |\vec{C}|^2$.



  Hence, $|\vec{B}| = |\vec{C}|$. But recall that $\vec{B} = \overrightarrow{OB}$, and hence $|\vec{B}|$ is the distance OB. Similarly, $|\vec{C}| = OC$. So we have proved that $OB = OC$.

  Similarly, O is equidistant from A and C.

  Thus, O is the circumcenter of $\triangle ABC$.

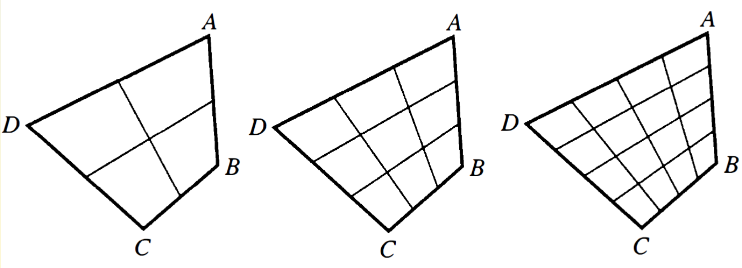
Subsection 4

Checkerboards

Checkerboards

- Let $ABCD$ be a convex quadrilateral, i.e., with all of its angles less than 180°.
- Divide each side of $ABCD$ into n equal parts, where n is some fixed positive integer, and join the corresponding points to form a crisscross pattern that we call an $n \times n$ **checkerboard**.

  Example: The figures show $2 \times 2, 3 \times 3$ and 4×4 checkerboards, all based on the same quadrilateral.



CrissCross Segments of a Checkerboard

- Consider a 2×2 checkerboard.

  We know that the midpoints of the four sides of the quadrilateral $ABCD$ are the vertices of a parallelogram.

  The two crossing line segments of the 2×2 checkerboard are the diagonals of this parallelogram, and hence they bisect each other.

  Thus, each of the six line segments that make up a 2×2 checkerboard is cut into two equal pieces.
- Similarly, each of the eight line segments that make up a 3×3 checkerboard is divided into three equal pieces.
- More generally, an $n \times n$ checkerboard is made up of $4 + 2(n-1)$ line segments, and it turns out that each of these segments is divided into n equal pieces.

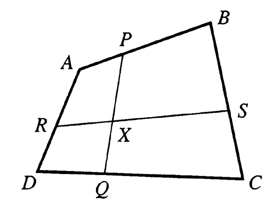
  By the definition of a checkerboard, we know that each side of the original quadrilateral is divided into n equal pieces; the surprise is that the $2n-2$ crisscross segments are also equally divided.

The CrissCross Property

Theorem

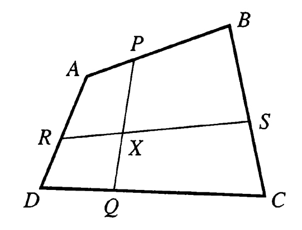
Each of the $2n+2$ line segments that comprise an $n \times n$ checkerboard is cut into n equal pieces.

- Suppose P is one of the division points on side AB and Q is the corresponding division point on side DC.
  Then PQ is one of the crisscross line segments, and we have $\frac{AP}{AB} = \frac{k}{n} = \frac{DQ}{DC}$, where k is an integer with $0 < k < n$.



  Similarly, suppose R and S are corresponding division points on sides AD and BC. Thus, RS is a crisscross line segment and we have $\frac{AR}{AD} = \frac{\ell}{n} = \frac{BS}{BN}$, where ℓ is an integer with $0 < \ell < n$.
  We need to show that PQ cuts RS at a point that lies exactly $\frac{k}{n}$ in of the way from R to S as we move along RS and that this intersection point lies exactly $\frac{\ell}{n}$ of the way from P to Q along PQ.

The CrissCross Property (Cont'd)

- 

We write $\alpha = \frac{k}{n}$ and $\beta = \frac{\ell}{n}$. We get for P, Q, R and S: $\vec{P} = (1-\alpha)\vec{A} + \alpha\vec{B}$, $\vec{Q} = (1-\alpha)\vec{D} + \alpha\vec{C}$, $\vec{R} = (1-\beta)\vec{A} + \beta\vec{D}$ and $\vec{S} = (1-\beta)\vec{B} + \beta\vec{C}$. Let X be the point on RS that we expect is the point where PQ crosses RS.

In other words, X is the point that lies α of the way from R to S along RS. Similarly, let Y be the point on PQ that we expect lies on RS. So Y lies β of the way from P to Q along PQ. Our goal is to show that X and Y are the same point:

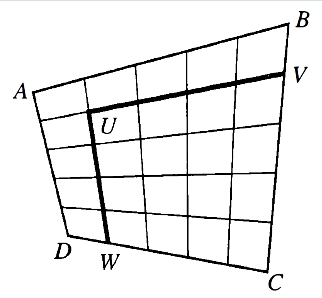
$\vec{X} = (1-\alpha)\vec{R} + \alpha\vec{S} = (1-\alpha)((1-\beta)\vec{A} + \beta\vec{D}) + \alpha((1-\beta)\vec{B} + \beta\vec{C}) =$
$(1-\alpha)(1-\beta)\vec{A} + \alpha(1-\beta)\vec{B} + \alpha\beta\vec{C} + (1-\alpha)\beta\vec{D}$.

$\vec{Y} = (1-\beta)\vec{P} + \beta\vec{Q} = (1-\beta)((1-\alpha)\vec{A} + \alpha\vec{B}) + \beta((1-\alpha)\vec{D} + \alpha\vec{C}) =$
$(1-\alpha)(1-\beta)\vec{A} + \alpha(1-\beta)\vec{B} + \alpha\beta\vec{C} + (1-\alpha)\beta\vec{D}$.

Thus $X = Y$ must be the point of intersection of PQ and RS.

Deleting a Row and a Column

- The figure shows a 5×5 checkerboard $ABCD$. We focus on the smaller quadrilateral $UVCW$. We know that all of the pieces on each crisscross line of the original checkerboard are equal. So we see that UV and UW are each divided into four equal pieces. Thus, $UVCW$ is a 4×4 checkerboard.

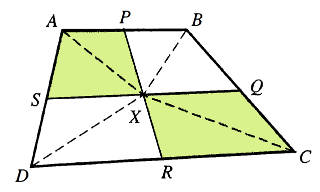


- The same thing works in general: We can create an $(n-1) \times (n-1)$ checkerboard from an $n \times n$ checkerboard by deleting the first row and first column of boxes.

Areas of the Squares Along a Diagonal

Proposition

Let $ABCD$ be a 2×2 checkerboard, where two of the four boxes have been shaded. Then the shaded area is exactly half of the total area of the checkerboard.



- Let P, Q, R and S be the midpoints of the sides of $ABCD$, and X the point where PR meets QS. Draw the line segments joining X to A, B, C and D. This partitions the total into four triangular pieces: $\triangle AXB$, $\triangle BXC$, $\triangle CXD$, and $\triangle DXA$. It suffices to show that exactly half of the area of each of these four triangles is shaded. But $AP = PB$. Thus $\triangle APX$ and $\triangle BPX$ have equal bases AP and PB, and they have equal altitudes. It follows that $\triangle APX$ and $\triangle BPX$ have equal areas. Thus, exactly half of the area of $\triangle AXB$ is shaded. A similar argument works for each of the other three triangles.

Introducing the $n \times n$ Case

- If we shade the boxes along the diagonal of any $n \times n$ checkerboard, we will prove that the total area of the n shaded boxes is exactly $\frac{1}{n}$ of the area of the entire checkerboard.
- In this way, we have shaded exactly one n-th of the n^2 boxes, but, since, in general, the boxes do not all have equal areas, this certainly does not show that we have shaded one n-th of the area:
  - The case $n = 2$ is exactly the preceding proposition;
  - The case $n = 1$ is a triviality with no content.
- In fact, we need not restrict ourselves to diagonal boxes:

  If we shade any n of the n^2 boxes, subject only to the condition that no two of the shaded boxes lie in the same row or column, then exactly one n-th of the entire area will be shaded.

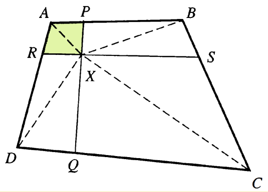
Introducing the $n \times n$ Case

Theorem

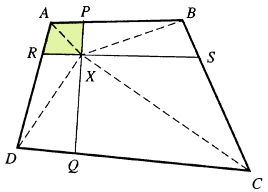
Suppose that we are given an arbitrary $n \times n$ checkerboard $ABCD$ with area K_{ABCD}. Writing d to denote the total area of the n boxes along the diagonal of this checkerboard, we have $d = \frac{1}{n} K_{ABCD}$.

- The theorem holds when $n = 1$. Assume $n \geq 2$. and that the theorem holds for all smaller values of n. In particular, the area of the $n-1$ diagonal boxes of any $(n-1) \times (n-1)$ checkerboard is exactly $\frac{1}{n-1}$ of the total area of that checkerboard.

  Let PQ and RS be the leftmost and uppermost of the crisscross lines of the $n \times n$ checkerboard $ABCD$ and let X be the point where these lines meet. Thus, $APXR$ is the uppermost of the n diagonal boxes whose total area d we need to compute. There are $n-1$ more shaded boxes, all lying inside $XSCQ$.



Introducing the $n \times n$ Case (Cont'd)



- Quadrilateral $XSCQ$ is an $(n-1)\times(n-1)$ checkerboard. By the inductive hypothesis, the area of the $n-1$ diagonal boxes inside quadrilateral $XSCQ$ is $\frac{1}{n-1}K_{XSCQ}$. Thus, the total shaded diagonal area d is given by the formula $d = K_{APXR} + \frac{1}{n-1}K_{XSCQ}$.

  We want to show that $d = \frac{1}{n}K_{ABCD}$, i.e., $nd = K_{ABCD}$. To accomplish this, we join X to each of the points A, B, C and D. Since $AP = \frac{1}{n}AB$ and $AR = \frac{1}{n}AD$, we see that $K_{AXP} = \frac{1}{n}K_{AXB}$ and $K_{AXR} = \frac{1}{n}K_{AXD}$. Adding these and multiplying by n, we get $nK_{APXR} = K_{ABXD}$. Similarly, since $QC = \frac{n-1}{n}DC$ and $SC = \frac{n-1}{n}BC$, we get $K_{XQC} = \frac{n-1}{n}K_{XDC}$ and $K_{XSC} = \frac{n-1}{n}K_{KBC}$. If we add these and multiply by n, we get $nK_{XSCQ} = (n-1)K_{DXBC}$. Finally, we get $nd = nK_{APXR} + \frac{nK_{XSCQ}}{n-1} = K_{ABXD} + \frac{(n-1)K_{DXBC}}{n-1} = K_{ABXD} + K_{DXBC} = K_{ABCD}$.

Subsection 5

A Bit of Trigonometry

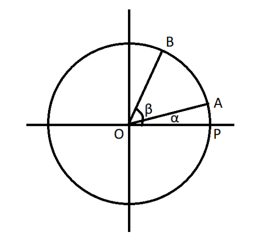
The Sine and Cosine of the Sum of Two Angles

Theorem

The following formulas hold for all angles α and β.

a. $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$.
b. $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$.

- Let O be the origin, let P be the point $(1,0)$, and let A and B be the points on the unit circle such that $\angle POA=\alpha$ and $\angle POB=\beta$. The coordinates of A are $(\cos(\alpha),\sin(\alpha))$ and the coordinates of B are $(\cos(\beta),\sin(\beta))$. So we can write $\overrightarrow{OA}=(\cos(\alpha),\sin(\alpha))$ and $\overrightarrow{OB}=(\cos(\beta),\sin(\beta))$.



We showed that the dot product of two vectors is equal to the product of their lengths times the angle between them.

The Sine and Cosine of the Sum (Cont'd)

- We have $|\overrightarrow{OA}| = 1$, $|\overrightarrow{OB}| = 1$ and the angle between these vectors is $\alpha - \beta$. Hence $\overrightarrow{OA} \cdot \overrightarrow{OB} = \cos(\alpha - \beta)$. By the definition of the dot product, $\overrightarrow{OA} \cdot \overrightarrow{OB} = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$. Thus, we conclude that $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.

  By substituting $-\beta$ for β, we get

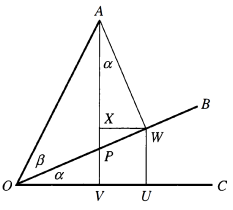
  $$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

  To prove (b), we compute that

  $$\begin{aligned} \sin(\alpha + \beta) &= \cos(90^\circ - \alpha - \beta) \\ &= \cos(90^\circ - \alpha)\cos(\beta) + \sin(90^\circ - \alpha)\sin(\beta) \\ &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta). \end{aligned}$$

A Geometric Proof for $0 < \alpha + \beta < 90°$

- Start with line OC, and draw OB and OA so that $\angle BOC = \alpha$ and $\angle AOB = \beta$. Drop perpendiculars AW and AV from A to OB and OC. Drop, also, perpendiculars WX and WU from W to AV and OC. We see that $\angle WAP$ and $\angle VOP$ are complementary to equal vertical angles $\angle APW = \angle OPV$. Thus, $\angle WAP = \angle VOP = \alpha$. Assume that the length OA is 1 unit. Then we have



$$\begin{aligned}
\cos(\alpha+\beta) &= OV = OU - VU = OW\cos(\alpha) - XW \\
&= \cos(\beta)\cos(\alpha) - AW\sin(\alpha) \\
&= \cos(\beta)\cos(\alpha) - \sin(\beta)\sin(\alpha); \\
\sin(\alpha+\beta) &= AV = XV + AX \\
&= OW\sin(\alpha) + AW\cos(\alpha) \\
&= \cos(\beta)\sin(\alpha) + \sin(\beta)\cos(\alpha).
\end{aligned}$$

Subsection 6

Linear Operators

Operators

- An **operator** is a function T that yields a vector whenever we plug in a vector, i.e., if $\vec{v}$ is any vector, then $T(\vec{v})$ is some vector determined by $\vec{v}$ according to some specific rule.

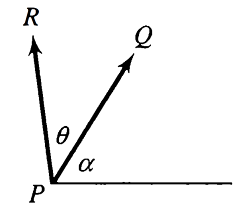
  Example: The operator
  $$T(\vec{v}) = -\vec{v}$$
  reverses the direction of all arrows representing vectors.

  Equivalently, T rotates all arrows by $180°$.

  Example: More generally, given any number θ, we can consider the operator T that rotates arrows representing vectors counterclockwise through θ degrees.

The Rotation Operators

- If $\vec{v} = (a, b)$, we want to express the coordinates c and d of the vector $T(\vec{v}) = (c, d)$ in terms of the coordinates a and b and the angle of rotation θ.

  Suppose that $\vec{v} = \overrightarrow{PQ}$ and $T(\vec{v}) = \overrightarrow{PR}$, so that $\angle QPR = \theta$.

  Let α be the angle between $\overrightarrow{PQ}$ and the horizontal vector $(0, 1)$. Let $r = |\vec{v}| = \overrightarrow{PQ}$. Then $a = r\cos(\alpha)$ and $b = r\sin(\alpha)$. The angle between $\overrightarrow{PR}$ and the horizontal is $\alpha + \theta$, and the length $PR = PQ = r$. It follows that $(c, d) = T(\vec{v}) = \overrightarrow{PR} = (r\cos(\alpha+\theta), r\sin(\alpha+\theta))$. By the theorem,
  $c = r\cos(\alpha+\theta) = r(\cos(\alpha)\cos(\theta) - \sin(\alpha)\sin(\theta)) = a\cos(\theta) - b\sin(\theta)$,
  $d = r\sin(\alpha+\theta) = r(\cos(\alpha)\sin(\theta) + \sin(\alpha)\cos(\theta)) = a\sin(\theta) + b\cos(\theta)$.

  This transformation can be written $(c, d) = (a, b)\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$.

Linear Operators

- An operator T is linear if, for all vectors $\vec{v}$ and $\vec{w}$ and for all scalars z,
  - $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$;
  - $T(z\vec{v}) = zT(\vec{v})$.

  Example: To check that our rotation operator is linear, we choose arbitrary vectors $\vec{v}$ and $\vec{w}$, and, denoting A the matrix of sines and cosines, compute:

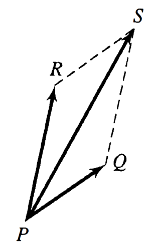
$$T(\vec{v}+\vec{w}) = (\vec{v}+\vec{w})A = \vec{v}A + \vec{w}A = T(\vec{v}) + T(\vec{w}).$$

  Also, if z and $\vec{v}$ are an arbitrary scalar and an arbitrary vector, we have

$$T(z\vec{v}) = (z\vec{v})A = z(\vec{v}A) = zT(\vec{v}).$$

Representation of the Sum of Two Vectors

- It is also possible to see geometrically why the rotation operator T is linear.
- Represent the vectors $\vec{v}, \vec{w}$ as arrows all having the same tail P, say $\vec{v} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{PR}$. Form the parallelogram $PQSR$. We then have $\overrightarrow{PR} = \overrightarrow{QS}$. Hence



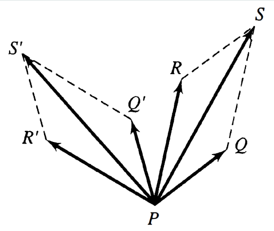
$$\overrightarrow{PQ} + \overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS}.$$

To add two vectors represented by arrows with a common tail, we complete the parallelogram and the arrow along the diagonal of the parallelogram represents the sum of the two original vectors.

Linearity of Rotation: A Different View

- To see that rotation is linear we need to show, first, that rotating the sum is the same as the sum of the rotated vectors:

  We drew the parallelogram $PQSR$, and then we rotate the entire configuration counterclockwise through θ degrees about point P.

  The result of this rotation is parallelogram $PQ'S'R'$, and it should be clear that $T(\overrightarrow{PQ}) = \overrightarrow{PQ'}$, $T(\overrightarrow{PR}) = \overrightarrow{PR'}$ and $T(\overrightarrow{PS}) = \overrightarrow{PS'}$.

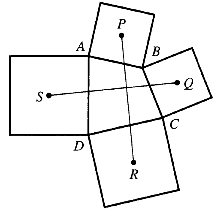
  We can now see that
  $T(\overrightarrow{PQ} + \overrightarrow{PR}) = T(\overrightarrow{PS}) = \overrightarrow{PS'} = \overrightarrow{PQ'} + \overrightarrow{PR'} = T(\overrightarrow{PQ}) + T(\overrightarrow{PR})$. Thus, the operator T respects vector addition.

  To see that the rotation operator T also respects scalar multiplication, and hence is linear, observe that if we stretch a vector and then rotate it, the result is the same as that obtained by first rotating and then stretching the same vector.

Property of a Quadrilateral

Proposition

Outward-facing squares are drawn on the sides of an arbitrary quadrilateral $ABCD$. If P, Q, R and S are the centers of these four squares, then line segments PR and SQ are equal and perpendicular.



- Let T be the linear operator corresponding to a $90°$ counterclockwise rotation. It suffices to show $T(\overrightarrow{PR}) = \overrightarrow{SQ}$. We express P, Q, R and S in terms of A, B, C and D. Let U be the midpoint of AB. Note that $PU = AU$ and PU is perpendicular to AU. Thus, $T(\overrightarrow{AU}) = \overrightarrow{UP}$. By the linearity of T, $T(\vec{U}) - T(\vec{A}) = T(\vec{U} - \vec{A}) = T(\overrightarrow{AU}) = \overrightarrow{UP} = \vec{P} - \vec{U}$. This yields $\vec{P} = \vec{U} + T(\vec{U}) - T(\vec{A})$. We also know $\vec{U} = \frac{1}{2}(\vec{A} + \vec{B})$. Thus, $\vec{P} = \frac{1}{2}(\vec{A} + \vec{B} + T(\vec{A}) + T(\vec{B})) - T(\vec{A}) = \frac{1}{2}(\vec{A} + \vec{B} - T(\vec{A}) + T(\vec{B}))$.

Property of a Quadrilateral (Cont'd)

- Marching around the quadrilateral, replacing A by B, B by C, C by D, and D by A, we get

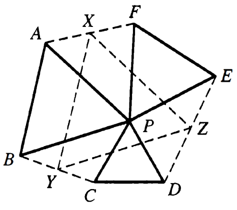
$$\vec{P} = \frac{1}{2}(\vec{A}+\vec{B}-T(\vec{A})+T(\vec{B})), \quad \vec{Q} = \frac{1}{2}(\vec{B}+\vec{C}-T(\vec{B})+T(\vec{C})),$$
$$\vec{R} = \frac{1}{2}(\vec{C}+\vec{D}-T(\vec{C})+T(\vec{D})), \quad \vec{S} = \frac{1}{2}(\vec{D}+\vec{A}-T(\vec{D})+T(\vec{A})).$$

  Next, we compute
  $\overrightarrow{PR} = \vec{R}-\vec{P} = \frac{1}{2}(\vec{C}+\vec{D}-T(\vec{C})+T(\vec{D})-\vec{A}-\vec{B}+T(\vec{A})-T(\vec{B}))$.
  To compute $T(\overrightarrow{PR})$, we must apply T to the right side of this equation. So we need to know how to compute $T(T(\vec{v}))$, where $\vec{v}$ is an arbitrary vector. Since T is a 90° rotation, we get $T(T(\vec{v})) = -\vec{v}$. Using this fact, together with the linearity of T, we obtain
  $T(\overrightarrow{PR}) = \frac{1}{2}(T(\vec{C})+T(\vec{D})+\vec{C}-\vec{D}-T(\vec{A})-T(\vec{B})-\vec{A}+\vec{B})$. Finally, note $\overrightarrow{SQ} = \vec{Q}-\vec{S} = \frac{1}{2}(\vec{B}+\vec{C}-T(\vec{B})+T(\vec{C})-\vec{D}-\vec{A}+T(\vec{D})-T(\vec{A}))$.
  This is identical with the formula for $T(\overrightarrow{PR})$.

Equilateral Triangles Sharing a Vertex

Proposition

Equilateral triangles $\triangle PAB$, $\triangle PCD$ and $\triangle PEF$ share a vertex. The remaining six vertices of these three triangles are joined in pairs by line segments FA, BC and DE and points X, Y and Z are the midpoints of these three segments. Then $\triangle XYZ$ is equilateral.

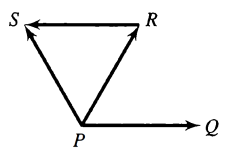


- Let T be the linear operator that rotates vectors counterclockwise through $60°$. Choose the origin at P so that $\overrightarrow{PA} = \vec{A}$, $\vec{B} = T(\vec{A})$, $\vec{D} = T(\vec{C})$ and $\vec{F} = T(\vec{E})$. We express X, Y and Z in terms of A, C and E. It suffices to show $T(\overrightarrow{ZX}) = \overrightarrow{ZY}$.
  Note $\vec{X} = \frac{1}{2}(\vec{A} + \vec{F}) = \frac{1}{2}(\vec{A} + T(\vec{E}))$. Similarly, $\vec{Y} = \frac{1}{2}(\vec{C} + T(\vec{A}))$. and $\vec{Z} = \frac{1}{2}(\vec{E} + T(\vec{C}))$. Therefore, $\overrightarrow{ZX} = \vec{X} - \vec{Z} = \frac{1}{2}(\vec{A} + T(\vec{E}) - \vec{E} - T(\vec{C}))$.
  We now must apply T, and show the result equal to
  $\overrightarrow{ZY} = \vec{Y} - \vec{Z} = \frac{1}{2}(\vec{C} + T(\vec{A}) - \vec{E} - T(\vec{C}))$.

Equilateral Triangles Sharing a Vertex (Cont'd)

- We got $\overrightarrow{ZX} = \frac{1}{2}(\vec{A} + T(\vec{E}) - \vec{E} - T(\vec{C}))$. We now must apply T, and show the result equal to $\overrightarrow{ZY} = \frac{1}{2}(\vec{C} + T(\vec{A}) - \vec{E} - T(\vec{C}))$.
  We first obtain a formula for $T(T(\vec{v}))$, where $\vec{v}$ is an arbitrary vector.

  Assume $\overrightarrow{PQ} = \vec{v}$, $\overrightarrow{PR} = T(\vec{v})$ and $\vec{PS} = T(T(\vec{v}))$. Then $PR = PS$ and $\angle RPS = 60°$. It follows that $\triangle RPS$ is equilateral. Thus, $RS = RP = PQ$, and $\angle SRP = 60° = \angle RPQ$. We conclude that RS is parallel and equal to PQ, i.e., $\overrightarrow{RS} = -\overrightarrow{PQ} = -\vec{v}$.



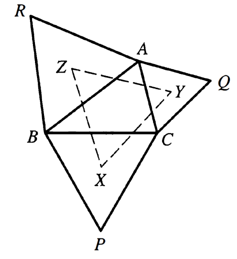
  We, thus, have $T(T(\vec{v})) = T(T(\overrightarrow{PQ})) = \overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS} = T(\vec{v}) - \vec{v}$.
  We now have: $T(\overrightarrow{ZX}) = \frac{1}{2}T(\vec{A} + T(\vec{E}) - \vec{E} - T(\vec{C})) =$
  $\frac{1}{2}(T(\vec{A}) + T(T(\vec{E})) - T(\vec{E}) - T(T(\vec{C}))) =$
  $\frac{1}{2}(T(\vec{A}) + T(\vec{E}) - \vec{E} - T(\vec{E}) - T(\vec{C}) + \vec{C}) = \frac{1}{2}(T(\vec{A}) - \vec{E} - T(\vec{C}) + \vec{C})$.

Napoleon Bonaparte's Theorem

Theorem

If we construct outward-pointing equilateral triangles on the three sides of an arbitrary given triangle, then the triangle formed by the centroids of the three equilateral triangles is equilateral.

- Given $\triangle ABC$, we construct equilateral $\triangle BCP$, $\triangle CAQ$ and $\triangle ABR$, with centroids X, Y and Z. We express X, Y and Z in terms of A, B and C. We then show $T(\overrightarrow{XY}) = \overrightarrow{XZ}$, where T is the operator that rotates vectors 60° counterclockwise. Since $\triangle BCP$ is equilateral, we see that $T(\overrightarrow{CB}) = \overrightarrow{CP}$. Thus, $\vec{P} - \vec{C} = T(\vec{B} - \vec{C})$.



  The linearity of T and some algebra yield $\vec{P} = \vec{C} - T(\vec{C}) + T(\vec{B})$. But $\vec{X} = \frac{1}{3}(\vec{B} + \vec{C} + \vec{P})$. So $\vec{X} = \frac{1}{3}(\vec{B} + 2\vec{C} - T(\vec{C}) + T(\vec{B}))$.

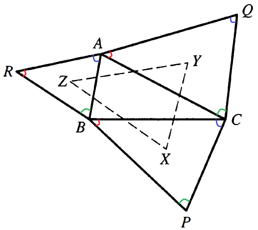
Napoleon Bonaparte's Theorem (Cont'd)

- We obtained $\vec{X} = \frac{1}{3}(\vec{B} + 2\vec{C} - T(\vec{C}) + T(\vec{B}))$.
  Similarly, we get
  $$\vec{Y} = \frac{1}{3}(\vec{C} + 2\vec{A} - T(\vec{A}) + T(\vec{C})), \quad \vec{Z} = \frac{1}{3}(\vec{A} + 2\vec{B} - T(\vec{B}) + T(\vec{A})).$$
  These equations yield
  $\overrightarrow{XY} = \vec{Y} - \vec{X} = \frac{1}{3}(\vec{C} + 2\vec{A} - T(\vec{A}) + T(\vec{C}) - \vec{B} - 2\vec{C} + T(\vec{C}) - T(\vec{B})) = \frac{1}{3}(2\vec{A} - \vec{B} - \vec{C} - T(\vec{A}) - T(\vec{B}) + 2T(\vec{C}))$ and
  $\overrightarrow{XZ} = \vec{Z} - \vec{X} = \frac{1}{3}(\vec{A} + 2\vec{B} - T(\vec{B}) + T(\vec{A}) - \vec{B} - 2\vec{C} + T(\vec{C}) - T(\vec{B})) = \frac{1}{3}(\vec{A} + \vec{B} - 2\vec{C} + T(\vec{A}) - 2T(\vec{B}) + T(\vec{C}))$.
  To compute $T(\overrightarrow{XY})$, recall that $T(T(\vec{v})) = T(\vec{v}) - \vec{v}$. Therefore,
  $T(\overrightarrow{XY}) = \frac{1}{3}(2T(\vec{A}) - T(\vec{B}) - T(\vec{C}) - T(\vec{A}) + \vec{A} - T(\vec{B}) + \vec{B} + 2T(\vec{C}) - 2\vec{C}) = \frac{1}{3}(\vec{A} + \vec{B} - 2\vec{C} + T(\vec{A}) - 2T(\vec{B}) + T(\vec{C}))$.

Extension of Napoleon's Theorem

Theorem

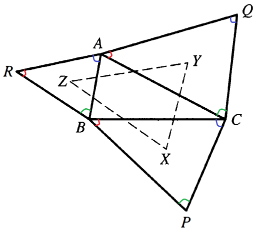
Suppose that three similar outward-pointing triangles are constructed on the sides of an arbitrary $\triangle ABC$, where $\triangle PCB \sim \triangle CQA \sim \triangle BAR$. If X, Y and Z are, respectively, the centroids of these three similar triangles, then $\triangle XYZ$ is similar to each of them.



- We define a linear operator T as:
  - a counterclockwise rotation by $\theta = \angle CPB = \angle QCA = \angle ABR$;
  - followed by multiplication by the scalar $z = \frac{PB}{PC}$.

By the definition of T, $T(\overrightarrow{PC}) = \overrightarrow{PB}$. Since $\triangle PCB \sim \triangle CQA$, $\frac{PC}{CQ} = \frac{PB}{CA}$. So $\frac{CA}{CQ} = \frac{PB}{PC} = z$. It follows that if we rotate the vector $\overrightarrow{CQ}$ counterclockwise through θ and then multiply by the scalar z, the result is the vector $\overrightarrow{CA}$. Thus, $T(\overrightarrow{CQ}) = \overrightarrow{CA}$. Similarly, $T(\overrightarrow{BA}) = \overrightarrow{BR}$.

Extension of Napoleon's Theorem (Cont'd)



- We show that $T(\overrightarrow{XY}) = \overrightarrow{XZ}$. It will then follow that $\angle YXZ = \theta = \angle CPB$. We also know that $\frac{XZ}{XY} = z = \frac{PB}{PC}$. Hence $\frac{PC}{XY} = \frac{PB}{XZ}$. Thus, $\triangle XYZ \sim \triangle PCB$ by the SAS similarity criterion.

  We have the following equations:

$$T(\vec{C}) - T(\vec{P}) = T(\overrightarrow{PC}) = \overrightarrow{PB} = \vec{B} - \vec{P},$$
$$T(\vec{Q}) - T(\vec{C}) = T(\overrightarrow{CQ}) = \overrightarrow{CA} = \vec{A} - \vec{C},$$
$$T(\vec{A}) - T(\vec{B}) = T(\overrightarrow{BA}) = \overrightarrow{BR} = \vec{R} - \vec{B}.$$

  But $\vec{Y} = \frac{1}{3}(\vec{C} + \vec{Q} + \vec{A})$, $\vec{X} = \frac{1}{3}(\vec{P} + \vec{C} + \vec{B})$, and $\vec{Z} = \frac{1}{3}(\vec{B} + \vec{A} + \vec{R})$. By adding the preceding three equations and multiplying by $\frac{1}{3}$, we obtain $T(\vec{Y}) - T(\vec{X}) = \vec{Z} - \vec{X}$. Thus, $T(\overrightarrow{XY}) = \overrightarrow{XZ}$, as desired.