Introduction to Graph Theory

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 351

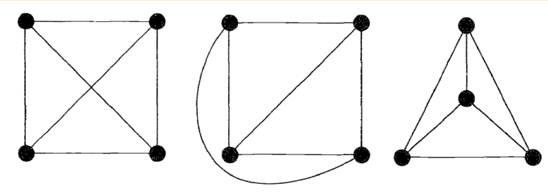
Subsection 1

Planar Graphs

Planar and Plane Graphs

- A **planar graph** is a graph that can be drawn in the plane without crossings, i.e., so that no two edges intersect geometrically except at a vertex to which both are incident.
- Any such drawing is a **plane drawing**.
- For convenience, we often use the abbreviation **plane graph** for a plane drawing of a planar graph.

  Example: From the three drawings of the planar graph K_4,



  only the second and third are plane graphs.
- K. Wagner (1936) and I. Fary (1948) showed that:

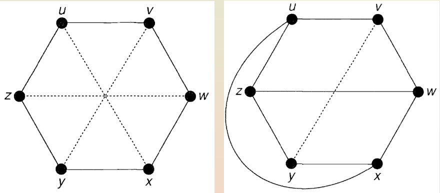
  Every simple planar graph can be drawn with straight lines.

Non-Planarity of $K_{3,3}$

Theorem

$K_{3,3}$ and K_5 are non-planar.

- Suppose first that $K_{3,3}$ is planar.



Since $K_{3,3}$ has a cycle $u \to v \to w \to x \to y \to z \to u$ of length 6, any plane drawing must contain this cycle drawn in the form of a hexagon, as on the right.

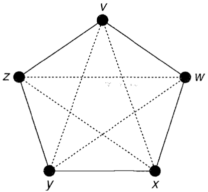
Now the edge wz must lie either wholly inside the hexagon or wholly outside it.

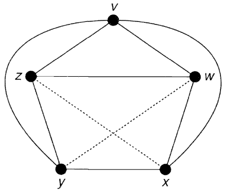
  - Assume wz lies inside the hexagon (the other case is similar). Since the edge ux must not cross the edge wz, it must lie outside the hexagon. It is then impossible to draw the edge vy, as it would cross either ux or wz. This yields a contradiction.

Non-Planarity of K_5

- Now suppose that K_5 is planar.

  K_5 has a cycle $v \to w \to x \to y \to z \to v$ of length 5.





  Any plane drawing must contain this cycle drawn in the form of a pentagon, as on the right.

  The edge wz must lie either wholly inside the pentagon or wholly outside it.

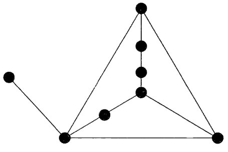
  - We deal with the case in which wz lies inside the pentagon (other case is similar).

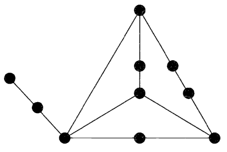
    Since the edges vx and vy do not cross the edge wz, they must both lie outside the pentagon. But the edge xz cannot cross the edge vy and so must lie inside the pentagon. Similarly the edge wy must lie inside the pentagon. But then, the edges wy and xz must cross. This yields a contradiction.

Kuratowski's Theorem

- Every subgraph of a planar graph is planar.
- Every graph with a non-planar subgraph must be non-planar.
- Thus, any graph with $K_{3,3}$ or K_5 as a subgraph is non-planar.
- $K_{3,3}$ and K_5 are the "building blocks" for non-planar graphs, in the sense that a non-planar graph must "contain" at least one of them:
- Define two graphs to be **homeomorphic** if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges.

  Example: Any two cycle graphs are homeomorphic. The graphs on the right are also.





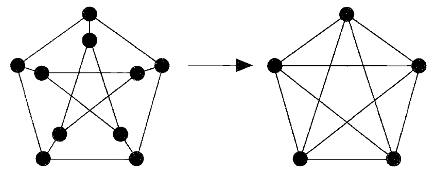
Theorem (Kuratowski, 1930)

A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

A Second Criterion for Planarity

- Define a graph H to be **contractible** to K_5 or $K_{3,3}$ if we can obtain K_5 or $K_{3,3}$ by successively contracting edges of H.

  Example: The Petersen graph is contractible to K_5, since the five "spokes" joining the inner and outer 5-cycles can be contracted to obtain K_5.



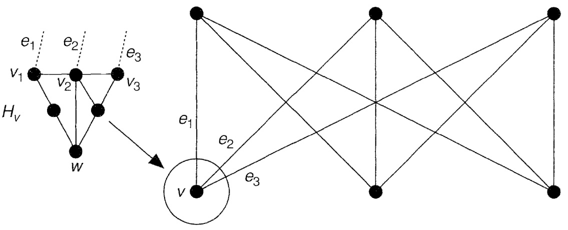
Theorem

A graph is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$.

$\Leftarrow$: Assume first that the graph G is non-planar. Then, by Kuratowski's theorem, G contains a subgraph H homeomorphic to K_5 or $K_{3,3}$. Successively contract edges of H that are incident to a vertex of degree 2. Then H is contracted to K_5 or $K_{3,3}$.

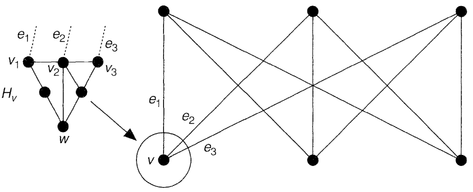
A Second Criterion for Planarity (The Converse)

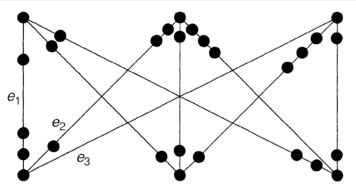
$\Rightarrow$: Now assume that G contains a subgraph H contractible to $K_{3,3}$, and let the vertex v of $K_{3,3}$ arise from contracting the subgraph H_v of H.



The vertex v is incident in $K_{3,3}$ to three edges e_1, e_2 and e_3. When regarded as edges of H, these edges are incident to three (not necessarily distinct) vertices v_1, v_2 and v_3 of H_v. If v_1, v_2 and v_3 are distinct, then we can find a vertex w of H_v and three paths from w to these vertices, intersecting only at w. (A similar construction applies if the vertices are not distinct, the paths degenerating in this case to single vertices.)

A Second Criterion for Planarity (Converse Cont'd)



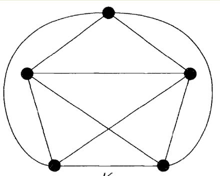


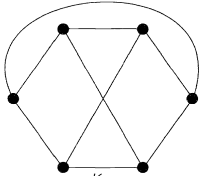
It follows that we can replace the subgraph H_v by a vertex w and three paths leading out of it. If this construction is carried out for each vertex of $K_{3,3}$, and the resulting paths joined up with the corresponding edges of $K_{3,3}$, then the resulting subgraph is homeomorphic to $K_{3,3}$. It follows from Kuratowski's theorem that G is non-planar.

A similar argument can be carried out if G contains a subgraph contractible to K_5. The details are more complicated, as the subgraph obtained by this process can be homeomorphic to either K_5 or $K_{3,3}$.

The Crossing Number

- If we try to draw K_5 or $K_{3,3}$ on the plane, then there must be at least one crossing of edges, since these graphs are not planar.
- However, we do not need more than one crossing.
  We say that K_5 and $K_{3,3}$ have *crossing number* 1.





- The **crossing number** $\text{cr}(G)$ of a graph G is the minimum number of crossings that can occur when G is drawn in the plane.

  Thus, the crossing number measures how "unplanar" G is.

  Example: The crossing number of any planar graph is 0;
  $\text{cr}(K_5) = \text{cr}(K_{3,3}) = 1$.
- "crossing" always refers to the *intersection of just two edges*, since crossings of three or more edges are not permitted.

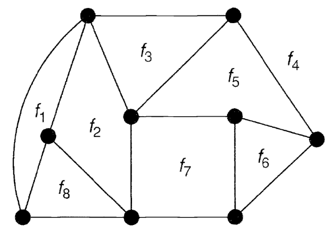
Subsection 2

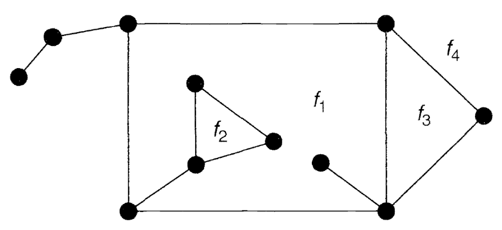
Euler's Formula

Faces of a Graph

- If G is a planar graph, then any plane drawing of G divides the set of points of the plane not lying on G into regions, called **faces**.

  Example: The plane graphs shown below



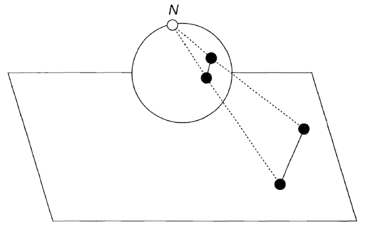


  have eight faces and four faces, respectively.

  Note that, in each case, the face f_4 is unbounded; it is called the **infinite face**.

Switching the Infinite Face

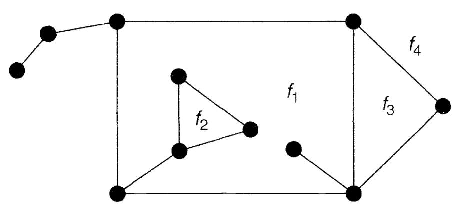
- Any face can be chosen as the infinite face:
  - Map the graph onto the surface of a sphere by stereographic projection;

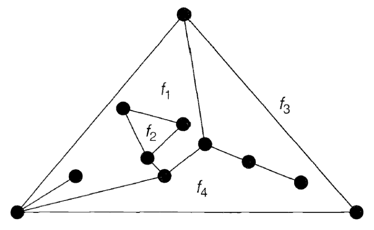


  - Rotate the sphere so that the point of projection (the north pole) lies inside the face we want as the infinite face;
  - Project the graph down onto the plane tangent to the sphere at the south pole;
  - The chosen face is now the infinite face.

Example

- Consider again the graph on the left





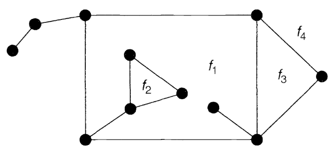
The right figure shows a representation of the previous graph in which the infinite face is f_3.

Euler's Formula

Theorem (Euler, 1750)

Let G be a plane drawing of a connected planar graph, and let n, m and f denote, respectively, the number of vertices, edges and faces of G. Then $n - m + f = 2$.

Example: $n = 11$, $m = 13$, $f = 4$, and $n - m + f = 11 - 13 + 4 = 2$.



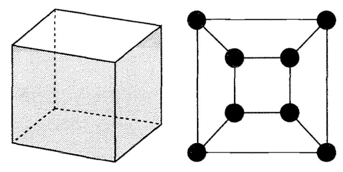
The proof by induction on the number of edges of G:

- If $m = 0$, then $n = 1$ (since G is connected) and $f = 1$ (the infinite face). The theorem is therefore true in this case.
- Suppose the theorem holds for all graphs with at most $m - 1$ edges. Let G be a graph with m edges.
  - If G is a tree, then $m = n - 1$ and $f = 1$. So that $n - m + f = 2$.
  - If G is not a tree, let e be an edge in some cycle of G. Then $G - e$ is a connected plane graph with n vertices, $m - 1$ edges and $f - 1$ faces. Thus, $n - (m - 1) + (f - 1) = 2$, by the hypothesis. So, $n - m + 1 = 2$.

Euler's Formula and Polyhedra

- Euler's formula is often called "Euler's polyhedron formula", since it relates the numbers of vertices, edges and faces of a convex polyhedron.

  Example: For a cube we have $n = 8$, $m = 12$, $f = 6$ and $n - m + f = 8 - 12 + 6 = 2$.



- The connection is established as follows:
  - Project the polyhedron out onto its circumsphere;
  - Use stereographic projection to project it down onto the plane.

  The resulting graph is a **polyhedral graph**, i.e., a 3-connected plane graph in which each face is bounded by a polygon.

Corollary

Let G be a polyhedral graph, with number of vertices, edges and faces, respectively, n, m and f. Then $n - m + f = 2$.

Euler's Formula for Disconnected Graphs

Corollary

Let G be a plane graph with n vertices, m edges, f faces and k components. Then $n - m + f = k + 1$.

- Suppose G has k components $G_1, \ldots, G_k$.
  Assume component G_i has n_i vertices, m_i edges and f_i faces.
  Then, we have the following relations:

$$\begin{array}{rcl} n_1 + n_2 + \cdots + n_k & = & n; \\ m_1 + m_2 + \cdots + m_k & = & m; \\ f_1 + f_2 + \cdots + f_k & = & f + (k-1). \end{array}$$

  Using Euler's formula for each of the components, we get:

$$\begin{gathered} (n_1 - m_1 + f_1) + (n_2 - m_2 + f_2) + \cdots + (n_k - m_k + f_k) = 2k \\ (n_1 + \cdots + n_k) - (m_1 + \cdots + m_k) + (f_1 + \cdots + f_k) = 2k \\ n - m + f + (k-1) = 2k \\ n - m + f = k + 1. \end{gathered}$$

Inequalities for Connected Simple Graphs

Corollary

(i) If G is a connected simple planar graph with n (≥ 3) vertices and m edges, then $m \leq 3n - 6$.

(ii) If, in addition, G has no triangles, then $m \leq 2n - 4$.

(i) Suppose we have a plane drawing of G. Each face is bounded by at least three edges. Each edge bounds two faces. Thus, by counting up the edges around each face, we get $3f \leq 2m$. So $f \leq \frac{2}{3}m$. Apply Euler's formula:

$$2 = m - n + f \leq n - m + \frac{2}{3}m = n - \frac{1}{3}m.$$

Multiply both sides by 3: $6 \leq 3n - m$. So $m \leq 3n - 6$.

(ii) This part follows in a similar way, except that the inequality $3f \leq 2m$ is replaced by $4f \leq 2m$.

Second Proof of Non-Planarity of K_5 and $K_{3,3}$

Corollary

K_5 and $K_{3,3}$ are non-planar.

- Suppose K_5 is planar. Recall that $n = 5$ and $m = 10$. Applying the inequality $m \leq 3n - 6$, we get $10 \leq 3 \cdot 5 - 6 = 9$. This is a contradiction.
- Suppose $K_{3,3}$ is planar. Recall that $n = 6$ and $m = 9$. Note that $K_{3,3}$ does not have any triangles. Applying the inequality $m \leq 2n - 4$, we get $9 \leq 2 \cdot 6 - 4 = 8$. This is a contradiction.

Planarity and Minimum Degree of a Vertex

Theorem

Every simple planar graph contains a vertex of degree at most 5.

- Without loss of generality we can assume that the graph is connected and has at least three vertices.

  Suppose each vertex has degree at least 6.

  With the above notation, we have the inequality

  $$6n \leq 2m.$$

  So $3n \leq m$.

  Thus, by the corollary,

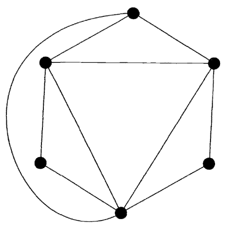
  $$3n \leq m \leq 3n - 6 \quad \Rightarrow \quad 0 \leq -6.$$

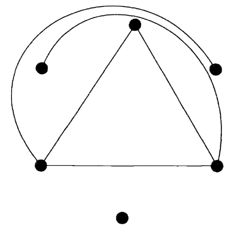
  This yields a contradiction.

Thickness of a Graph

- We define the **thickness** $t(G)$ of a graph G to be the smallest number of planar graphs that can be superimposed to form G.
- Like the crossing number, the thickness is a measure of how "unplanar" a graph is.

  Example: The thickness of a planar graph is 1. The thickness of K_5 and $K_{3,3}$ is 2.





The thickness of K_6 is 2.

Lower Bounds on the Thickness of a Graph

Theorem

Let G be a simple graph with n (≥ 3) vertices and m edges. Then the thickness $t(G)$ of G satisfies $t(G) \geq \lceil \frac{m}{3n-6} \rceil$ and $t(G) \geq \lfloor \frac{m+3n-7}{3n-6} \rfloor$.

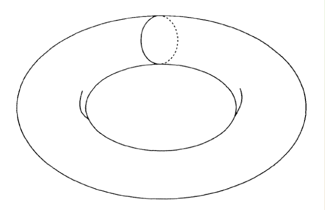
- Suppose G has n vertices and m edges.
  Each of the $k = t(G)$ layers $G_1, \ldots, G_k$ has n vertices and, say, m_i edges, with $m_1 + m_2 + \cdots + m_k = m$.
  Since each layer is planar, we must have
  $$m_1 \leq 3n - 6, m_2 \leq 3n - 6, \ldots, m_k \leq 3n - 6.$$
  So $m_1 + \cdots + m_k \leq t(G)(3n - 6)$, i.e., $m \leq t(G)(3n - 6)$.
  This gives $t(G) \geq \frac{m}{3n-6}$. Since $t(G)$ is an integer, $t(G) \geq \lceil \frac{m}{3n-6} \rceil$.
  The second part follows from the first by using the relation $\lceil \frac{a}{b} \rceil = \lfloor \frac{a+b-1}{b} \rfloor$, where a and b are positive integers.

Subsection 3

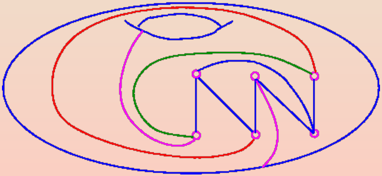
Graphs on Other Surfaces

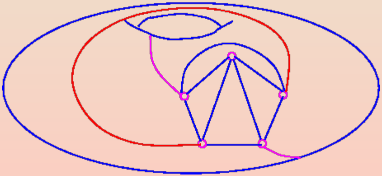
Surfaces Other than the Plane

- We considered graphs drawn in the plane or (equivalently) on the surface of a sphere.
- We may draw graphs on other surfaces, such as the torus.



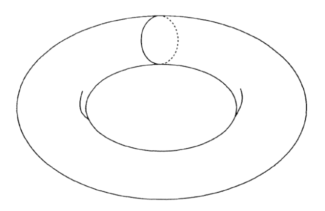
Example: K_5 and $K_{3,3}$ can be drawn without crossings on the surface of a torus.

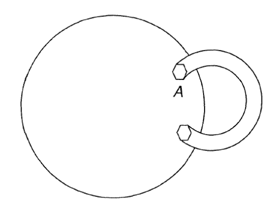




The Genus of a Surface and the Genus of a Graph

- The torus can be thought of as a sphere with one "handle".





- A surface is of **genus** g if it is topologically homeomorphic to a sphere with g handles (intuitively the surface of a doughnut with g holes in it).

  Example: The genus of a sphere is 0, and that of a torus is 1.
- A graph that can be drawn without crossings on a surface of genus g, but not on one of genus $g-1$, is a **graph of genus** g.

  Example: K_5 and $K_{3,3}$ are graphs of genus 1 (also called **toroidal graphs**).

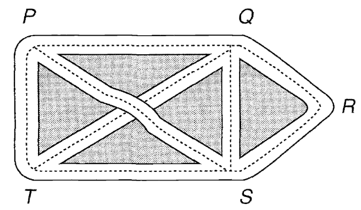
Upper Bound on the Genus of a Graph

Theorem

The genus of a graph does not exceed the crossing number.

- We draw the graph on the surface of a sphere so that the number of crossings is as small as possible, and is therefore equal to the crossing number c.

  At each crossing, we construct a "bridge" and run one edge over the bridge and the other under it.



  Since each bridge can be thought of as a handle, we have drawn the graph on the surface of a sphere with c handles. It follows that the genus does not exceed c.

Kuratowski's and Euler's Theorems for Graphs of Genus g

- There is currently no complete analogue of Kuratowski's theorem for surfaces of genus g.
- Robertson and Seymour have proved that there exists a finite collection of "forbidden" subgraphs of genus g, for each value of g, corresponding to the forbidden subgraphs K_5 and $K_{3,3}$ for graphs of genus 0.
- In the case of Euler's formula, we define a **face** of a graph of genus g in the obvious way.

Theorem

Let G be a connected graph of genus g with n vertices, m edges and F faces. Then $n - m + f = 2 - 2g$.

Lower Bound on the Genus of a Graph

Corollary

The genus $g(G)$ of a simple graph G with n (≥ 4) vertices and m edges satisfies the inequality $g(G) \geq \lceil \frac{m-3n}{6} + 1 \rceil$.

- Since each face is bounded by at least three edges, we have $3f \leq 2m$. Thus, $f \leq \frac{2}{3}m$. Now we use $n - m + f = 2 - 2g$.

$$\begin{aligned} n - m + \tfrac{2}{3}m &\geq 2 - 2g \\ n - \tfrac{1}{3}m &\geq 2 - 2g \\ 2g &\geq \tfrac{1}{3}m - n + 2 \\ g &\geq \tfrac{1}{6}m - \tfrac{1}{2}n + 1 \\ g &\geq \tfrac{m-3n}{6} + 1. \end{aligned}$$

Since g is an integer, $g \geq \lceil \frac{m-3n}{6} + 1 \rceil$.

Genus of Complete Graphs

- The complete graph K_n has n vertices and $\frac{n(n-1)}{2}$ edges. By the corollary, $g \geq \lceil \frac{m-3n}{6} + 1 \rceil$:

$$g(K_n) \geq \left\lceil \frac{\frac{n(n-1)}{2} - 3n}{6} + 1 \right\rceil$$
$$g(K_n) \geq \lceil \frac{n^2 - n - 6n}{12} + 1 \rceil$$
$$g(K_n) \geq \lceil \frac{n^2 - 7n + 12}{12} \rceil$$
$$g(K_n) \geq \lceil \frac{(n-3)(n-4)}{12} \rceil.$$

- Heawood asserted in 1890 that the inequality $g(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ is an equality.

Theorem (Ringel and Youngs, 1968)

$g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$.

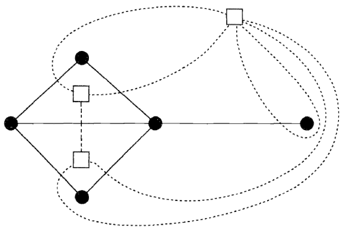
Subsection 4

Dual Graphs

The Dual Graph of a Planar Graph

- Given a plane drawing of a planar graph G, we construct another graph G^*, called the (**geometric**) **dual** of G:
  (i) inside each face f of G we choose a point v^*; these points are the vertices of G^*;
  (ii) corresponding to each edge e of G we draw a line e^* that crosses e (but no other edge of G), and joins the vertices v^* in the faces f adjoining e; these lines are the edges of G^*.

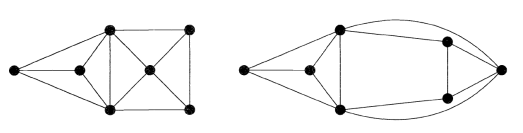
  Example: The vertices v^* of G^* are represented by small squares.
  The edges e of G are shown as solid lines.
  The edges e^* of G^* are shown as dotted lines.



- An end-vertex or a bridge of G gives rise to a loop of G^*.
- If two faces of G have more than one edge in common, then G^* has multiple edges.

Relations Between Duals

- Any two geometric duals of G must be isomorphic.
  This is why G^* is called "the dual of G" instead of "a dual of G".
- On the other hand, if G is isomorphic to H, it does not necessarily follow that G^* is isomorphic to H^*:



- If G is both plane and connected, then G^* is plane and connected:

Lemma

Let G be a plane connected graph with n vertices, m edges and f faces, and let its geometric dual G^* have n^* vertices, m^* edges and f^* faces. Then $n^* = f$, $m^* = m$ and $f^* = n$.

- The first two relations are direct consequences of the definition of G^*. The third relation follows on substituting these two relations into Euler's formula, applied to both G and G^*.

The Dual of the Dual

- Since the dual G^* of a plane graph G is also a plane graph, we can repeat the above construction to form the dual G^{**} of G^*.
- If G is connected, then the relationship between G^{**} and G is particularly simple:

Theorem

If G is a plane connected graph, then G^{**} is isomorphic to G.

- The result follows immediately, since the construction that gives rise to G^* from G can be reversed to give G from G^*.

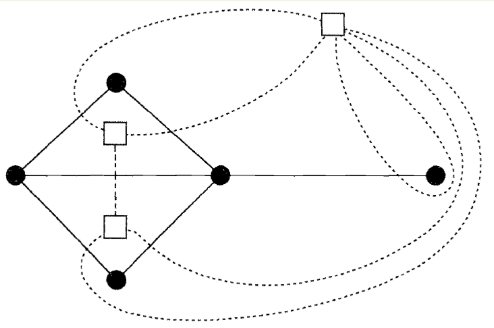
  We need to check only that a face of G^* cannot contain more than one vertex of G. Letting n^{**} be the number of vertices of G^{**}, we get:

$$n^{**} = f^* = n.$$

  So each face of G^* contains exactly one vertex of G.

Example

- The graph G is the dual of the graph G^*.



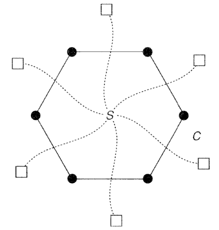
Geometric Duals, Cycles and Cutsets

Theorem

Let G be a planar graph and let G^* be a geometric dual of G. Then a set of edges in G forms a cycle in G if and only if the corresponding set of edges of G^* forms a cutset in G^*.

- We can assume that G is a connected plane graph. If C is a cycle in G, then C encloses one or more finite faces of C. Thus, it contains in its interior a non-empty set S of vertices of G^*.

  It follows that those edges of G^* that cross the edges of C form a cutset of G^* whose removal disconnects G^* into two subgraphs, one with vertex set S and the other containing those vertices that do not lie in S.

  The converse implication is similar.

Geometric Duals, Cutsets and Cycles

Corollary

A set of edges of G forms a cutset in G if and only if the corresponding set of edges of G^* forms a cycle in G^*.

- The result follows on applying the preceding theorem to G^*:

  A set of edges in G^* forms a cycle in G^*
  iff the corresponding set of edges of G^{**} forms a cutset in G^{**}.

  But G^{**} is isomorphic to G.

  So, we get

  A set of edges in G^* forms a cycle in G^*
  iff the corresponding set of edges of G forms a cutset in G.

Subsection 5

Infinite Graphs

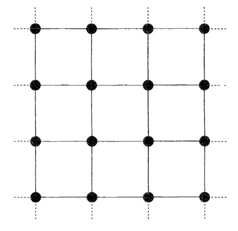
Infinite Graphs

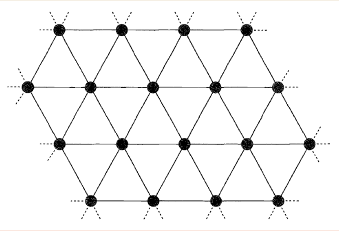
- An **infinite graph** G consists of:
  - An infinite set $V(G)$ of elements called **vertices**;
  - An infinite family $E(G)$ of unordered pairs of elements of $V(G)$ called **edges**.
- If $V(G)$ and $E(G)$ are both countably infinite, then G is a **countable graph**.
- We exclude the possibility of:
  - $V(G)$ being infinite but $E(G)$ finite, as such objects are merely finite graphs together with infinitely many isolated vertices;
  - $E(G)$ being infinite but $V(G)$ finite, as such objects are essentially finite graphs but with infinitely many loops or multiple edges.
- Many of our earlier definitions (**adjacent**, **incident**, **isomorphic**, **subgraph**, **connected**, **planar**, etc.) extend to infinite graphs.

Degrees in Infinite Graphs

- The **degree** of a vertex v of an infinite graph is the cardinality of the set of edges incident to v, and may be finite or infinite.
- An infinite graph, each of whose vertices has finite degree, is **locally finite**.

  Examples: The infinite square lattice and the infinite triangular lattice are both locally finite infinite graphs.





- We similarly define a **locally countable** infinite graph to be one in which each vertex has countable degree.

Connected Locally Countable Infinite Graphs

Theorem

Every connected locally countable infinite graph is a countable graph.

- Let v be any vertex of such an infinite graph. Let A_1 be the set of vertices adjacent to v, A_2 be the set of all vertices adjacent to a vertex of A_1, and so on. By hypothesis, A_1 is countable. Since the union of a countable collection of countable sets is countable, so are $A_2, A_3, \ldots$. Hence $\{v\}, A_1, A_2, \ldots$ is a sequence of sets whose union is countable and contains every vertex of the infinite graph, by connectedness. This yields the result.

Corollary

Every connected locally finite infinite graph is a countable graph.

Types of Paths in Infinite Graphs

- In an infinite graph G, there are three different types of walk:
  - (i) A **finite walk** is defined as for finite graphs;
  - (ii) A **one-way infinite walk** with initial vertex v_0 is an infinite sequence of edges of the form $v_0v_1, v_1v_2, \ldots$;
  - (iii) A **two-way infinite walk** is an infinite sequence of edges of the form $\ldots, v_{-2}v_{-1}, v_{-1}v_0, v_0v_1, v_1v_2, \ldots$;
- **One-way** and **two-way infinite trails** and **paths** are defined analogously, as are the **length** of a path and the **distance** between vertices.

König's Lemma

Theorem (König's Lemma, 1927)

Let G be a connected locally finite infinite graph. Then, for any vertex v of G, there exists a one-way infinite path with initial vertex v.

- Eor each vertex z other than v, there is a non-trivial path from v to z. It follows that there are infinitely many paths in G with initial vertex v. Since the degree of v is finite, infinitely many of these paths must start with the same edge. If vv_1 is such an edge, then we can repeat this procedure for the vertex v_1 and thus obtain a new vertex v_2 and a corresponding edge v_1v_2. By carrying on in this way, we obtain the one-way infinite path $v \to v_1 \to v_2 \to \cdots$.

Planarity and Infinite Graphs

Theorem

If G be a countable graph, every finite subgraph of which is planar, then G is planar.

- Since G is countable, its vertices may be listed as $v_1, v_2, v_3, \ldots$. Construct a strictly increasing sequence $G_1 \subset G_2 \subset G_3 \subset \cdots$ of subgraphs of G, where G_k is the subgraph whose vertices are $v_1, \ldots, v_k$ and whose edges those of G joining two of these vertices. Graph G_i can be drawn in the plane in only a finite number $m(i)$ of topologically distinct ways.

  We use this to construct another infinite graph H.

Planarity and Infinite Graphs (Cont'd)

- We construct the infinite graph H as follows:
  - The vertices w_{ij}, $i \geq 1$, $1 \leq j \leq m(i)$, of H correspond to the drawings of the graphs G_i;
  - The edges of H join those vertices w_{ij} and $w_{k\ell}$, for which $k = i + 1$ and the plane drawing corresponding to $w_{k\ell}$ extends the drawing corresponding to w_{ij}.

  Since H is connected and locally finite, by König's Lemma, H contains a one-way infinite path.

  Since G is countable, this infinite path gives a plane drawing of G.

Eulerian Infinite Graphs

- We call a connected infinite graph G **Eulerian** if there exists a two-way infinite trail that includes every edge of G.

  Such an infinite trail is a two-way **Eulerian trail**.
- These definitions require G to be countable.
- The following theorems give additional necessary conditions for an infinite graph to be Eulerian:

Theorem

Let G be a connected countable graph which is Eulerian. Then:

(i) G has no vertices of odd degree;

(ii) For each finite subgraph H of G, the infinite graph $\overline{H}$ obtained by deleting from G the edges of H has at most two infinite connected components;

(iii) If, in addition, each vertex of H has even degree, then $\overline{H}$ has exactly one infinite connected component.

Eulerian Infinite Graphs: Proof of Necessary Conditions

(i) Suppose that P is an Eulerian trail. Then, by the argument given in the proof of the finite case, each vertex of G must have either even or infinite degree.

(ii) Let P be split up into three subtrails P_-, P_0 and P_+ in such a way that P_0 is a finite trail containing every edge of H, and possibly other edges as well, and P_- and P_+ are one-way infinite trails. Then the infinite graph K formed by the edges of P_- and P_+ and the vertices incident to them, has at most two infinite components. Since $\overline{H}$ is obtained by adding only a finite set of edges to K, the result follows.

(iii) Let the initial and final vertices of P_0 be v and w. We wish to show that v and w are connected in $\overline{H}$.

- If $v = w$, this is obvious.
- Suppose $v \neq w$. If we remove the edges of H from P_0, the resulting graph has exactly two vertices v and w of odd degree. Therefore, v and w must belong to the same component, i.e., they are connected.

Characterization of Eulerian Infinite Graphs

- It turns out the preceding three conditions are also sufficient.

Theorem

If G is a connected countable graph, then G is Eulerian if and only if Conditions (i), (ii) and (iii) of the preceding theorem are satisfied.