Introduction to Graph Theory

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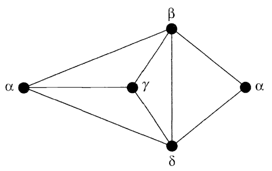
Subsection 1

Coloring Vertices

k-Colorability and Chromatic Number

- If G is a graph without loops, then G is k-**colorable** if we can assign one of k colors to each vertex so that adjacent vertices have different colors.
- If G is k-colorable, but not $(k-1)$-colorable, we say that G is k-**chromatic**, or that the **chromatic number** of G is k, and write $\chi(G) = k$.

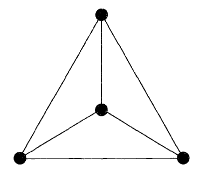
  Example: A graph G for which $\chi(G) = 4$; the colors are denoted by Greek letters. It is thus k-colorable if $k \geq 4$.

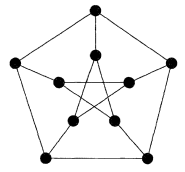


- We assume that all graphs are *simple*, since multiple edges do not affect colorability. We also assume that they are *connected*.

Some Examples

- $\chi(K_n) = n$.
- There are graphs with arbitrarily high chromatic number.
- $\chi(G) = 1$ if and only if G is a null graph.
- $\chi(G) = 2$ if and only if G is a non-null bipartite graph.
- Every tree and every cycle with an even number of vertices is 2-colorable.
- It is not known which graphs are 3-chromatic, but examples include:
  - The cycle graphs or wheels with an odd number of vertices;
  - The Petersen graph.





- The wheels with an even number of vertices are 4-chromatic.

Chromatic Number of an Arbitrary Graph

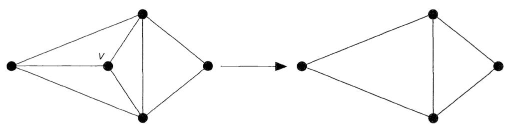
- If a graph has n vertices, then its chromatic number cannot exceed n.
- If the graph contains K_r as a subgraph, then its chromatic number cannot be less than r.

Theorem

If G is a simple graph with largest vertex-degree Δ, then G is $(\Delta + 1)$-colorable.

- By induction on the number of vertices of G.

  Let G be simple with n vertices. If we delete any vertex v and its incident edges, then the graph that remains is simple with $n - 1$ vertices and largest vertex-degree at most Δ.



  By the induction hypothesis, this graph is $(\Delta+1)$-colorable.

  A $(\Delta + 1)$-coloring for G is obtained by coloring v with a different color from the (at most Δ) vertices adjacent to v.

Brooks' Theorem

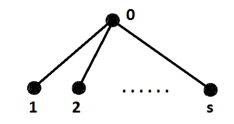
- By more careful treatment we can strengthen this theorem to obtain Brooks' Theorem whose proof is presented later:

Theorem (Brooks' Theorem, 1941)

If G is a simple connected graph which is not a complete graph, and if the largest vertex-degree of G is Δ (≥ 3), then G is Δ-colorable.

- By the previous theorem, every cubic graph is 4-colorable.
- By Brooks' Theorem, every connected cubic graph, other than K_4, is 3-colorable.
- If the graph has a few vertices of large degree, then these theorems are rather weak:

  Example: Brooks' Theorem asserts that, for any s, the graph $K_{1,s}$ is s-colorable. It is in fact 2-colorable.



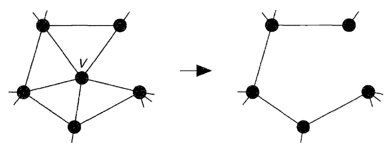
6-Colorability of Planar Graphs

Theorem

Every simple planar graph is 6-colorable.

- By induction on the number of vertices:
  - The result is trivial for simple planar graphs with at most six vertices.
  - Suppose that G is a simple planar graph with n vertices, and that all simple planar graphs with $n-1$ vertices are 6-colorable.

  By a preceding theorem, G contains a vertex v of degree at most 5. If we delete v and its incident edges, then the graph that remains has $n-1$ vertices and is thus 6-colorable.



  A 6-coloring of G is then obtained by coloring v with a color different from the (at most five) vertices adjacent to v.

The Five Color Theorem

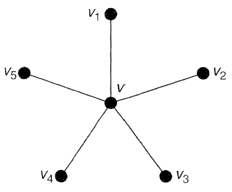
Theorem

Every simple planar graph is 5-colorable.

- By induction on the number of vertices:
  - The result is trivial for simple planar graphs with fewer than six vertices.
  - Suppose then that G is a simple planar graph with n vertices, and that all simple planar graphs with $n-1$ vertices are 5-colorable. As before, G contains a vertex v of degree at most 5. The deletion of v leaves a graph with $n-1$ vertices, which is thus 5-colorable. The goal is to color v with one of the five colors, so completing the 5-coloring of G.
    - If $\deg(v) < 5$, then v can be colored with any color not assumed by the (at most four) vertices adjacent to v, completing the proof in this case.
    - Suppose that $\deg(v) = 5$, and that the vertices $v_1, \ldots, v_5$ adjacent to v are arranged around v in clockwise order. If the vertices $v_1, \ldots, v_5$ are all mutually adjacent, then G contains the non-planar graph K_5 as a subgraph, which is impossible.

The Five Color Theorem (Cont'd)

- So at least two of the vertices v_i (say, v_1 and v_3) are not adjacent:
  We now contract the two edges vv_1 and vv_3. The resulting graph is a planar graph with fewer than n vertices, and is thus 5-colorable.



  We next reinstate the two edges, giving both v_1 and v_3 the color originally assigned to v. A 5-coloring of G is then obtained by coloring v with a color different from the (at most four) colors assigned to the vertices v_i.

The Four-Color Theorem

- One of the most famous unsolved problems in mathematics had been the "four-colour problem", whether every simple planar graph is 4-colorable.
- This problem, first posed in 1852, was eventually settled by K. Appel and W. Haken in 1976.

The Four-Color Theorem (Appel and Haken, 1976)

Every simple planar graph is 4-colorable.

- Their proof, which took them several years and a substantial amount of computer time, relies on a complicated extension of the ideas in the proof of the five color theorem.

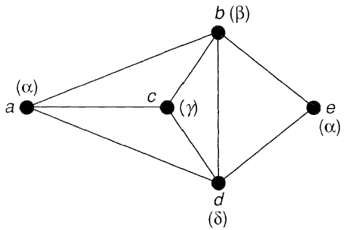
The Chemical Storage Problem

- Suppose chemicals a, b, c, d and e are to be stored in various areas of a warehouse. Some of these react when in contact, and so must be kept in separate areas. In the following table, an asterisk indicates those pairs of chemicals that must be separated. How many areas are needed?

	a	b	c	d	e
a	—	*	*	*	—
b	*	—	*	*	*
c	*	*	—	*	—
d	*	*	*	—	*
e	—	*	—	*	—

  To answer this, we draw a graph:

  - Its vertices correspond to the five chemicals;
  - Two vertices are adjacent whenever the corresponding chemicals are to be kept apart.



  We color the vertices as shown.

  The colors correspond to the areas needed for safe storage.

Subsection 2

Brooks' Theorem

Brooks' Theorem

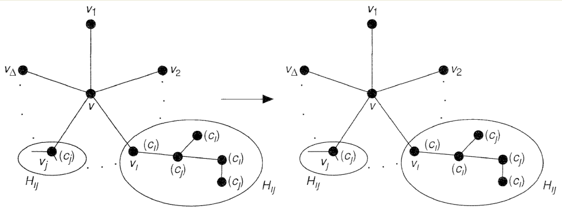
Theorem

If G is a simple connected graph which is not a complete graph, and if the largest vertex degree of G is Δ (≥ 3), then G is Δ-colorable.

- The proof is by induction on the number of vertices of G. Suppose that G has n vertices.
  - If any vertex of G has degree less than Δ, then we can complete the proof by imitating the proof of the preceding theorem.
  - We may thus suppose that G is regular of degree Δ.
    If we delete a vertex v and its incident edges, then the graph that remains has $n-1$ vertices and the largest vertex degree is at most Δ. By the induction hypothesis, this graph is Δ-colorable. Our aim is now to color v with one of the Δ colors. We can assume that the vertices $v_1, \ldots, v_\Delta$, adjacent to v, are arranged around v in clockwise order, and that they are colored with distinct colors $c_1, \ldots, c_\Delta$, since, otherwise, there would be a spare color that could be used to color v.

Brooks' Theorem: The Components

- We define H_{ij} ($i \neq j$, $1 \leq i, j \leq \Delta$) to be the subgraph of G whose vertices are those colored c_i or c_j and whose edges are those joining a vertex colored c_i and a vertex colored c_j.
  - If the vertices v_i and v_j lie in different components of H_{ij}, then we can interchange the colors of all the vertices in the component of H_{ij} containing v_i.

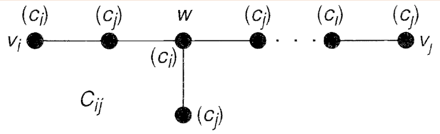


    The result of this recoloring is that v_i and v_j both have color c_j. This enables v to be colored c_i.
  - We may thus assume that, given any i and j, v_i and v_j are connected by a path that lies entirely in H_{ij}. We denote the component of H_{ij} containing v_i and v_j by C_{ij}.

Brooks' Theorem: The Paths

- If v_i is adjacent to more than one vertex with color c_j, then there is a color (other than c_i) that is not assumed by any vertex adjacent to v_i. In this case, v_i can be recolored using this color, enabling v to be colored with color c_i.
- If this does not happen, then we can use a similar argument to show that every vertex of C_{ij} (other than v_i and v_j) must have degree 2.

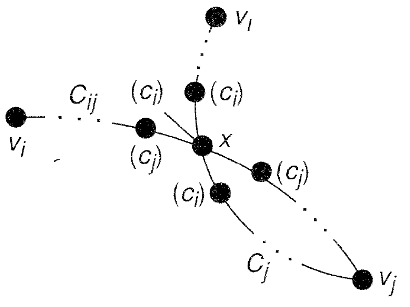
  For, if w is the first vertex of the path from v_i to v_j with degree greater than 2, then w can be recolored with a color other than c_i, c_j, thereby destroying the property that v_i and v_j are connected by a path lying entirely in C_{ij}.

  We can thus assume that, for any i and j, the component C_{ij} consists only of a path from v_i to v_j.

Brooks' Theorem: Pairs of Path

- We can also assume that two paths of the form C_{ij} and $C_{j\ell}$ (where $i \neq \ell$) intersect only at v_j, since any other point of intersection x can be recolored with a color different from c_i, c_j or c_ℓ.

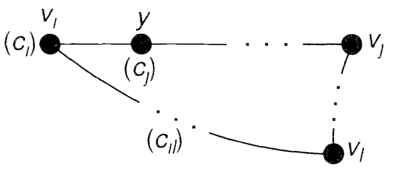


This would contradict the fact that v_i and v_j are connected by a path.

Brooks' Theorem: Conclusion

- To complete the proof, we choose two vertices v_i and v_j that are not adjacent, and let y be the vertex with color c_j that is adjacent to v_i.

  If $C_{i\ell}$ is a path (for some $\ell \neq j$), then we can interchange the colors of the vertices in this path without affecting the coloring of the rest of the graph.



  But if we carry out this interchange, then y would be a vertex common to the paths C_{ij} and $C_{j\ell}$ which is a contradiction. This proves the theorem.

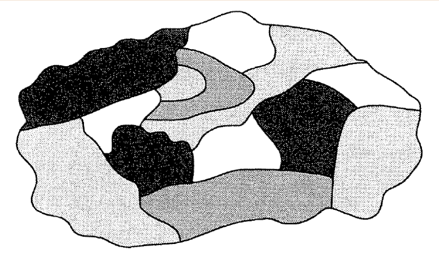
Subsection 3

Coloring Maps

Coloring a Map

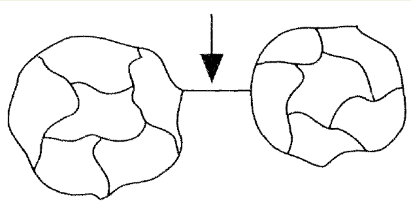
- Given a map containing several countries, the goal is to find the number of colors needed to color them so that no two countries with a boundary line in common share the same color.
- The most familiar form of the four-color theorem is the statement that every map can be colored with only four colors.

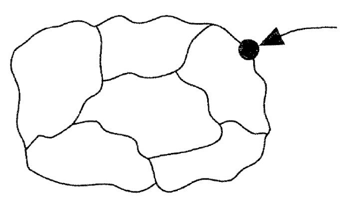
  Example: The figure shows a map that has been colored with four colors (shades of grey):



Formal Definition of Maps

- Since the two colors on either side of an edge must be different, we need to exclude maps containing a bridge:





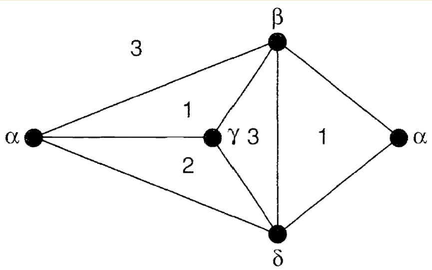
- We also exclude vertices of degree 2, as they can easily be eliminated.
- We define a **map** to be a 3-connected plane graph.

  Thus a map contains no cutsets with 1 or 2 edges, and in particular no vertices of degree 1 or 2.

Face and Vertex Colorability

- A map is k-**colorable(f)** if its faces can be colored with k colors so that no two faces with a boundary edge in common have the same color.
- To avoid confusion, we use k-**colorable(v)** to mean k-colorable in the usual sense.

  Example: The map in the figure is 3-colorable(f) and 4-colorable(v).



Criterion for 2-Colorability(f)

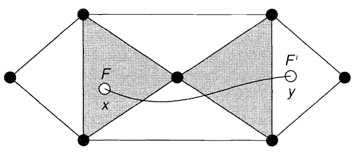
Theorem

A map G is 2-colorable(f) if and only if G is an Eulerian graph.

$\Rightarrow$: For each vertex v of G, the faces surrounding v must be even in number, since they can be colored with two colors. It follows that each vertex has even degree. So G is Eulerian.

$\Leftarrow$: If G is Eulerian, we color its faces in two colors as follows. Choose any face F and color it red. Draw a curve from a point x in F to a point in each other face, passing through no vertex of G.

- If such a curve crosses an even number of edges, color the face red;
- otherwise, color it blue.



Each vertex has an even number of edges incident with it. Thus, a "cycle" of two such curves crosses an even number of edges of G. Hence, the coloring is well defined.

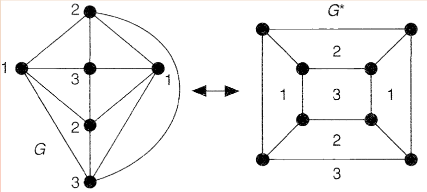
Geometric Duality and Colorability

Theorem

Let G be a plane graph without loops, and let G^* be a geometric dual of G. Then G is k-colorable(v) if and only if G^* is k-colorable(f).

$\Rightarrow$: We can assume that G is simple and connected, so that G^* is a map. If we have a k-coloring(v) for G, then we can k-color the faces of G^* so that each face inherits the color of the unique vertex that it contains.

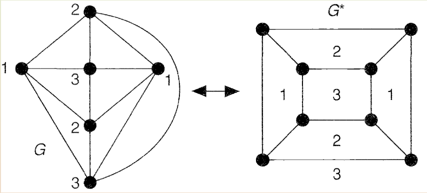
No two adjacent faces of G^* can have the same color because the vertices of G that they contain are adjacent in G and so are differently colored. Thus, G^* is k-colorable(f).



Geometric Duality and Colorability

$\Leftarrow$: Suppose now that we have a k-coloring(f) of G^*.

Then we can k-color the vertices of G so that each vertex inherits the color of the face containing it. No two adjacent vertices of G have the same color, by reasoning similar to the above. Thus, G is k-colorable(v).



- The theorem asserts that we can dualize any theorem on the coloring of the vertices of a planar graph to give a theorem on the coloring of the faces of a map, and conversely.

2-Colorability and 4-colorability Revisited

Theorem

A map G is 2-colorable(f) if and only if G is an Eulerian graph.

- Second Proof: The dual of an Eulerian planar graph is a bipartite planar graph, and conversely. Now note that a connected planar graph without loops is 2-colorable(v) iff it is bipartite.

Corollary

The four-color theorem for maps is equivalent to the four color theorem for planar graphs.

$\Rightarrow$: We may assume that G is a simple connected plane graph. Then its geometric dual G^* is a map. By assumption G^* is 4-colorable(f). By duality, G is 4-colorable(v).

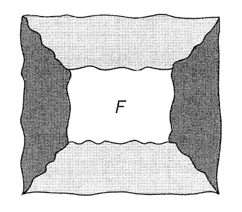
$\Leftarrow$: Let G be a map and G^* its dual. Then G^* is a simple planar graph. By assumption G^* is 4-colorable(v). So G is 4-colorable(f).

3-Colorability of Cubic Maps

Theorem

Let G be a cubic map. Then G is 3-colorable(f) if and only if each face is bounded by an even number of edges.

- $\Rightarrow$: Given any face F of G, the faces of G that surround F must alternate in color. Thus, there must be an even number of them. So each face is bounded by an even number of edges.



- $\Leftarrow$: We prove the dual result:

  If G is a simple connected plane graph in which each face is a triangle and each vertex has even degree (that is, G is Eulerian), then G is 3-colorable(v).

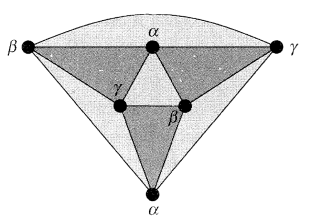
3-Colorability of Cubic Maps (Cont'd)

- Denote the three colors by α, β and γ.

  Since G is Eulerian, its faces can be colored with two colors, red and blue.

  A 3-coloring of the vertices of G is then obtained as follows:

  - Color the vertices of any red face so that the colors α, β and γ appear in clockwise order;
  - Color the vertices of any blue face so that these colors appear in anti-clockwise order.



  This vertex coloring can be extended to the whole graph.

Relaxing the Cubic Hypothesis

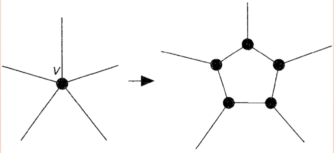
Theorem

In order to prove the four-color theorem, it is sufficient to prove that each cubic map is 4-colorable(f).

- By the first corollary, it is sufficient to prove that the 4-colorability(f) of every cubic map implies the 4-colorability(f) of any map.

  Let G be any map. If G has any vertices of degree 2, then we can remove them without affecting the coloring. It remains only to eliminate vertices of degree 4 or more.

  But if v is such a vertex, then we can stick a "patch" over v: Repeating this for all such vertices, we obtain a cubic map that is 4-colorable(f) by hypothesis.



  The required 4-coloring of the faces of G is then obtained by shrinking each patch to a single vertex and reinstating each vertex of degree 2.

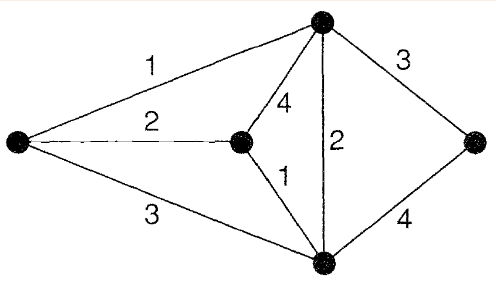
Subsection 4

Coloring Edges

Edge Colorability and Chromatic Index

- A graph G is k-**colorable(e)** (or k-**edge colorable**) if its edges can be colored with k colors so that no two adjacent edges have the same color.
- If G is k-colorable(e) but not $(k-1)$-colorable(e), we say that the **chromatic index** of G is k, and write $\chi'(G) = k$.

  Example: The figure shows a graph G for which $\chi'(G) = 4$.



Vizing's Theorem

- If Δ is the largest vertex degree of G, then $\chi'(G) \geq \Delta$.
- The following result, known as Vizing's theorem, gives very sharp bounds for the chromatic index of a simple graph G:

Theorem (Vizing, 1964)

If G is a simple graph with largest vertex-degree Δ, then $\Delta \leq \chi'(G) \leq \Delta + 1$.

- It is not known which graphs have chromatic index Δ and which have chromatic index $\Delta + 1$.
- For particular types of graphs, this classification can easily be found.

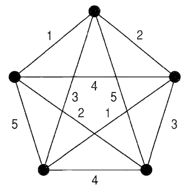
  Example: $\chi'(C_n) = 2$ or 3, depending on whether n is even or odd.
  $\chi'(W_n) = n - 1$ if $n \geq 4$.

Chromatic Index of Complete Graphs: n Odd

Theorem

$\chi'(K_n) = n$ if n is odd ($n \neq 1$) and $\chi'(K_n) = n - 1$ if n is even.

- The result is trivial if $n = 2$. Assume that $n \geq 3$.
  - If n is odd, then we can n-color the edges of K_n by placing the vertices of K_n in the form of a regular n-gon:
    - coloring the edges around the boundary with a different color for each edge;
    - then coloring each remaining edge with the color used for the boundary edge parallel to it.

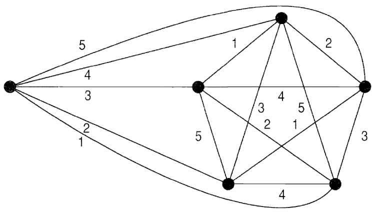


K_n is not $(n-1)$-colorable(e): K_n has n vertices and $\frac{n(n-1)}{2}$ edges. So the largest possible number of edges of the same color is $\frac{n-1}{2}$. It follows that K_n has at most $\frac{n-1}{2} \cdot \chi'(K_n)$ edges. This gives $\chi'(K_n) \geq n$.

Chromatic Index of Complete Graphs: n Even

- If n is even, then we first obtain K_n by joining the complete graph K_{n-1} to a single vertex.

  If we now color the edges of K_{n-1} as above, then there is one color missing at each vertex, and these missing colors are all different. We complete the edge coloring of K_n by coloring the remaining edges with these missing colors.

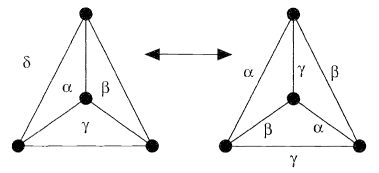


Four Color Theorem and Edge Colorings

Theorem

The four-color theorem is equivalent to the statement that $\chi'(G) = 3$ for each cubic map G.

$\Rightarrow$: Suppose that we have a 4-coloring of the faces of G, where the colors are denoted by $\alpha = (1,0)$, $\beta = (0,1)$, $\gamma = (1,1)$ and $\delta = (0,0)$. We can then construct a 3-coloring of the edges of G by coloring each edge e with the color obtained by adding together (modulo 2) the colors of the two faces adjoining e.



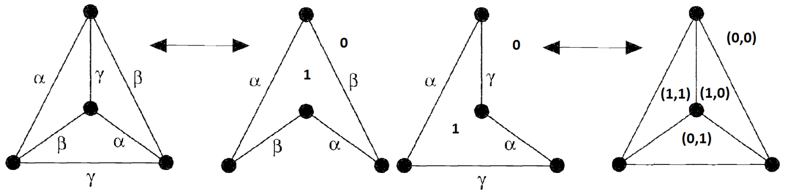
For example, if e adjoins two faces colored α and γ, then e is colored β, since $(1,0) + (1,1) = (0,1)$. Note that the color δ cannot occur in this edge coloring, since the two faces adjoining each edge must be distinct. Moreover, no two adjacent edges can share the same color. We thus have the required edge coloring.

Four Color Theorem and Edge Colorings (Cont'd)

$\Leftarrow$: Suppose now that we have a 3-coloring of the edges of G. Then there is an edge of each color at each vertex.

- The subgraph determined by those edges colored α or β is regular of degree 2. So we can color its faces with two colors, 0 and 1.
- Similarly, we can color the faces of the subgraph determined by those edges colored α or γ with the colors 0 and 1.

Thus, we can assign to each face of G two coordinates (x, y), where each of x and y is 0 or 1.



Since the coordinates assigned to two adjacent faces of G must differ in at least one place, we get a 4-coloring of the faces of G.

The Chromatic Index of a Bipartite Graph

Theorem (König 1916)

If G is a bipartite graph with largest vertex-degree Δ, then $\chi'(G) = \Delta$.

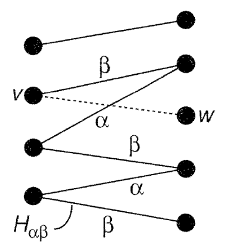
- By induction on the number of edges of G, we show that if all but one of the edges have been colored with at most Δ colors, then there is a Δ-coloring of the edges of G.

  Suppose that each edge of G has been colored, except for the edge vw. Then there is at least one color missing at the vertex v, and at least one color missing at the vertex w.
  - If some color is missing from both v and w, then we color the edge vw with this color.
  - If this is not the case, then let α be a color missing at v, and β be a color missing at w.

The Chromatic Index of a Bipartite Graph (Cont'd)

- We let α be a color missing at v, and β be a color missing at w.

  Let $H_{\alpha\beta}$ be the connected subgraph of G consisting of the vertex v and those edges and vertices of G that can be reached from v by a path consisting entirely of edges colored α or β.

  Since G is bipartite, the subgraph $H_{\alpha\beta}$ cannot contain the vertex w. So we can interchange the colors α and β in this subgraph without affecting w or the rest of the coloring.

  The edge vw can now be colored β, thereby completing the coloring of the edges of G.

Corollary

$\chi'(K_{r,s}) = \max(r, s)$.

Subsection 5

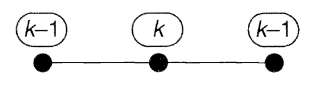
Chromatic Polynomials

The Chromatic Function of a Simple Graph

- Let G be a simple graph, and let $P_G(k)$ be the number of ways of coloring the vertices of G with k colors so that no two adjacent vertices have the same color.

  P_G is called the **chromatic function** of G.

  Example: If G is the tree shown on the right, then $P_G(k) = k(k-1)^2$, since the middle vertex can be colored in k ways, and then the end-vertices can each be colored in any of $k-1$ ways.



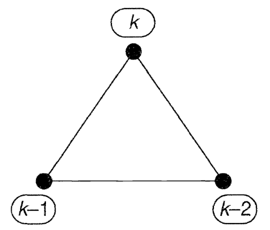
- If T is any tree with n vertices, then

$$P_T(k) = k(k-1)^{n-1}.$$

The Chromatic Function of a Complete Graph

- If G is the complete graph K_3, then

$$P_G(k) = k(k-1)(k-2).$$



- This can be extended to

$$P_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1).$$

- It is clear that:
  - if $k < \chi(G)$, then $P_G(k) = 0$;
  - if $k \geq \chi(G)$, then $P_G(k) > 0$.
- The four-color theorem is equivalent to the statement:

  If G is a simple planar graph, then $P_G(4) > 0$.

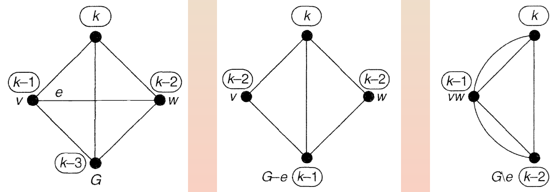
Calculating the Chromatic Function

- A systematic method for obtaining the chromatic function of a simple graph in terms of the chromatic functions of null graphs is available:

Theorem

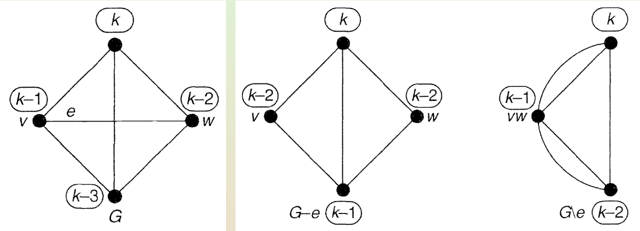
Let G be a simple graph, and let $G - e$ and G/e be the graphs obtained from G by deleting and by contracting an edge e, respectively. Then $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$.

Example: Let G be the following graph:



The graphs $G - e$ and G/e are shown on the right. By the theorem $k(k-1)(k-2)(k-3) = [k(k-1)(k-2)^2] - [k(k-1)(k-2)]$.

Calculating the Chromatic Function (Cont'd)



- Let $e = vw$. We compute $P_{G-e}(k)$ as follows:
  - The number of k-colorings of $G - e$ in which v and w have different colors is unchanged if the edge e is drawn joining v and w. So, it is equal to $P_G(k)$.
  - The number of k-colorings of $G - e$ in which v and w have the same color is unchanged if v and w are identified. It is therefore equal to $P_{G/e}(k)$.

  So, the total number $P_{G-e}(k)$ of k-colorings of $G - e$ is $P_G(k) + P_{G/e}(k)$. This gives $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$.

Chromatic Functions of Simple Graphs

Corollary

The chromatic function of a simple graph is a polynomial.

- We continue the above procedure by choosing edges in $G - e$ and G/e and deleting and contracting them. We then repeat the procedure for these four new graphs, and so on. The process terminates when no edges remain, i.e., when each remaining graph is a null graph.

  The chromatic function of a null graph is a polynomial ($= k^r$, where r is the number of vertices).

  So by repeated application of the theorem, the chromatic function of the graph G must be a sum of polynomials.

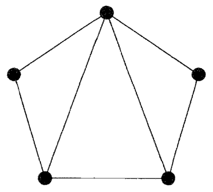
  Hence, it must itself be a polynomial.

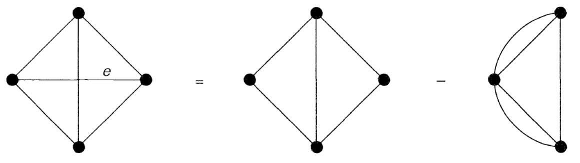
The Chromatic Polynomial of a Simple Graph

- We can call $P_G(k)$ the **chromatic polynomial** of G.
  - If G has n vertices, then $P_G(k)$ is of degree n, since no new vertices are introduced at any stage.
  - Since the construction yields only one null graph on n vertices, the coefficient of k^n is 1.
  - The coefficients alternate in sign.
  - The coefficient of k^{n-1} is $-m$, where m is the number of edges of G.
  - Since we cannot color a graph if no colors are available, the constant term of any chromatic polynomial is 0.

Example

- We find the chromatic polynomial of the graph G on the left.

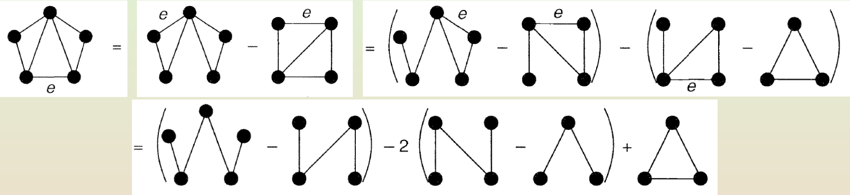




- We then verify that this polynomial has the form $k^5 - 7k^4 + ak^3 - bk^2 + ck$, where a, b and c are positive constants.
- It is convenient at each stage to draw the graph itself, rather than write its chromatic polynomial:
  For example, instead of writing
  $$P_G(k) = P_{G-e}(k) - P_{G/e}(k),$$
  we "draw" the equation of the right of the figure above.

Example (Cont'd)

- With this convention, and ignoring multiple edges, we have:



Thus,

$$\begin{aligned} P_G(k) &= k(k-1)^4 - 3k(k-1)^3 + 2k(k-1)^2 + k(k-1)(k-2) \\ &= k(k^4 - 4k^3 + 6k^2 - 4k + 1) - 3k(k^3 - 3k^2 + 3k - 1) \\ &\quad + 2k(k^2 - 2k + 1) + k(k^2 - 3k + 2) \\ &= k^5 - 4k^4 + 6k^3 - 4k^2 + k - 3k^4 + 9k^3 - 9k^2 + 3k \\ &\quad + 2k^3 - 4k^2 + 2k + k^3 - 3k^2 + 2k \\ &= k^5 - 7k^4 + 18k^3 - 20k^2 + 8k. \end{aligned}$$

This has the required form $k^5 - 7k^4 + ak^3 - bk^2 + ck$, where a, b and c are positive constants.