Introduction to Graph Theory

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LSSU Math 351

1. Digraphs
   - Definitions
   - Eulerian Digraphs and Tournaments
   - Markov Chains

Subsection 1

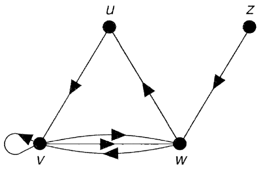
Definitions

Digraphs

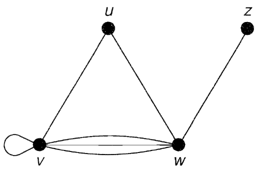
- A **directed graph**, or **digraph**, D consists of:
  - a non-empty finite set $V(D)$ of elements called **vertices**;
  - a finite family $A(D)$ of ordered pairs of elements of $V(D)$ called **arcs**.

  We call $V(D)$ the **vertex set** and $A(D)$ the **arc family** of D.
- An arc (v, w) is usually abbreviated to vw.

  Example: $V(D) = \{u, v, w, z\}$; $A(D)$ consists of the arcs uv, vv, vw (twice), wv, wu and zw, the ordering of the vertices in an arc being indicated by an arrow.

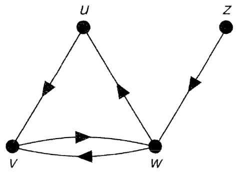


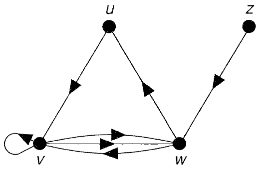
- If D is a digraph, the graph obtained from D by "removing the arrows" (that is, by replacing each arc of the form vw by a corresponding edge vw) is the **underlying graph** of D.

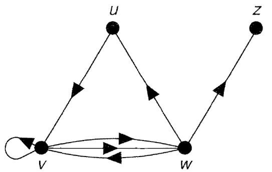


Simple Digraphs, Isomorphism, Adjacency and Incidence

- D is a **simple digraph** if the arcs of D are all distinct, and if there are no loops (arcs of the form vv).
- The underlying graph of a simple digraph need not be a simple graph!





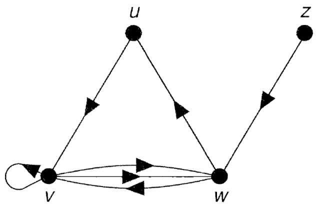


- Two digraphs are **isomorphic** if there is an isomorphism between their underlying graphs that preserves the ordering of the vertices in each arc. The digraphs on the right are not isomorphic.
- Two vertices v and w of a digraph D are **adjacent** if there is an arc in $A(D)$ of the form vw or wv. v and w are **incident** to such an arc.
- If D has vertex set $\{v_1, \ldots, v_n\}$, the **adjacency matrix** of D is the $n \times n$ matrix $A = (a_{ij})$, where a_{ij} is the number of arcs from v_i to v_j.

Walks, Trails and Paths

- A **walk** in a digraph D is a finite sequence of arcs of the form $v_0v_1, v_1v_2, \ldots, v_{m-1}v_m$.
- We sometimes write this sequence as $v_0 \to v_1 \to \cdots \to v_m$ and speak of a **walk from** v_0 **to** v_m.
- In an analogous way, we can define **directed trails**, **directed paths** and **directed cycles** or, simply, **trails**, **paths** and **cycles**, if there is no possibility of confusion.

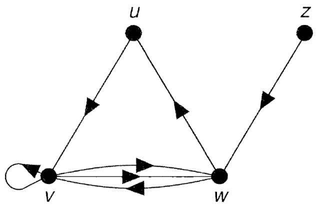
  Example: Although a trail cannot contain a given arc vw more than once, it can contain both vw and wv.
  In the figure $z \to w \to v \to w \to u$ is a trail.



Connectedness and Strong Connectedness

- A digraph D is **connected** if it cannot be expressed as the union of two digraphs, defined in the obvious way.
- This is equivalent to saying that the underlying graph of D is a connected graph.
- D is **strongly connected** if, for any two vertices v and w of D, there is a path from v to w.
- Every strongly connected digraph is connected, but not all connected digraphs are strongly connected.

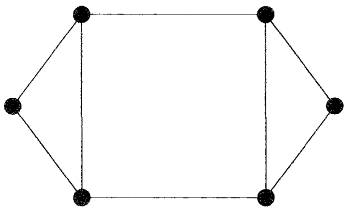
  Example: The connected digraph on the right is not strongly connected since there is no path from v to z.

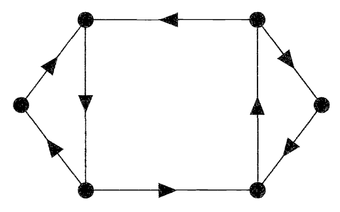


Orientable Graphs

- Define a graph G to be **orientable** if each edge of G can be directed so that the resulting digraph is strongly connected.

  Example: If G is the graph shown on the left,

  then G is orientable, since its edges can be directed to give the strongly connected digraph on the right.
- Any Eulerian graph is orientable, since we simply follow any Eulerian trail, directing the edges in the direction of the trail.

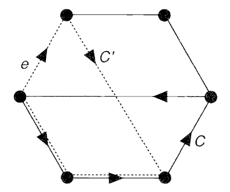
Characterization of Orientable Graphs

Theorem

Let G be a connected graph. Then G is orientable if and only if each edge of G is contained in at least one cycle.

- The necessity of the condition is clear. To prove the sufficiency, we choose any cycle C and direct its edges cyclically.
  - If each edge of G is contained in C, then the proof is complete.
  - If not, we choose any edge e that is not in C but which is adjacent to an edge of C.

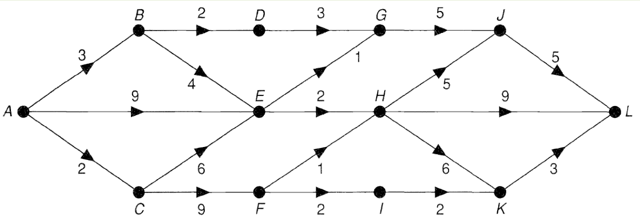
    By hypothesis, e is contained in some cycle C'. Direct the edges of C' cyclically, except for those edges that have already been directed, i.e., that also lie in C. The resulting digraph is strongly connected.



    We proceed in this way, at each stage directing at least one new edge, until all edges are directed. Since the digraph remains strongly connected at each stage, the result follows.

Dividing a Job into a Number of Activities

- Consider a "weighted digraph", or **activity network**, in which each arc represents the length of time taken for an activity.

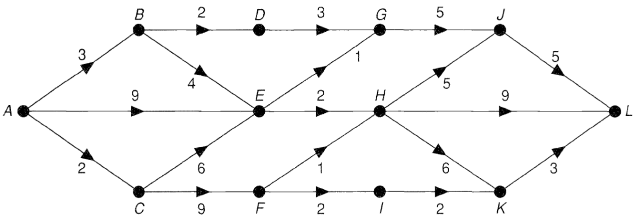


The vertex A represents the beginning of the job, and the vertex L represents its completion.

- The entire job cannot be completed until each path from A to L has been traversed.
- Thus, we would like to find the longest path from A to L.
- This is accomplished by using a technique known as **program evaluation and review technique** (**PERT**).

Applying PERT

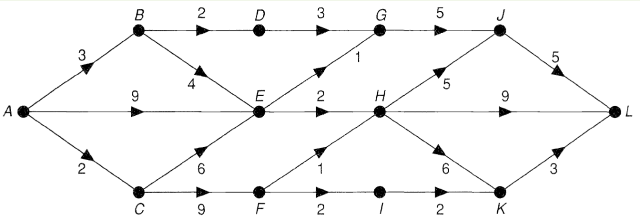
- **PERT** is similar to the technique used for the shortest path problem, except that, as we move across the digraph from left to right, we associate with each vertex V a number $\ell(V)$ indicating the length of the longest path from A to V.



vertex A: $\ell(A) = 0$;
vertex B: $\ell(B) = \ell(A) + 3 = 3$;
vertex C: $\ell(C) = \ell(A) + 2 = 2$;
vertex D: $\ell(D) = \ell(B) + 2 = 5$;
vertex E: $\ell(E) = \max\{\ell(A) + 9, \ell(B) + 4, \ell(C) + 6\} = 9$;

Applying PERT (Cont'd)

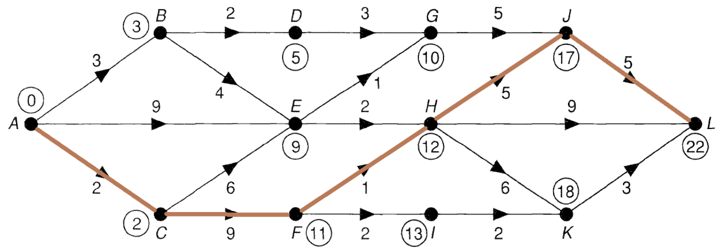
- We have $\ell(A) = 0, \ell(B) = 3, \ell(C) = 2, \ell(D) = 5, \ell(E) = 9$.



vertex F: $\ell(F) = \ell(C) + 9 = 11$;
vertex G: $\ell(G) = \max\{\ell(D) + 3, \ell(E) + 1\} = 10$;
vertex H: $\ell(H) = \max\{\ell(E) + 2, \ell(F) + 1\} = 12$;
vertex I: $\ell(I) = \ell(F) + 2 = 13$;
vertex J: $\ell(J) = \max\{\ell(G) + 5, \ell(H) + 5\} = 17$;
vertex K: $\ell(K) = \max\{\ell(H) + 6, \ell(I) + 2\} = 18$;
vertex L: $\ell(L) = \max\{\ell(H) + 9, \ell(J) + 5, \ell(K) + 3\} = 22$.

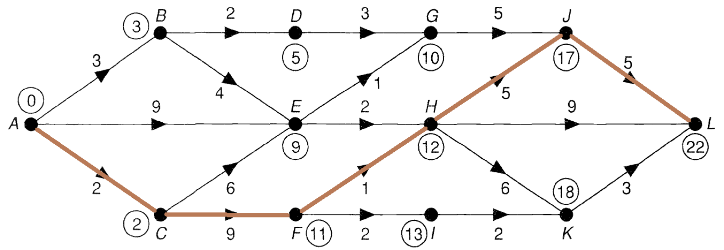
The Critical Path

- Thus, the longest path has length 22 and the job cannot be completed until time 22.



- This longest path is called a **critical path**, since any delay in an activity on this path creates a delay in the whole job.

Latest Completion Times



- We find the latest time by which any given operation must be completed if the work is not to be delayed by working back from L.
- For L to be completed by time 22:
  - K must be reached by time $22 - 3 = 19$;
  - I must be reached by time $19 - 2 = 17$;
  - H must be reached by time $\min\{17 - 5, 22 - 9, 19 - 6\} - 12$;

  $\vdots$

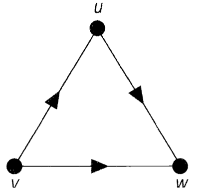
Subsection 2

Eulerian Digraphs and Tournaments

Eulerian Digraphs

- A connected digraph D is **Eulerian** if there exists a closed trail containing every arc of D. Such a trail is an **Eulerian trail**.

  Example: The digraph shown

  is not Eulerian, although its underlying graph is an Eulerian graph.
- A necessary condition for a digraph to be Eulerian is that the digraph is strongly connected.

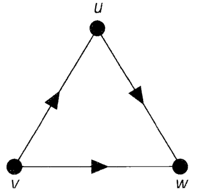
Out-degrees and In-degrees

- The **out-degree** of a vertex v of D is the number of arcs of the form vw, and is denoted by $\text{outdeg}(v)$.
- The **in-degree** of v is the number of arcs of D of the form wv, and is denoted by $\text{indeg}(v)$.
- **The Handshaking Dilemma**: The sum of the out-degrees of all the vertices of D is equal to the sum of their in-degrees (each arc of D contributes exactly 1 to each sum).

Characterization of Eulerian Digraphs

- A **source** of D is a vertex with in-degree 0.
- A **sink** of D is a vertex with out-degree 0.

  Example: In the digraph shown,



  v is a source and w is a sink.
- Any Eulerian digraph with at least one arc has no sources or sinks.

Theorem

A connected digraph is Eulerian if and only if for each vertex v of D, $\text{outdeg}(v) = \text{indeg}(v)$.

Hamiltonicity

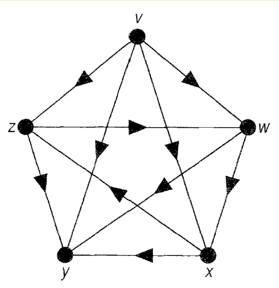
- A digraph D is **Hamiltonian** if there is a cycle that includes every vertex of D.
- A *non-Hamiltonian* digraph that contains a path passing through every vertex is **semi-Hamiltonian**.

Theorem (Ghouila-Houri)

Let D he a strongly connected digraph with n vertices. If $\text{outdeg}(v) \geq \frac{n}{2}$ and $\text{indeg}(v) \geq \frac{n}{2}$, for each vertex v, then D is Hamiltonian.

Tournaments

- A **tournament** is a digraph in which any two vertices are joined by exactly one arc.



- Such a digraph can be used to record the result of a tennis tournament, or any other game in which draws are not allowed.

Tournaments and Hamiltonicity

Theorem (Rédei and Camion)

(i) Every non-Hamiltonian tournament is semi-Hamiltonian;

(ii) Every strongly connected tournament is Hamiltonian.

(i) We prove the result by induction on the number of vertices.

The statement is clearly true if the tournament has fewer than four vertices.

Assume that every non-Hamiltonian tournament on n vertices is semi-Hamiltonian.

Let T be a non-Hamiltonian tournament on $n+1$ vertices. Let T' be the tournament on n vertices obtained by removing from T a vertex v and its incident arcs. By the induction hypothesis, T' has a semi-Hamiltonian path $v_1 \to v_2 \to \cdots \to v_n$.

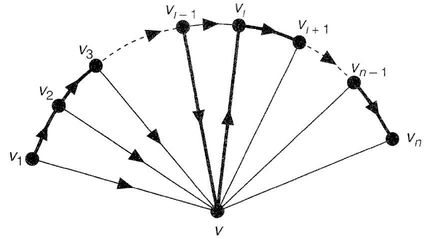
We next consider three cases.

Tournaments and Hamiltonicity (Cont'd)

(i) (1) Suppose vv_1 is an arc in T. Then the required path is $v \to v_1 \to v_2 \cdots \to v_n$.

(2) Suppose vv_1 is not an arc in T. Then $v_1 v$ is an arc.

(a) Suppose there exists an i such that vv_i is an arc in T. Then choosing i to be the first such, the required path is



$v_1 \to v_2 \to \cdots \to v_{i-1} \to v \to v_i \to \cdots \to v_n$.

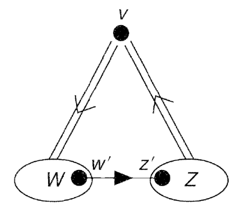
(b) Suppose there is no arc in T of the form vv_i. Then the required path is $v_1 \to v_2 \to \cdots \to v_n \to v$.

Tournaments and Hamiltonicity: Part (ii)

(ii) We prove that a strongly connected tournament T on n vertices contains cycles of length $3, 4, \ldots, n$.

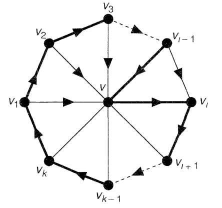
- We show, first, that T contains a cycle of length 3: Let v be any vertex of T, and set:
  - W be the set of all vertices w such that vw is an arc in T;
  - Z be the set of all vertices z such that zv is an arc.

  Since T is strongly connected, W and Z must both be non-empty, and there must be an arc in T of the form $w'z'$, where w' is in W and z' is in Z. The required cycle of length 3 is then $v \to w' \to z' \to v$.



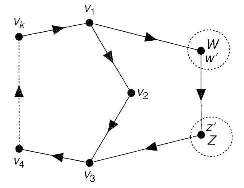
Tournaments and Hamiltonicity: Part (ii) Cont'd

- We show that, if there is a cycle of length k, where $k < n$, then there is one of length $k+1$. Let $v_1 \to \cdots \to v_k \to v_1$ be such a cycle.
  - Suppose there exists a vertex v not contained in this cycle, such that there exist arcs in T of the form vv_i and of the form v_jv. Then there must be a vertex v_i, such that both $v_{i-1}v$ and vv_i are arcs in T. The cycle is $v_1 \to v_2 \to \cdots \to v_{i-1} \to v \to v_i \to \cdots \to v_k \to v_1$.



  - If no such vertex exists, then the set of vertices not contained in the cycle may be divided into two disjoint sets W and Z, where:
    - W is the set of vertices w such that v_iw is an arc for each i;
    - Z is the set of vertices z such that zv_i is an arc for each i.

  Since T is strongly connected, W and Z must both be non-empty, and there must be an arc in T of the form $w'z'$, where w' is in W and z' is in Z. The required cycle is then $v_1 \to w' \to z' \to v_3 \to \cdots \to v_k \to v_1$.

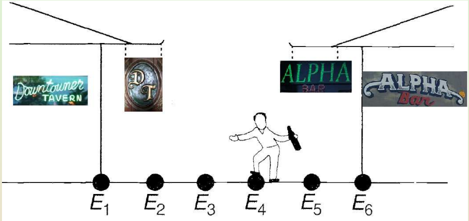


Subsection 3

Markov Chains

One Dimensional Random Walk

- Johnny is standing between two pubs "The DT" and "The Alpha".



- Every minute he either staggers four meters towards the first pub (with probability $\frac{1}{2}$) or towards the second pub (with probability $\frac{1}{3}$) or he stays where he is (with probability $\frac{1}{6}$)
- Such a procedure is called a **one-dimensional random walk**.
- We assume that the two pubs are "absorbing", in the sense that if he arrives at either of them he stays there.
- Given the distance between the two pubs and his initial position, we would like to find out:
  - Which pub he is more likely to reach first;
  - How long he is likely to take getting there.

Probability Vectors

- Suppose the two pubs are 20 meters apart, and that Johnny is initially 8 meters from "The Alpha".
- Denote the places at which he can stop by $E_1, \ldots, E_6$, where E_1 and E_6 are the two pubs.
- His initial position E_4 can be described by the vector $\mathbf{x} = (0, 0, 0, 1, 0, 0)$, in which the ith component is the probability that he is initially at E_i.
- The probabilities of his position after one minute are given by the vector $(0, 0, \frac{1}{2}, \frac{1}{6}, \frac{1}{3}, 0)$.
- After two minutes, they are given by by $(0, \frac{1}{4}, \frac{1}{6}, \frac{13}{36}, \frac{1}{9}, \frac{1}{9})$.
- A convenient way to calculate the probability of his being at a given place after k minutes is to introduce the transition matrix.

The Transition Matrix

- Let p_{ij} be the probability that he moves from E_i to E_j in one minute; e.g., $p_{23} = \frac{1}{3}$ and $p_{24} = 0$.
- These probabilities p_{ij} are called the **transition probabilities**;
- The 6×6 matrix $\mathbf{P} = (p_{ij})$ is the **transition matrix**.

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

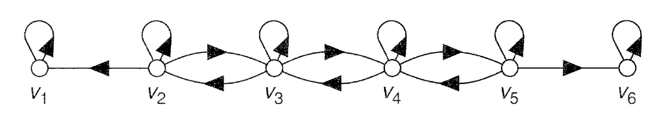
- Each entry of $\mathbf{P}$ is non-negative and the sum of the entries in each row is 1.
- If $\mathbf{x}$ is the initial row vector defined above, then the probabilities of his position after one minute are given by the row vector $\mathbf{xP}$, and after k minutes by the vector $\mathbf{xP}^k$.

  In other words, the ith component of $\mathbf{xP}^k$ represents the probability that he is at E_i after k minutes have elapsed.

Probability Vectors, Transition Matrices and Chains

- A **probability vector** is a row vector whose entries are all non-negative and have sum 1.
- A **transition matrix** is a square matrix, each of whose rows is a probability vector.
- A finite **Markov chain** (or simply, a **chain**) consistis of an $n \times n$ transition matrix $\mathbf{P}$ and a $1 \times n$ row vector $\mathbf{x}$.
- The positions E_i are the **states** of the chain.
- The Markov clain is represented by its **associated digraph**:
  - Its vertices correspond to the states;
  - Its arcs represent one-time-step transitions between states.

  Thus, if each state E_i is represented by a vertex v_i, then we obtain the required digraph by drawing an arc from v_i to v_j if and only if $p_{ij} \neq 0$.



Additional Example

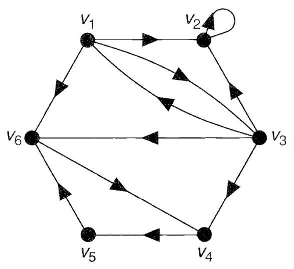
- If we are given a Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{12} & 0 & \frac{1}{12} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \text{ then}$$

  - its associated adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix};$$

  - its associated digraph is:



- We can get from a state E_i to a state E_j in a Markov chain if and only if there is a path from v_i to v_j in the associated digraph.
- The least possible time to do so is the length of the shortest path.

Irreducible Chains, Persistent and Transient States

- A Markov chain in which we can get from any state to any other is called an **irreducible chain**.
- A Markov chain is irreducible if and only if its associated digraph is strongly connected.
- We distinguish between those states to which we keep on returning however long we continue, and those that we visit a few times and then never return to:
  - If on starting at E_i the probability of returning to E_i at some later stage is 1, then E_i is a **persistent state**;
  - Otherwise E_i is **transient**.

  Example: In the pub problem, E_1 and E_6 are persistent and the other states are transient.
- A state E_i is persistent if and only if the existence of a path from v_i to v_j in the associated digraph implies the existence of a path from v_j to v_i.
- A state from which we can get to no other state is **absorbing**.

Periodicity

- A state E_i of a Markov chain is **periodic of period** t ($t \neq 1$) if it is possible to return to E_i only after a period of time that is a multiple of t.

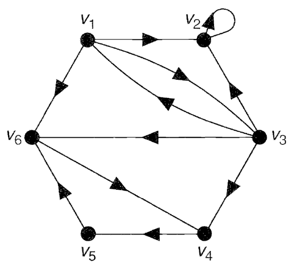
  If no such t exists, then E_j is **aperiodic**.
- Every state E_i for which $p_{ii} \neq 0$ is aperiodic; e.g., every absorbing state is aperiodic.

  Example: In the pub problem, every state is aperiodic.
- In digraph terms, a state E_i is periodic of period t if and only if in the associated digraph the length of each closed trail containing v_i is a multiple of t.

Example

- Consider again the Markov chain whose transition matrix and associated digraph are given below:

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{12} & 0 & \frac{1}{12} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



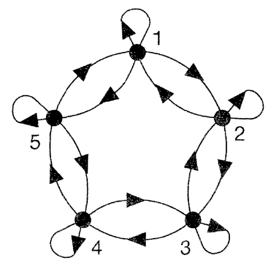
- States E_1 and E_3 are periodic of period 2.
- States E_4, E_5 and E_6 are periodic of period 3.
- So, the absorbing state E_2 is the only aperiodic state.

Ergodicity

- A state is **ergodic** if it is both persistent and aperiodic.
- If every state is ergodic then the chain is an **ergodic chain**.
  Example: A game is played with a die by five people around a circular table. If the player with the die throws:
  - An odd number, he passes the die to the player on his left;
  - A 2 or 4, he passes it to the player on his right;
  - A 6, he keeps the die and throws again.

(i) The corresponding transition matrix and its associated digraph are shown below:

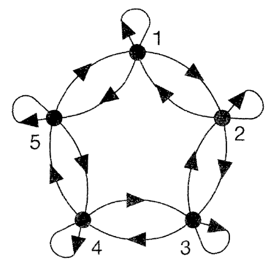
$$\mathbf{P} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$$



Example (Cont'd)

- We got transition matrix and associate digraph:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$$



(ii) Each state is:
  - Persistent, since whenever there is a path to some other state, there is also a path leading from the other state to itself.
  - Aperiodic, since $p_{ii} = \frac{1}{6} \neq 0$, for all i.

Therefore, the corresponding Markov chain is ergodic.