Introduction to Graph Theory

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LSSU Math 351

Subsection 1

Hall's Marriage Theorem

The Marriage Problem

- There is a finite set of girls, each of whom knows several boys.

  Under what conditions can all the girls marry the boys in such a way that each girl marries a boy she knows?

  Example: Suppose there are four girls $\{g_1, g_2, g_3, g_4\}$ and five boys $\{b_1, b_2, b_3, b_4, b_5\}$, and the friendships are as shown below:

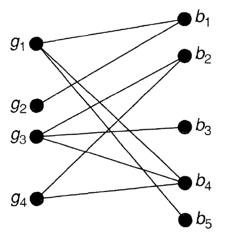
girl	boys known by girl		
g_1	b_1	b_4	b_5
g_2	b_1		
g_3	b_2	b_3	b_4
g_4	b_2	b_4	

  Then a possible solution is given by the following pairings: $g_1 \mapsto b_4$, $g_2 \mapsto b_1$, $g_3 \mapsto b_3$ and $g_4 \mapsto b_2$.

Marriage Problem and Bipartite Graphs

- This problem can be represented by a bipartite graph G:
  - The vertex set is divided into two disjoint sets V_1 and V_2, corresponding to the girls and boys;
  - Each edge joins a girl to a boy she knows.

girl	boys known by girl		
g_1	b_1	b_4	b_5
g_2	b_1		
g_3	b_2	b_3	b_4
g_4	b_2	b_4	



- A **complete matching** from V_1 to V_2 in a bipartite graph $G(V_1, V_2)$ is a one-one correspondence between the vertices in V_1 and a subset of the vertices in V_2, such that corresponding vertices are joined.
- **The marriage problem**: if $G = G(V_1, V_2)$ is a bipartite graph, when does there exist a complete matching from V_1 to V_2 in G?

The Marriage Condition and Hall's Theorem

- For the marriage problem to have a solution, it must satisfy the **marriage condition**:

  Every k girls must know collectively at least k boys, for all integers k satisfying $1 \leq k \leq m$, where m denotes the total number of girls.
- This is necessary since, if it were not true for a given set of k girls, then we could not marry the girls in that set, let alone the others.
- Hall's "marriage" theorem asserts that the marriage condition is also sufficient.

Theorem (Hall, 1935)

A necessary and sufficient condition for a solution of the marriage problem is that each set of k girls collectively knows at least k boys, for $1 \leq k \leq m$.

- The theorem has many other applications.

  For example, it gives a necessary and sufficient condition for the solution of a job assignment problem in which applicants must be assigned to jobs for which they are variously qualified.

The Proof of Halmos and Vaughan

- To prove sufficiency, we use induction on m.

  The theorem is true if $m = 1$.

  Assume that the theorem is true if the number of girls is less than m.

  Suppose there are m girls. There are two cases:

  (i) Suppose every k girls ($k < m$) collectively know at least $k + 1$ boys. Take any girl and marry her to any boy she knows. The original condition remains true for the other $m - 1$ girls. Thus, they can be married by induction.

  (ii) Suppose there is a set of k girls ($k < m$) who collectively know exactly k boys. These k girls can be married by induction to the k boys. This leaves $m - k$ girls still to be married. But any collection of h of these $m - k$ girls, for $h \leq m - k$, must know at least h of the remaining boys, since otherwise these h girls, together with the above collection of k girls, would collectively know fewer than $h + k$ boys, contrary to hypothesis. It follows that the original condition applies to the $m - k$ girls. Thus, they can also be married by induction.

Hall's Theorem for Bipartite Graphs

- We can state Hall's theorem in the language of complete matchings in a bipartite graph.

Corollary

Let $G = G(V_1, V_2)$ be a bipartite graph, and for each subset A of V_1, let $\varphi(A)$ be the set of vertices of V_2 that are adjacent to at least one vertex of A. Then a complete matching from V_1 to V_2 exists if and only if $|A| \leq |\varphi(A)|$, for each subset A of V_1.

- The proof is just a translation into graph terminology of the preceding proof.

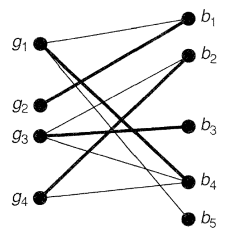
Subsection 2

Transversal Theory

Matching Problem Revisited

- Recall the sets of boys known by the four girls were

girl	boys known by girl		
g_1	b_1	b_4	b_5
g_2	b_1		
g_3	b_2	b_3	b_4
g_4	b_2	b_4	



- Using set notation, the sets of boys that each of the four girls knows are:

$$\{b_1, b_4, b_5\}, \{b_1\}, \{b_2, b_3, b_4\}, \{b_2, b_4\}.$$

- A solution of the marriage problem was obtained by finding four distinct b's, e.g. b_4, b_1, b_3, b_2, one from each of these sets of boys.

Transversals and Partial Transversals

- If E is a non-empty finite set, and if $\mathcal{F} = (S_1, \ldots, S_m)$ is a family of (not necessarily distinct) non-empty subsets of E, then a **transversal** of $\mathcal{F}$ is a set of m distinct elements of E, one chosen from each set S_i.

  Example: Suppose that $E = \{1, 2, 3, 4, 5, 6\}$, and let $S_1 = S_2 = \{1, 2\}$, $S_3 = S_4 = \{2, 3\}$, $S_5 = \{1, 4, 5, 6\}$.

  Then it is impossible to find five distinct elements of E, one from each subset S_i, in other words, the family $\mathcal{F} = (S_1, \ldots, S_5)$ has no transversal.

  However, the subfamily $\mathcal{F}' = (S_1, S_2, S_3, S_5)$ has a transversal, e.g., $\{1, 2, 3, 4\}$.
- We call a transversal of a subfamily of $\mathcal{F}$ a **partial transversal** of $\mathcal{F}$.

  Example: In the example, $\mathcal{F}$ has several partial transversals, such as $\{1, 2, 3, 6\}$, $\{2, 3, 6\}$, $\{1, 5\}$, and $\emptyset$.
- Note that any subset of a partial transversal is a partial transversal.

Hall's Theorem for Transversals

- To reveal conditions under which a given family of subsets of a set has a transversal, we may connect the transversal to the marriage problem:
  - Take E to be the set of boys;
  - Take S_i to be the set of boys known by girl g_i, for $1 \leq i \leq m$.

  A transversal in this case is simply a set of m boys, one corresponding to, and known by, each girl.
- Hall's Theorem gives a necessary and sufficient condition for a given family of sets to have a transversal.

Hall's Theorem for Transversals

Let E be a non-empty finite set, and let $\mathcal{F} = (S_1, \ldots, S_m)$ be a family of non-empty subsets of E. Then $\mathcal{F}$ has a transversal if and only if the union of any k of the subsets S_i contains at least k elements, $1 \leq k \leq m$.

Rado's Proof of Hall's Theorem

- To prove the sufficiency, we show that, if one of the subsets (S_1, say) contains more than one element, then we can remove an element from S_1 without altering the condition. By repeating this procedure, we eventually reduce the problem to the case in which each subset contains only one element, and the proof is then trivial.
  It remains only to show the validity of this "reduction procedure". Suppose that S_1 contains elements x and y, the removal of either of which invalidates the condition. Then there are subsets A and B of $\{2, 3, \ldots, m\}$ with the property that $|P| \leq |A|$ and $|Q| \leq |B|$, where $P = \bigcup_{j \in A} S_j \cup (S_1 - \{x\})$ and $Q = \bigcup_{j \in B} S_j \cup (S_1 - \{y\})$. Then $|P \cup Q| = |\bigcup_{j \in A \cup B} S_j \cup S_1|$ and $|P \cap Q| \geq |\bigcup_{j \in A \cap B} S_j|$. The required contradiction now follows:

$$\begin{array}{rcl} |A| + |B| & \geq & |P| + |Q| = |P \cup Q| + |P \cap Q| \\ & \geq & |\bigcup_{j \in A \cup B} S_j \cup S_1| + |\bigcup_{j \in A \cap B} S_j| \\ & \geq & (|A \cup B| + 1) + |A \cap B| = |A| + |B| + 1. \end{array}$$

Transversals of Specific Size

Corollary

If E and $\mathcal{F}$ are as before, then $\mathcal{F}$ has a partial transversal of size t if and only if the union of any k of the subsets S_i contains at least $k+t-m$ elements.

- **Sketch**: The result follows on applying Hall's Theorem to the family $\mathcal{F}' = (S_1 \cup D, \ldots, S_m \cup D)$, where D is any set disjoint from E and containing $m-t$ elements. Note that $\mathcal{F}$ has a partial transversal of size t if and only if $\mathcal{F}$ has a transversal.

Corollary

If E and $\mathcal{F}$ are as before, and if X is any subset of E, then X contains a partial transversal of $\mathcal{F}$ of size t if and only if, for each subset A of $\{1, \ldots, m\}$, $|(\bigcup_{j \in A} S_j) \cap X| \geq |A| + t - m$.

- **Sketch**: The result follows on applying the previous corollary to the family $\mathcal{F}_X = (S_1 \cap X, \ldots, S_m \cap X)$.

Subsection 3

Applications of Hall's Theorem

Latin Rectangles and Latin Squares

- An $m \times n$ **latin rectangle** is an $m \times n$ matrix $\mathbf{M} = (m_{ij})$ whose entries are integers satisfying:
  - (i) $1 \leq m_{ij} \leq n$;
  - (ii) no two entries in any row or in any column are equal.
- Note that (i) and (ii) imply that $m \leq n$.
- If $m = n$, then the latin rectangle is a **latin square**.

Example: A 3×5 latin rectangle $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$ and a 5×5 latin square: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 4 & 3 & 5 & 2 & 1 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$.

Extending a Latin Rectangle to a Latin Square

Theorem

Let **M** be an $m \times n$ latin rectangle with $m < n$. Then **M** can he extended to a latin square by the addition of $n - m$ new rows.

- We prove that **M** can be extended to an $(m+1) \times n$ latin rectangle. By repeating the procedure, we eventually obtain a latin square. Let $E = \{1, 2, \ldots, n\}$ and $\mathcal{F} = (S_1, \ldots, S_n)$, where S_i is the set consisting of those elements of E that do not occur in the ith column of **M**. We prove that $\mathcal{F}$ has a transversal. Then, the elements in this transversal form the additional row. By Hall's Theorem, it is sufficient to show that the union of any k of the S_i contains at least k distinct elements. Note such a union contains $(n - m)k$ elements altogether, including repetitions. Thus, if there were fewer than k distinct elements, then at least one of them would have to appear more than $n - m$ times. Since each element occurs exactly $n - m$ times, we have the required contradiction.

Incidence Matrices of Transversals and $(0,1)$-Matrices

- Consider a set $E = \{e_1, \ldots, e_n\}$.
- Let $\mathcal{F} = (S_1, \ldots, S_m)$ be a family of non-empty subsets of E.
- The **incidence matrix** of the family $\mathcal{F}$ is the $m \times n$ matrix $\mathbf{A} = (a_{ij})$ in which $a_{ij} = \begin{cases} 1, & \text{if } e_j \in S_i \\ 0, & \text{otherwise} \end{cases}$.
- Such a matrix, in which each entry is 0 or 1, is called a $(0,1)$-**matrix**.
- The **term rank** of $\mathbf{A}$ is the largest number of 1s of $\mathbf{A}$, no two of which lie in the same row or column.
- Clearly, $\mathcal{F}$ has a transversal if and only if the term rank of its incidence matrix $\mathbf{A}$ is m.
- Moreover, the term rank of $\mathbf{A}$ is precisely the number of elements in a partial transversal of largest possible size.

The König-Egerváry Theorem

Theorem (König-Egerváry, 1931)

The term rank of a $(0,1)$-matrix $\mathbf{A}$ is equal to the minimum number μ of rows and columns that together contain all the 1s of $\mathbf{A}$.

Example: Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$.

This is the incidence matrix of the following family of subsets of the set $E = \{1,2,3,4,5,6\}$:

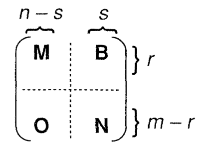
$$\mathcal{F} = (S_1, S_2, S_3, S_4, S_5),$$
$$S_1 = S_2 = \{1,2\}, \quad S_3 = S_4 = \{2,3\}, \quad S_5 = \{1,4,5,6\}.$$

Clearly the term rank and μ are both 4.

Proof of the König-Egerváry Theorem

- It is clear that the term rank cannot exceed μ.

  To prove equality, we can suppose that all the 1s of **A** are contained in r rows and s columns, where $r + s = \mu$, and that the order of the rows and columns is such that **A** contains, in the bottom left-hand corner, an $(m - r) \times (n - s)$ submatrix consisting entirely of 0s.



  If $i \leq r$, let S_i be the set of integers $j \leq n - s$, such that $a_{ij} = 1$. By the minimality of μ, the union of any k of the S_i contains at least k integers. Hence the family $\mathcal{F} = (S_1, \ldots, S_r)$ has a transversal.

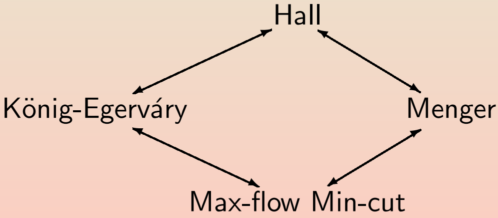
  It follows that the submatrix **M** of **A** contains a set of r 1s, no two of which lie in the same row or column.

  Similarly, the matrix **N** contains a set of s 1s with the same property.

  Hence **A** contains a set of $r + s$ 1s, no two of which lie in the same row or column. This shows that μ cannot exceed the term rank.

Some Equivalences

- We proved the König-Egerváry Theorem using Hall's Theorem.
- It is even easier to prove Hall's Theorem using the König-Egerváry Theorem.
- It follows that these two theorems are, in some sense, equivalent.
- Later we prove Menger's Theorem and the Max-flow Min-cut Theorem and, in fact, we have



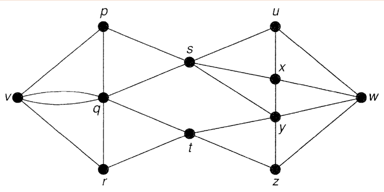
Subsection 4

Menger's Theorem

Edge-Disjoint and Vertex-Disjoint Paths

- A set of paths connecting two given vertices v and w in a graph G are called **edge-disjoint paths** if no two of them have an edge in common.
- We are interested in the maximum number of edge-disjoint paths from v to w in a given graph.
- A set of paths from v to w in a graph G are called **vertex-disjoint paths** if no two of them have a vertex in common, except, of course, v ad w.
- We may also ask for the maximum number of vertex-disjoint paths from v to w.

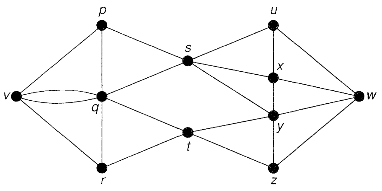
  Example: In the pictured graph there are four edge-disjoint paths and two vertex-disjoint ones.



Disconnecting and Separating Sets

- Assume that G is a connected graph and that v and w are distinct vertices of G.
- A vw-**disconnecting set** of G is a set E of edges of G such that each path from v to w includes an edge of E.
- Note that a vw-disconnecting set is a disconnecting set of G.
- A vw-**separating set** of G is a set S of vertices, other than v or w, such that each path from v to w passes through a vertex of S.

  Example: The sets $E_1 = \{ps, qs, ty, tz\}$ and $E_2 = \{uw, xw, yw, zw\}$ are vw-disconnecting sets, and $V_1 = \{s, t\}$ and $V_2 = \{p, q, y, z\}$ are vw-separating sets.



- If E is a vw-disconnecting set with k edges, then the number of edge-disjoint paths cannot exceed k (otherwise some edge in E would be included in more than one path).

Menger's Theorem

Menger's Theorem (Ford and Fulkerson, 1955)

The maximum number of edge-disjoint paths connecting two distinct vertices v and w of a connected graph is equal to the minimum number of edges in a vw-disconnecting set.

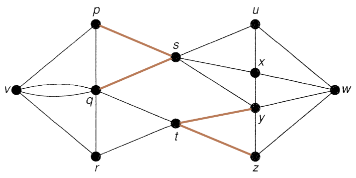
- We saw that the maximum number of edge-disjoint paths connecting v and w cannot exceed the minimum number of edges in a vw-disconnecting set.

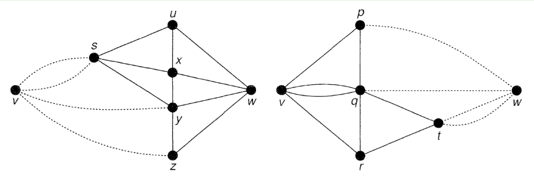
  We use induction on the number of edges of the graph G to prove that these numbers are equal.

  Suppose the theorem is true for all graphs with fewer than m edges. Let G be a graph with m edges

  (i) Suppose there exists a vw-disconnecting set E of minimum size k, such that not all of its edges are incident to v, and not all are incident to w. The removal from G of the edges in E leaves two disjoint subgraphs V and W containing v and w, respectively.

Menger's Theorem: Case (i)





- We now define two new graphs G_1 and G_2 as follows:
  - G_1 is obtained from G by contracting every edge of V (that is, by shrinking V down to v);
  - G_2 is obtained by similarly contracting every edge of W.

  The graphs G_1 and G_2 have fewer edges than G.

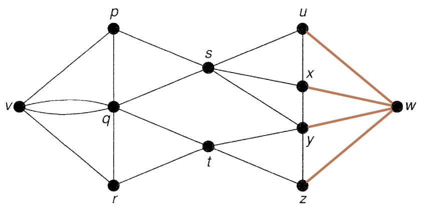
  E is a vw-disconnecting set of minimum size for both G_1 and G_2.

  Thus, the induction hypothesis gives us k edge-disjoint paths in G_1 from v to w, and similarly for G_2.

  The required k edge-disjoint paths in G are obtained by combining these paths in the obvious way.

Menger's Theorem: Case (ii)

(ii) Now suppose that each vw-disconnecting set of minimum size k consists only of edges that are all incident to v or all incident to w.



We can assume that each edge of G is contained in a vw-disconnecting set of size k, since otherwise its removal would not affect the value of k and the induction hypothesis would suffice.

We could use the induction hypothesis to obtain k edge-disjoint paths: If P is any path from v to w, then P must consist of either one or two edges. It can thus contain at most one edge of any vw-disconnecting set of size k. Remove from G the edges of P. By the induction hypothesis, we obtain a graph with at least $k-1$ edge-disjoint paths. These paths, together with P, give the required k paths in G.

Number of Vertex-Disjoint Paths

- The original theorem of Menger actually gives the number of vertex-disjoint paths from v to w.

Theorem (Menger, 1927)

The maximum number of vertex-disjoint paths connecting two distinct non-adjacent vertices v and w of a graph is equal to the minimum number of vertices in a vw-separating set.

- We immediately deduce the following necessary and sufficient conditions for a graph to be k-connected and k-edge-connected:

Corollary

A graph G is k-edge-connected if and only if any two distinct vertices of G are connected by at least k edge-disjoint paths.

Corollary

A graph G with at least $k+1$ vertices is k-connected if and only if any two distinct vertices of G are connected by at least k vertex-disjoint paths.

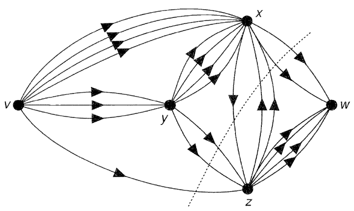
The Number of Arc-Disjoint Paths

- For the number of arc-disjoint paths from a vertex v to a vertex w in a digraph, we can take v to be a source and w to be a sink.
- Note that, in a digraph, a vw-**disconnecting set** is a set A of arcs such that each path from v to w includes an arc in A.

Theorem

The maximum number of arc-disjoint paths from a vertex v to a vertex w in a digraph is equal to the minimum number of arcs in a vw-disconnecting set.

Example: In the pictured digraph there are six arc-disjoint paths from v to w. A corresponding vw-disconnecting set consists of the arcs vz, xz, yz (twice) and xw (twice).



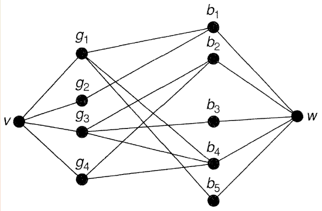
Menger's Theorem Implies Hall's Theorem

Theorem

Menger's Theorem implies Hall's Theorem.

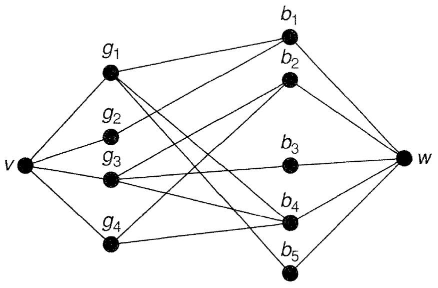
- Let $G = G(V_1, V_2)$ be a bipartite graph. We must prove that, if $|A| \leq |\varphi(A)|$, for each subset A of V_1, then there is a complete matching from V_1 to V_2.

  To do this, we apply the vertex form of Menger's theorem to the graph obtained by adjoining to G a vertex v adjacent to every vertex in V_1 and a vertex w adjacent to every vertex in V_2.



  Note that a complete matching from V_1 to V_2 exists if and only if the number of vertex-disjoint paths from v to w is equal to the number of vertices in V_1 ($= k$, say). Therefore, it is enough to show that every vw-separating set has at least k vertices.

Menger's Theorem Implies Hall's Theorem



Claim: Every vw-separating set has at least k vertices.

Let S be a vw-separating set consisting of a subset A of V_1 and a subset B of V_2. Since $A \cup B$ is a vw-separating set, no edge can join a vertex of $V_1 - A$ to a vertex of $V_2 - B$. Hence $\varphi(V_1 - A) \subseteq B$. It follows that $|V_1 - A| \leq |\varphi(V_1 - A)| \leq |B|$. So

$$|S| = |A| + |B| \geq |A| + |V_1 - A| = |V_1| = k.$$

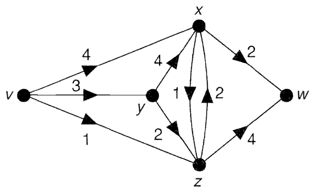
Subsection 5

Network Flows

Networks, Capacities and Degrees

- A **network** N is be a weighted digraph, i.e., a digraph to each arc a of which is assigned a non-negative real number $c(a)$ called its **capacity**.
- The **out-degree** $\text{outdeg}(x)$ of a vertex x is the sum of the capacities of the arcs of the form xz, and the **in-degree** $\text{indeg}(x)$ is the sum of the capacities of the arcs of the form zx.

  Example: In the network of the figure



$$\begin{aligned} \text{outdeg}(v) &= 8; \\ \text{indeg}(x) &= 10. \end{aligned}$$

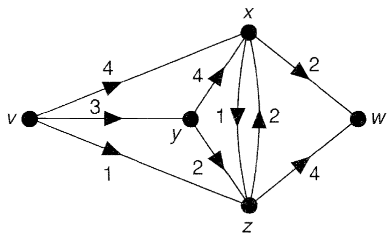
- **The Handshaking Dilemma**: The sum of the out-degrees of the vertices of a network is equal to the sum of the in-degrees.

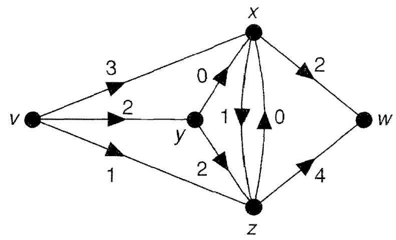
Flows in Networks

- A **flow** in a network is a function φ that assigns to each arc a a non-negative real number $\varphi(a)$, called the **flow in** a, so that:
  - (i) for each arc a, $\varphi(a) \leq c(a)$;
  - (ii) the out-degree and in-degree of each vertex, other than v or w, are equal.
- In terms of flows, these conditions say the following:
  - (i) The flow in any arc cannot exceed its capacity;
  - (ii) The "total flow" into each vertex, other than v or w, equals the "total flow" out of it.

An Example of a Flow

- Consider the network shown on the left.





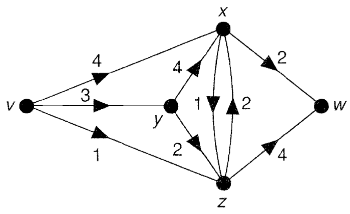
The right figure shows a flow in the network.

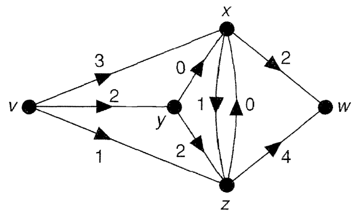
- An arc a for which $\varphi(a) = c(a)$ is called **saturated**.
- Otherwise it is called **unsaturated**.

Value of a Flow and Maximum Flow

- The Handshaking Dilemma implies that the sum of the flows in the arcs out of v is equal to the sum of the flows in the arcs into w; this sum is called the **value of the flow**.
- We are mainly interested in flows whose value is as large as possible - the **maximum flows**.

  Example: The flow of the figure on the right





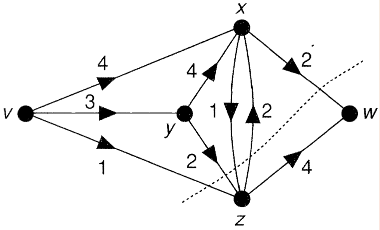
  is a maximum flow for the network on the left, and its value is 6.
- A network can have several different maximum flows, but their values must be equal.

Cuts, Capacity of a Cut and Minimum Cuts

- A **cut** in a network N is a set A of arcs such that each path from v to w includes an arc in A.
- Thus, a cut in a network is a vw-disconnecting set in the corresponding digraph D.
- The **capacity of a cut** is the sum of the capacities of the arcs in the cut.
- A **minimum cut** is a cut whose capacity is as small as possible.

Example: In the figure a minimum cut consists of the arcs vz, xz, yz and xw, but not the arc zx;
The capacity of this cut is

$$1 + 1 + 2 + 2 = 6.$$



The Max-flow Min-cut Theorem

- The value of any flow cannot exceed the capacity of any cut.
- So the value of any maximum flow cannot exceed the capacity of any minimum cut.
- These last two numbers are always equal, a famous result known as the **max-flow min-cut theorem** (Ford and Fulkerson, 1955):

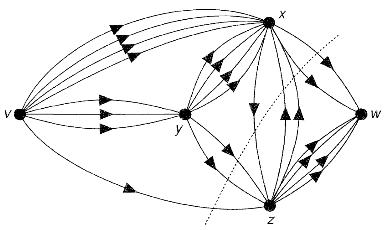
Theorem (Max-flow Min-cut Theorem)

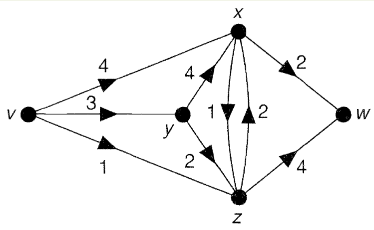
In any network, the value of any maximum flow is equal to the capacity of any minimum cut.

- When applying this theorem, it is often simplest to find a flow and a cut such that the value of the flow equals the capacity of the cut. It follows from the theorem that the flow must be a maximum flow and that the cut must be a minimum cut.
- If all the capacities are integers, then the value of a maximum flow is also an integer.

Equivalence of Max-Flow Min-Cut with Menger's Theorem

- Suppose first that the capacity of each arc is an integer. Then the network can be regarded as a digraph D whose capacities represent the number of arcs connecting the various vertices:





  - The value of a maximum flow is the total number of arc-disjoint paths from v to w in D.
  - The capacity of a minimum cut is the minimum number of arcs in a vw-disconnecting set of D.

Thus, in this case, the result follows from the directed version of Menger's Theorem.

Extension to Rationals and Reals

- The extension of this result to networks in which the capacities are rational numbers is effected by:
  - Multiplying these capacities by a suitable integer d to make them integers;
  - Exploiting the previous case;
  - Dividing by d.
- Finally, if some capacities are irrational, then we approximate them as closely as we please by rationals and use the above result.

  By choosing these rationals carefully, we can ensure that the value of any maximum flow and the capacity of any minimum cut are altered by an amount that is as small as we wish.

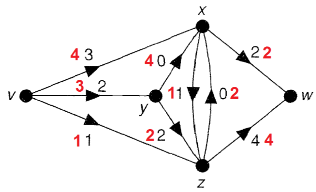
Direct Proof of the Max-Flow Min Cut Theorem

- Since the value of any maximum flow cannot exceed the capacity of any minimum cut, it is sufficient to prove the existence of a cut whose capacity is equal to the value of a given maximum flow.

  Let φ be a maximum flow. We define two sets V and W of vertices of the network. If G is the underlying graph, then:
  - A vertex z is contained in V if and only if there exists in G a path $v = v_0 \to v_1 \to v_2 \to \cdots \to v_{m-1} \to v_m = z$, such that each edge $v_i v_{i+1}$ corresponds either to an unsaturated arc $v_i v_{i+1}$ or to an arc $v_{i+1} v_i$ that carries a non-zero flow;
  - The set W consists of all those vertices that do not lie in V.

Example: In the figure we have:

$$\begin{array}{rcl} V & = & \{v, x, y\}; \\ W & = & \{z, w\}. \end{array}$$



Direct Proof of the Max-Flow Min Cut Theorem (Cont'd)

- Clearly, v is contained in V.

  Claim: W contains the vertex w.

  If this is not so, then w lies in V. Hence there exists in G a path $v \to v_1 \to v_2 \to \cdots \to v_{m-1} \to w$ of the above type.

  We now choose a positive number ε that does not exceed:
  - the amount needed to saturate any unsaturated arc $v_i v_{i+1}$;
  - the flow in any arc $v_{i+1} v_i$ that carries a non-zero flow.

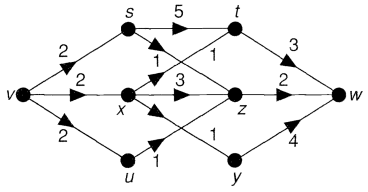
  If we increase by ε the flow in all arcs of the first type and decrease by ε the flow in all arcs of the second type, then we increase the value of φ by ε. This contradicts the maximality of φ. So w lies in W.

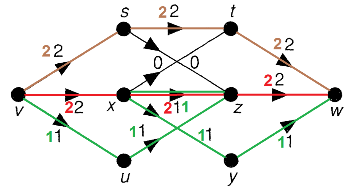
  Finally, let E be the set of all arcs of the form xz, where x is in V and z is in W. Clearly E is a cut. Moreover, each arc xz of E is saturated and each arc zx carries a zero flow, since otherwise z would also be an element of V. Thus, the capacity of E must equal the value of φ. So E is the required minimum cut.

Flow-Augmenting Paths

- A **flow-augmenting path** from v to w, is a path consisting entirely of unsaturated arcs xz and arcs zx that carry a non-zero flow.

  Example: Consider the network on the left.





  Starting with the zero flow, we construct flow-augmenting paths:
  - $v \to s \to t \to w$ along which value of flow can be increased by 2;
  - $v \to x \to z \to w$ along which value of flow can be increased by 2;
  - $v \to u \to z \to x \to y \to w$ along which value can be increased by 1.

  The resulting flow of value 5 is shown on the right.

  The network has a cut of capacity 5.

  So the above flow is a maximum flow and the cut is a minimum cut.