Introduction to Graph Theory

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Subsection 1

Introducing Matroids

Spanning Trees, Bases and Transversals

- Recall that a *spanning tree* in a connected graph G is a connected subgraph of G containing no cycles and including every vertex of G.
  - Note that a spanning tree cannot contain another spanning tree as a proper subgraph.
  - Moreover, if B_1 and B_2 are spanning trees of G and e is an edge of B_1, then there is an edge f in B_2, such that $(B_1 - \{e\}) \cup \{f\}$ (the graph obtained from B_1 on replacing e by f) is also a spanning tree of G.
- Analogous results hold in the theory of vector spaces and in transversal theory:
  - If V is a vector space and if B_1 and B_2 are bases of V and e is an element of B_1, then we can find an element f of B_2 such that $(B_1 - \{e\}) \cup \{f\}$ is also a basis of V.
  - If E is a set of points and $\mathcal{F}$ a collection of subsets of E, T_1 and T_2 are transversals of $\mathcal{F}$ and x an element of T_1, there exists an element y of T_2, such that $(T_1 - \{x\}) \cup \{y\}$ is also a transversal of $\mathcal{F}$.

Matroids

- A **matroid** M consists of:
  - A non-empty finite set E;
  - A non-empty collection $\mathcal{B}$ of subsets of E, called **bases**,

  satisfying the following properties:

  $\mathcal{B}$(i) No base properly contains another base;

  $\mathcal{B}$(ii) If B_1 and B_2 are bases and if e is any element of B_1, then there is an element f of B_2, such that $(B_1 - \{e\}) \cup \{f\}$ is also a base.
- By repeatedly using property $\mathcal{B}$(ii), we can easily show that any two bases of a matroid M have the same number of elements.

  This number is called the **rank** of M.

Cycle Matroids and Vector Matroids

- A matroid can be associated with any graph G by letting:
  - E be the set of edges of G;
  - $\mathcal{B}$ consist of the sets of edges of the spanning forests of G.

  This matroid is called the **cycle matroid** of G and is denoted by $M(G)$.
- Let E be a finite set of vectors in a vector space V.

  We can define a matroid on E by taking as bases all linearly independent subsets of E that span the same subspace as E.

  A matroid obtained in this way is called a **vector matroid**.

Independent Subsets in a Matroid

- A subset of E is **independent** if it is contained in some base of the matroid M.

  Example:
  - For a vector matroid, a subset of E is independent whenever its elements are linearly independent as vectors in the vector space.
  - For the cycle matroid $M(G)$ of a graph G, the independent sets are those sets of edges of G that contain no cycle.

    In other words, the independent sets are the edge sets of the forests contained in G.
- The bases of M are the maximal independent sets (that is, those independent sets contained in no larger independent set).
- So a matroid is uniquely defined by specifying its independent sets.

Matroids in terms of Independent Sets

- A **matroid** M consists of:
  - A non-empty finite set E;
  - A non-empty collection $\mathcal{I}$ of subsets of E, called **independent sets**,

  satisfying the following properties:

  $\mathcal{I}$(i) Any subset of an independent set is independent;

  $\mathcal{I}$(ii) If I and J are independent sets with $|J| > |I|$, then there is an element e, contained in J but not in I, such that $I \cup \{e\}$ is independent.
- With this definition, a **base** is a maximal independent set.
- Property $\mathcal{I}$(ii) can then be used repeatedly to show that any independent set can be extended to a base.
- If $M = (E, \mathcal{I})$ is a matroid defined in terms of its independent sets, then:
  - A subset of E is **dependent** if it is not independent;
  - A minimal dependent set is called a **cycle**.

  Example: If $M(G)$ is the cycle matroid of a graph G, then the cycles of $M(G)$ are precisely the cycles of G.

Matroids in terms of Cycles

- Let M be a matroid over a set E.
- A subset of E is independent if and only if it contains no cycles.
- It follows that a matroid can also be defined in terms of its cycles.
- A **matroid** M consists of:
  - A nonempty finite set E;
  - A collection $\mathcal{C}$ of non-empty subsets of E, called **cycles**,

  satisfying the following properties:

  $\mathcal{C}$(i) No cycle properly contains another cycle;

  $\mathcal{C}$(ii) If C_1 and C_2 are two distinct cycles each containing an element e, then there exists a cycle in $C_1 \cup C_2$ that does not contain e.

Matroids in terms of Rank

- If $M = (E, \mathcal{I})$ is a matroid defined in terms of its independent sets, and if A is a subset of E, then the **rank** of A, denoted by $r(A)$, is the size of the largest independent set contained in A.
- The previously defined rank of M is equal to $r(E)$.
- Since a subset A of E is independent if and only if $r(A) = |A|$, we can define a matroid in terms of its rank function:

Theorem

A matroid consists of:

- A non-empty finite set E;
- An integer valued function r defined on the set of subsets of E,

satisfying:

r(i) $0 \leq r(A) \leq |A|$, for each subset A of E;

r(ii) if $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$;

r(iii) for any $A, B \subseteq E$, $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

The Rank of a Matroid (Cont'd)

- Assume M is a matroid defined in terms of its independent sets.
  - r(i) Since the rank of A is the number of the largest independent set in A, we have $0 \leq r(A) \leq |A|$.
  - r(ii) Suppose $A \subseteq B$. The rank of A is the number of the largest independent set in A. But that set is also an independent set in B. So the number of its elements is at most equal to the rank of B. We get $r(A) \leq r(B)$.
  - r(iii) Let X be a base (a maximal independent subset) of $A \cap B$.
    Since X is an independent subset of A, X can be extended to a base Y of A.
    Since Y is an independent subset of A, it can be extended to a base Z of $A \cup B$.
    But, then, $X \cup (Z - Y)$ is an independent subset of B.
    Thus, we have

$$r(B) \geq r(X \cup (Z - Y)) = |X| + |Z| - |Y| = r(A \cap B) + r(A \cup B) - r(A).$$

The Rank of a Matroid (Cont'd)

- Let $M = (E, r)$ be a matroid defined in terms of a rank function r. Define a subset A of E to be independent if and only if $r(A) = |A|$.
  - To prove property $\mathcal{I}$(i), suppose A is independent and $B \subseteq A$. By definition, $r(A) = |A|$. Assume $r(B) < |B|$. Then we have

$$\begin{gathered} r(B \cup (A - B)) + r(B \cap (A - B)) \leq r(B) + r(A - B) \\ r(A) + 0 < |B| + |A - B| \\ r(A) < |A|. \end{gathered}$$

    This contradicts the hypothesis. So $r(B) = |B|$.
  - For $\mathcal{I}$(ii), let I and J be independent sets with $|J| > |I| = k$. Suppose that $r(I \cup \{e\}) = k$, for each element e that lies in J but not in I. If e and f are two such elements, then $r(I \cup \{e\} \cup \{f\}) \leq r(I \cup \{e\}) + r(I \cup \{f\}) - r(I) = k$. It follows that $r(I \cup \{e\} \cup \{f\}) = k$. We now continue this procedure, adding one new element of J at a time. Since at each stage the rank is k, we conclude that $r(I \cup J) = k$. Hence, (by r(ii)) $r(J) \leq k$, which is a contradiction. It follows that there exists an element f that lies in J but not in I, such that $r(I \cup \{f\}) = k + 1$.

Loops and Parallel Elements in a Matroid

- Let M be a matroid.
  - A **loop** of M is an element e of E satisfying $r(\{e\}) = 0$.
  - A pair of **parallel elements** of M is a pair $\{e, f\}$ of elements of E that satisfy $r(\{e, f\}) = 1$ and are not loops.

  Example: Suppose $M(G)$ is the cycle matroid of a graph G.
  - The loops of $M(G)$ correspond to loops in G.
  - The parallel elements of $M(G)$ correspond to multiple edges of G.

Subsection 2

Examples of Matroids

Trivial, Discrete and Uniform Matroids

- **Trivial matroids**: Given any non-empty finite set E, we can define on it a matroid whose only independent set is the empty set $\emptyset$.

  This matroid is the **trivial matroid** on E. It has rank 0.
- **Discrete matroids**: At the other extreme is the **discrete matroid** on E, in which every subset of E is independent.

  The discrete matroid on E has only one base, E itself.

  The rank of any subset A is the number of elements in A.
- **Uniform matroids**: The previous examples are special cases of the k-**uniform matroid** on E, whose bases are those subsets of E with exactly k elements.
  - The trivial matroid on E is 0-uniform;
  - The discrete matroid is $|E|$-uniform.

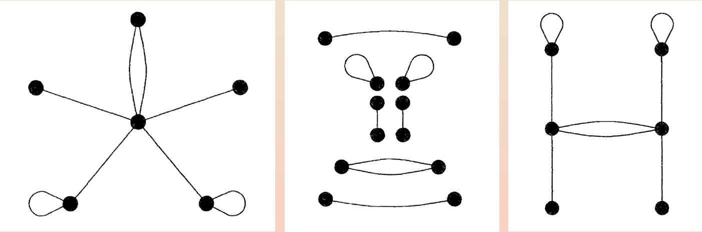
  Note that the independent sets are those subsets of E with not more than k elements.

  The rank of any subset A is either $|A|$ or k, whichever is smaller.

Isomorphic Matroids

- Two matroids M_1 and M_2 are **isomorphic** if there is a one-one correspondence between their underlying sets E_1 and E_2 that preserves independence.
- Thus, a set of elements of E_1 is independent in M_1 if and only if the corresponding set of elements of E_2 is independent in M_2.

  Example: The cycle matroids of the three graphs shown below are all isomorphic.



Matroid isomorphism preserves cycles, cutsets and number of edges, but not necessarily connectedness, the number of vertices, or degrees.

Graphic Matroids

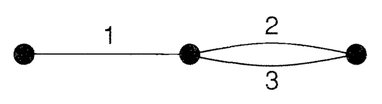
- Let G be a graph.
- We can define a matroid $M(G)$ (in terms of cycles):
  - E is the set of edges of G;
  - The cycles of $M(G)$ are the cycles of G.

  $M(G)$ is the **cycle matroid** of G.

  Its rank function is the cutset rank ξ.
- A matroid M is a **graphic matroid** if it is the cycle matroid of some graph, i.e., if there exists a graph G such that M is isomorphic to $M(G)$.

Example of a Graphic Matroid

- The matroid M on the set $\{1, 2, 3\}$ whose bases are $\{1, 2\}$, and $\{1, 3\}$ is a graphic matroid.
- It is isomorphic to the cycle matroid of the graph G shown below.



- The only cycle of $M(G)$ is $\{2, 3\}$;
- Thus, the independent sets (sets not containing a cycle) of $M(G)$ are

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\};$$

- So the bases (maximal independent sets) are

$$\{1, 2\} \quad \text{and} \quad \{1, 3\}.$$

Example of a Non-Graphic Matroid

- Consider the set $E = \{a, b, c, d\}$.
- The 2-uniform matroid M on E has bases

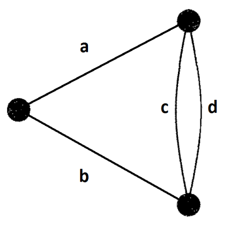
$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}.$$

Claim: M is not a graphic matroid, i.e., there is no graph G with M as its cycle matroid.

The cycles (minimal dependent sets) of M are

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}.$$

Suppose M were graphic. The first two cycles force the structure shown on the right.



But this graph has only three cycles:

$$\{a, b, c\}, \{a, b, d\}, \{c, d\}.$$

This, however, is a contradiction.

Cographic and Planar Matroids

- Let G be a graph.
- We can define a matroid $M^*(G)$ (in terms of cycles):
  - E is the set of edges of G;
  - The cycles of $M^*(G)$ are the cutsets of G.

  $M^*(G)$ is the **cutset matroid** of G.
- A set of edges of G is independent in $M^*(G)$ if and only if it contains no cutset of G.
- A matroid M is a **cographic matroid** if it is the cutset matroid of some graph, i.e., if there exists a graph G such that M is isomorphic to $M^*(G)$.
- A matroid that is both graphic and cographic is a **planar matroid**.

Bipartite and Eulerian Matroids

- A **bipartite matroid** is a matroid in which each cycle has an even number of elements.
- A matroid on a set E is an **Eulerian matroid** if E can be written as a union of disjoint cycles.
- Eulerian matroids and bipartite matroids are dual concepts, in a sense to be made precise later.

Representable Matroids

- Given a matroid M on a set E, we say that M is **representable over a field** F if there exist:
  - A vector space V over F;
  - A map φ from E to V, such that

    a subset A of E is independent in M if and only if φ is one-one on A and $\varphi(A)$ is linearly independent in V.
- This amounts to saying that, if we ignore loops and parallel elements, then M is isomorphic to a vector matroid defined in some vector space over F.
- We say that M is a **representable matroid** if there exists some field F such that M is representable over F.
- It turns out that some matroids are representable over every field (the **regular matroids**), some are representable over no field, and some are representable only over a restricted class of fields.

Binary Matroids

- A matroid is a **binary matroid** if it is representable over the field of integers modulo 2.

  Example: If G is any graph, then its cycle matroid $M(G)$ is a binary matroid.

  To see this, associate with each edge of G the corresponding column of the incidence matrix of G (rows labeled by vertices and columns by edges), regarded as a vector with components 0 or 1.

  If a set of edges of G forms a cycle, then the sum (modulo 2) of the corresponding vectors is 0.

Transversal Matroids

- Let E be a non-empty finite set.
- Let $\mathcal{F} = (S_1, \ldots, S_m)$ be a family of non-empty subsets of E.
- The **transversal matroid associated with** $\mathcal{F}$, denoted $M(\mathcal{F})$ or $M(S_1, \ldots, S_m)$, is the matroid on E whose independent sets are the partial transversals of $\mathcal{F}$.
- Any matroid obtained in this way (for suitable choices of E and $\mathcal{F}$) is a **transversal matroid**.

  Example: Consider the graphic matroid M, with $E = \{1, 2, 3\}$ and bases $\{1, 2\}$ and $\{1, 3\}$.

  This is a transversal matroid on the set $\{1, 2, 3\}$.

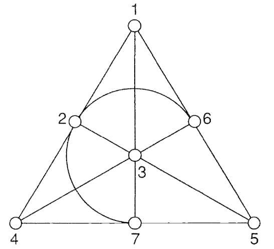
  In fact, its independent sets are the partial transversals of the family $\mathcal{F} = (S_1, S_2)$, where $S_1 = \{1\}$ and $S_2 = \{2, 3\}$.
- Note that the rank of a subset A of E is the size of the largest partial transversal contained in A.

The Fano Matroid

- The **Fano matroid** F is a matroid defined on $E = \{1, 2, 3, 4, 5, 6, 7\}$.

  Its bases are all those subsets of E with three elements, except $\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}$ and $\{7, 1, 3\}$.

  This matroid can be represented geometrically, as in the figure, where the bases are precisely those sets of three elements that do not lie on a line.



- F is binary and Eulerian.
- F is not graphic, cographic, transversal or regular.

Restrictions and Contractions

- If M is a matroid defined on a set E, and if A is a subset of E, then the **restriction** of M to A, denoted by $M \times A$, is the matroid whose cycles are precisely those cycles of M that are contained in A.
- Similarly, the **contraction** of M to A, denoted by $M \cdot A$, is the matroid whose cycles are the minimal members of the collection $\{C_i \cap A\}$, where C_i is a cycle of M.
- One can verify that these are indeed matroids, and that they correspond to the deletion and contraction of edges in a graph.
- A matroid obtained from M by restrictions and/or contractions is called a **minor** of M.
- It turns out that if M is graphic, cographic, binary and/or regular, then so is any minor of M, i.e., restrictions and contractions preserve these properties of matroids.

Subsection 3

Matroids and Graphs

The Dual Matroid

- Recall that we can form a matroid $M^*(G)$ on the set of edges of a graph G by taking as cycles of $M^*(G)$ the cutsets of G.
- Let M be a matroid on a set E, defined in terms of its rank function.
- Define the **dual matroid** M^* of M to be the matroid on E whose rank function r^* is given by the expression

$$r^*(A) = |A| + r(E - A) - r(E), \text{ for } A \subseteq E.$$

The Dual Matroid is a Matroid

Theorem

$M^* = (E, r^*)$ is a matroid on E.

- We must verify the properties r(i)-r(iii) for the function r^*.

r^*(i) By r(i), $r(E-A) \leq r(E)$. So $r(E-A) - r(E) \leq 0$. Thus, $r^*(A) = |A| + r(E-A) - r(E) \leq |A|$.
On the other hand, by r(iii), $r(E) + r(\emptyset) \leq r(A) + r(E-A)$. So we obtain $0 \leq r(A) + r(E-A) - r(E) \leq |A| + r(E-A) - r(E) = r^*(A)$.

r^*(ii) Let $A \subseteq B \subseteq E$. By r(iii), $r(E-A) + r(\emptyset) \leq r(B-A) + r(E-B)$. Thus, $r(E-A) - r(E-B) \leq r(B-A) \leq |B-A| = |B| - |A|$. So $r^*(A) = |A| + r(E-A) - r(E) \leq |B| + r(E-B) - r(E) \leq r^*(B)$.

r^*(iii) For any $A, B \subseteq E$,

$$\begin{aligned}
& r^*(A \cup B) + r^*(A \cap B) \\
&= |A \cup B| + |A \cap B| + r(E - (A \cup B)) + r(E - (A \cap B)) - 2r(E) \\
&= |A| + |B| + r((E-A) \cap (E-B)) + r((E-A) \cup (E-B)) - 2r(E) \\
&\leq |A| + |B| + r(E-A) + r(E-B) - 2r(E) \\
&= r^*(A) + r^*(B).
\end{aligned}$$

The Bases of the Dual Matroid

Theorem

The bases of M^* are precisely the complements of the bases of M.

- We show that, if B^* is a base of M^*, then $E - B^*$ is a base of M: Since B^* is independent in M^*, $|B^*| = r^*(B^*)$ $(= |B^*| + r(E - B^*) - r(E))$. Hence $r(E - B^*) = r(E)$. It remains only to prove that $E - B^*$ is independent in M. Since B^* is a base of M^*, $r^*(B^*) = r^*(E)$. Thus, $|B^*| + r(E - B^*) - r(E) = |E| - r(E)$. So $r(E - B^*) = |E| - |B^*| = |E - B^*|$. This shows that $E - B^*$ is independent in M.

  The converse result is obtained by reversing the argument.
- From this definition we obtain that:
  - Every matroid has a dual and this dual is unique.
  - The double-dual M^{**} is equal to M.

Duality of Cutset and Cycle Matroids of a Graph

- The cutset matroid $M^*(G)$ of a graph G is the dual of the cycle matroid $M(G)$.

Theorem

If G is a connected graph, then $M^*(G) = (M(G))^*$.

- Since the cycles of $M^*(G)$ are the cutsets of G, we must check that C^* is a cycle of $(M(G))^*$ if and only if C^* is a cutset of G.
  - Suppose first that C^* is a cutset of G. Assume C^* is independent in $(M(G))^*$. Then C^* can be extended to a base B^* of $(M(G))^*$. So $C^* \cap (E - B^*)$ is empty. But, since $E - B^*$ is a spanning forest. this is a contradiction. Thus, C^* is a dependent set in $(M(G))^*$. So it contains a cycle of $(M(G))^*$.
  - Suppose, conversely, that D^* is a cycle of $(M(G))^*$. Then D^* is not contained in any base of $(M(G))^*$. It follows that D^* intersects every base of $M(G)$, i.e., every spanning forest of G. Thus, D^* contains a cutset.

The "Co-Notation"

- We say that elements of a matroid M form a **cocycle** of M if they form a cycle of M^*.
- In view of the preceding theorem, the cocycles of the cycle matroid of a graph G are precisely the cutsets of G.
- We similarly define a **cobase** of M to be a base of M^*.
- Corresponding definitions apply for **corank**, **co-independent set**, etc.
- We also say that a matroid M is **cographic** if and only if its dual M^* is graphic.
- In view of the preceding theorem, this definition of "cographic" agrees with the one given in the previous section.

Illustrating the "Co-Notation"

- The "co-notation" allows to restrict to a single matroid M, without having to bring in M^*:

Theorem

Every cocycle of a matroid intersects every base.

- Let C^* be a cocycle of a matroid M. Suppose that there exists a base B of M with the property that $C^* \cap B$ is empty. Then C^* is contained in $E - B$. So C^* is a cycle of M^* which is contained in a base of M^*. This is a contradiction.

Corollary

Every cycle of a matroid intersects every cobase.

- Apply the result of the theorem to the matroid M^*.

Subsection 4

Matroids and Transversals

Transversal Matroids

- Let E be a non-empty finite set and $\mathcal{F} = (S_1, \ldots, S_m)$ a family of nonempty subsets of E.
- The **transversal matroid** $M(\mathcal{F}) = M(S_1, \ldots, S_m)$ is the matroid on E with independent sets the partial transversals of $\mathcal{F}$
- In this matroid, the rank of a subset A of E is the size of the largest partial transversal of $\mathcal{F}$ contained in A.

Transversals Containing a Given Subset

Proposition

A family $\mathcal{F}$ of subsets of E has a transversal containing a given subset A if and only if:

(i) $\mathcal{F}$ has a transversal;

(ii) A is a partial transversal of $\mathcal{F}$.

- These conditions are necessary.

  For sufficiency, observe that, since A is a partial transversal of $\mathcal{F}$, A is an independent set in the transversal matroid M determined by $\mathcal{F}$. So, it can be extended to a base of M. Since $\mathcal{F}$ has a transversal, every base of M must be a transversal of $\mathcal{F}$. The result follows immediately.

Rado's Theorem

- If $\mathcal{F}$ is a family of subsets of E, then Hall's theorem gives a necessary and sufficient condition for $\mathcal{F}$ to have a transversal.
- If we also have a matroid structure on E, there is a corresponding condition for the existence of an **independent transversal**, i.e., a transversal of $\mathcal{F}$ that is also an independent set in the matroid:

Theorem (Rado, 1942)

Let M be a matroid on a set E and let $\mathcal{F} = (S_1, \ldots, S_m)$ be a family of non-empty subsets of E. Then $\mathcal{F}$ has an independent transversal if and only if the union of any k of the subsets S_i contains an independent set of size at least k, for $1 \leq k \leq m$.

- The necessity is clear. For sufficiency, we show that if one of the subsets (S_1, say) contains more than one element, then we can remove an element from S_1 without altering the condition.
  By repeating this procedure, we eventually reduce the problem to the trivial case in which each subset contains only one element.

Rado's Theorem (Cont'd)

- We show the validity of the "reduction procedure":
  Suppose that S_1 contains elements x and y, the removal of either of which invalidates the condition. Then, there are subsets A and B of $\{2, 3, \ldots, m\}$ such that, if

$$P = \bigcup_{j \in A} S_j \cup (S_1 - \{x\}) \quad \text{and} \quad Q = \bigcup_{j \in B} S_j \cup (S_1 - \{y\}),$$

then $r(P) \leq |A|$ and $r(Q) \leq |B|$. It follows that

$$\begin{array}{rcl} r(P \cup Q) & = & r(\bigcup_{j \in A \cup B} S_j \cup S_1); \\ r(P \cap Q) & \geq & r(\bigcup_{j \in A \cap B} S_j). \end{array}$$

The required contradiction now follows, since

$$\begin{array}{rcl} |A| + |B| & \geq & r(P) + r(Q) \geq r(P \cup Q) + r(P \cap Q) \\ & \geq & |\bigcup_{j \in A \cup B} S_j \cup S_1| + |\bigcup_{j \in A \cap B} S_j| \\ & \geq & (|A \cup B| + 1) + |A \cap B| = |A| + |B| + 1. \end{array}$$

The Union of Matroids

- If $M_1, M_2, \ldots, M_k$ are matroids on the same set E, then we can define a new matroid $M_1 \cup \cdots \cup M_k$, called their **union**, by taking as independent sets all possible unions of an independent set in M_1, an independent set in M_2, ..., and an independent set in M_k.

Theorem

If $M_1, \ldots, M_k$ are matroids on a set E with rank functions $r_1, \ldots, r_k$, then the rank function r of $M_1 \cup \cdots \cup M_k$ is given by

$$r(X) = \min_{A \subseteq X} \{r_1(A) + \cdots + r_k(A) + |X - A|\}.$$

More on the Union of Matroids

Corollary

Let M be a matroid. Then M contains k disjoint bases if and only if for each subset A of E, $kr(A) + |E - A| \geq kr(E)$.

- M contains k disjoint bases if and only if the union of k copies of the matroid M has rank at least $kr(E)$. By the theorem, we must have

$$\min_{A \subseteq E} \{r(A) + \cdots + r(A) + |E - A|\} \geq kr(E).$$

So $kr(A) + |E - A| \geq kr(E)$, for all $A \subseteq E$.

Corollary

Let M be a matroid. Then E can be expressed as the union of k independent sets if and only if, for each subset A of E, $kr(A) \geq |A|$.

- In this case, the union of k copies of the matroid M has rank $|E|$. It follows from the theorem that $kr(A) + |E - A| \geq |E|$.

Application to Graphs using the Cycle Matrtoids

- By applying the corollaries to the cycle matroid $M(G)$ of a graph G, we easily obtain the following necessary and sufficient conditions for:
  - G to contain k edge-disjoint spanning forests;
  - G to split into k forests.
- Recall that the rank of a set of edges H in $M(G)$ is the cutset rank (number of edges in a spanning forest) of H, i.e., $r(H) = \xi(H)$.

Theorem

A graph G contains k edge-disjoint spanning forests if and only if, for each subgraph H of G, $k(\xi(G) - \xi(H)) \leq m(G) - m(H)$, where $m(H)$ and $m(G)$ denote the number of edges of H and G, respectively.

Theorem

A graph G splits into k forests if and only if, for each subgraph H of G, $k\xi(H) \geq m(H)$.