Intermediate Algebra

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LSSU Math 102

Subsection 1

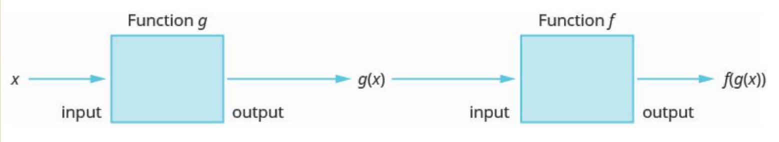
Finding Composite and Inverse Functions

We Shall Learn and Practice

- Find and evaluate composite functions.
- Determine whether a function is one-to-one.
- Find the inverse of a function.

Find and Evaluate Composite Functions

- In the operation of composition, the output of one function is the input of a second function.



- For functions f and g, the **composition** is written $f \circ g$ and is defined by
$$(f \circ g)(x) = f(g(x)).$$
- We read $f(g(x))$ as "f of g of x".

Example

- For functions $f(x) = 3x - 2$ and $g(x) = 5x + 1$, find (a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$ (c) $(f \cdot g)(x)$.

(a)

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(5x+1) = 3(5x+1) - 2 \\ &= 15x + 3 - 2 = 15x + 1;\end{aligned}$$

(b)

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(3x-2) = 5(3x-2) + 1 \\ &= 15x - 10 + 1 = 15x - 9;\end{aligned}$$

(c)

$$\begin{aligned}(f \cdot g)(x) &= f(x)g(x) = (3x-2)(5x+1) \\ &= 15x^2 + 3x - 10x - 2 = 15x^2 - 7x - 2.\end{aligned}$$

Example

- For functions $f(x) = x^2 - 9$, and $g(x) = 2x + 5$, find (a) $(f \circ g)(-2)$ (b) $(g \circ f)(-3)$ (c) $(f \circ f)(4)$.

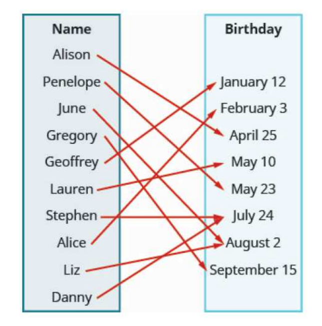
(a)
$$(f \circ g)(-2) = f(g(-2)) = f(1) = -8;$$

(b)
$$(g \circ f)(-3) = g(f(-3)) = g(0) = 5;$$

(c)
$$(f \circ f)(4) = f(f(4)) = f(7) = 40.$$

Determine Whether a Function is One-To-One

- A **function** is a relation that assigns to each element in its domain exactly one element in the range.



- A function is **one-to-one** if each value in the range has exactly one element in the domain mapping to it.
- Our example of the birthday relation is not a one-to-one function. Two people can share the same birthday.

Example

- For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.
  (a) $\{(-3,-6),(-2,-4),(-1,-2),(0,0),(1,2),(2,4),(3,6)\}$;
  (b) $\{(-4,8),(-2,4),(-1,2),(0,0),(1,2),(2,4),(4,8)\}$.

(a) This set represented a function, since there are no two ordered pairs with the same x-coordinate.

The function is one-to-one, since there are no two ordered pairs with the same y-coordinate.

(b) This set also represents a function, since there are no two ordered pairs with the same x-coordinate.

This function, however, is not one-to-one, since, e.g., 8, in the range, has two elements, -4 and 4, mapping to it.

The Horizontal Line Test

- **Vertical Line Test**:

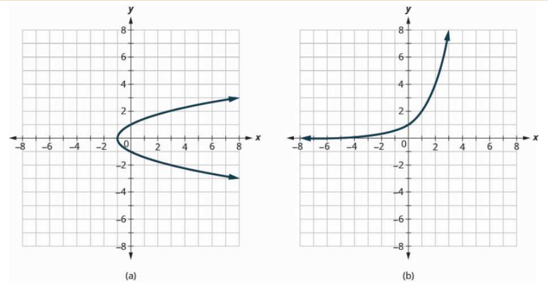
  If every vertical line intersects a graph in at most one point, then the graph is the graph of a function.

- **Horizontal Line Test**:

  If every horizontal line intersects the graph of a function in at most one point, then the function is a one-to-one function.

Example

- Determine (a) whether each graph is the graph of a function and, if so, (b) whether it is one-to-one.

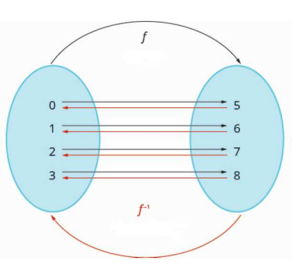


(a) This graph does not pass the vertical line test.
Hence, it is not the graph of a function.

(b) This graph passes the vertical line test. Thus it is the graph of a function. In addition, it passes the horizontal line test. Thus, it is the graph of a one-to-one function.

The Inverse of a Function

- Consider the one-to one function, f, represented by the ordered pairs $\{(0,5),(1,6),(2,7),(3,8)\}$.
- Going from f to the function consisting of the ordered pairs $\{(5,0),(6,1),(7,2),(8,3)\}$ is called "taking the inverse of f" and the new function is named f^{-1}.



- Notice that the ordered pairs of f and f^{-1} have their x-values and y-values reversed.
- The domain of f is the range of f^{-1} and the domain of f^{-1} is the range of f.
- If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x,y), then its **inverse function** $f^{-1}(x)$ is the set of ordered pairs (y,x).

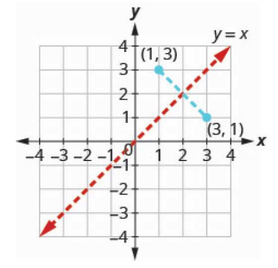
Example

- Find the inverse of $\{(0,4),(1,7),(2,10),(3,13)\}$. Determine the domain and range of the inverse function.

  The inverse of $\{(0,4),(1,7),(2,10),(3,13)\}$ is the function
  $$\{(4,0),(7,1),(10,2),(13,3)\}.$$
  The domain of the inverse function is
  $$\{4,7,10,13\}.$$
  The range of the inverse function is
  $$\{0,1,2,3\}.$$

Graphs of Inverse Functions

- We noted that if $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).
- So if a point (a, b) is on the graph of a function $f(x)$, then the ordered pair (b, a) is on the graph of $f^{-1}(x)$.

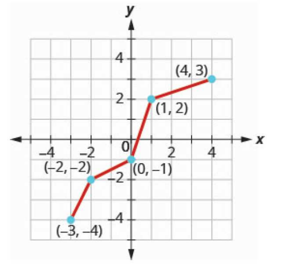


- The distance between any two pairs (a, b) and (b, a) is cut in half by the line $y = x$.

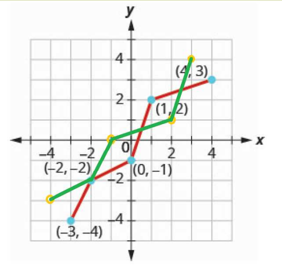
  So we say the points are **mirror images of each other through the line** $y = x$.
- Since every point on the graph of a function $f(x)$ is a mirror image of a point on the graph of $f^{-1}(x)$, we say the graphs are **mirror images of each other through the line** $y = x$.

Example

- Graph, on the same coordinate system, the inverse of the one-to-one function.

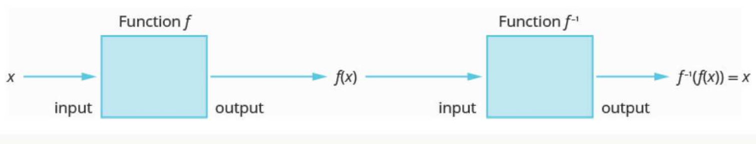


- Since the graph of f contains $(-3,-4),(-2,-2),(0,-1),(1,2),(4,3)$, that of f^{-1} must contain the points $(-4,-3),(-2,-2),(-1,0),(2,1),(3,4)$. Since it is the mirror image of that of f with respect to $y=x$, it must consist of straight line segments.



The Inversion Rules

- The inverse function "undoes" what the original function did to a value in its domain in order to get back to the original x-value.



- Thus, we have

$$f^{-1}(f(x)) = x, \text{ for all } x \text{ in the domain of } f;$$
$$f(f^{-1}(x)) = x, \text{ for all } x \text{ in the domain of } f^{-1}.$$

Example

- Verify that the functions are inverse functions. $f(x) = 4x - 3$ and $g(x) = \frac{x+3}{4}$.
  We check that $g(f(x)) = x$ and $f(g(x)) = x$.
  $$\begin{aligned} g(f(x)) &= g(4x-3) = \frac{(4x-3)+3}{4} = \frac{4x}{4} = x; \\ f(g(x)) &= f(\frac{x+3}{4}) = 4 \cdot \frac{x+3}{4} - 3 = x + 3 - 3 = x. \end{aligned}$$

How To Find The Inverse Of A One-To-One Function

- To find the inverse of a one-to-one function, follow the steps:
  - Substitute y for $f(x)$;
  - Interchange the variables x and y;
  - Solve for y;
  - Substitute $f^{-1}(x)$ for y;
  - Verify that the functions are inverses.

Example

- Find the inverse of the function $f(x) = 5x - 3$.

$$
\begin{aligned}
f(x) &= 5x - 3 \\
y &= 5x - 3 \\
x &= 5y - 3 \\
x + 3 &= 5y \\
\frac{x+3}{5} &= y \\
f^{-1}(x) &= \frac{x+3}{5}.
\end{aligned}
$$

Verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Example

- Find the inverse of the function $f(x) = \sqrt[5]{3x-2}$.

$$\begin{aligned} f(x) &= \sqrt[5]{3x-2} \\ y &= \sqrt[5]{3x-2} \\ x &= \sqrt[5]{3y-2} \\ x^5 &= 3y-2 \\ x^5+2 &= 3y \\ \frac{x^5+2}{3} &= y \\ f^{-1}(x) &= \frac{x^5+2}{3}. \end{aligned}$$

Verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Subsection 2

Evaluate and Graph Exponential Functions

We Shall Learn and Practice

- Graph exponential functions.
- Solve exponential equations.
- Use exponential models in applications.

Graph Exponential Functions

- An exponential function is a function of the form

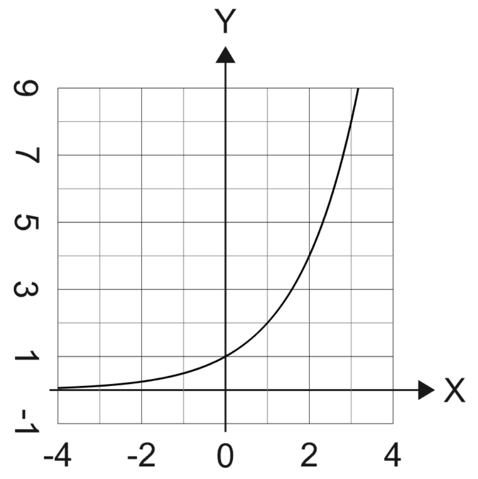
$$f(x) = a^x,$$

where $a > 0$ and $a \neq 1$.

Example

- Graph: $f(x) = 2^x$.

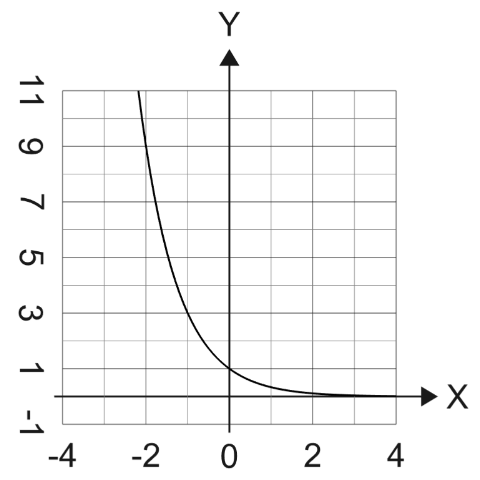
x	$f(x)$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4



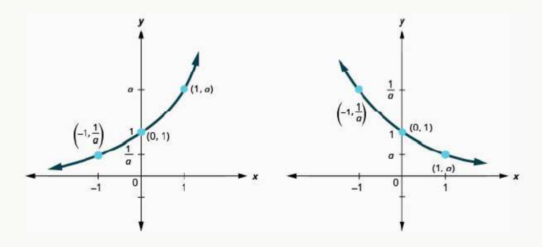
Example

- Graph: $f(x) = (\frac{1}{3})^x$.

x	$f(x)$
-2	9
-1	3
0	1
1	$\frac{1}{3}$
2	$\frac{1}{9}$



Properties of the Graph of $f(x) = a^x$

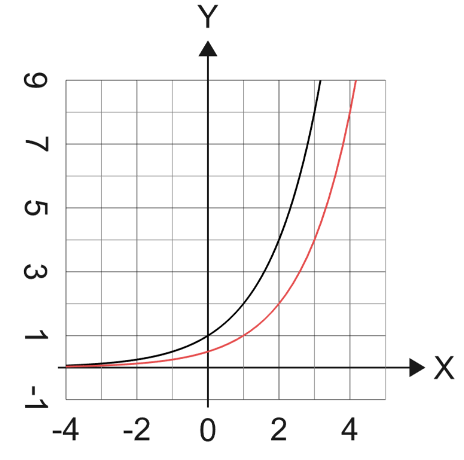


when $a > 1$		when $0 < a < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x-intercept	none	x-intercept	none
y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$	Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing

Example

- On the same coordinate system, graph: $f(x) = 2^x$ and $g(x) = 2^{x-1}$.

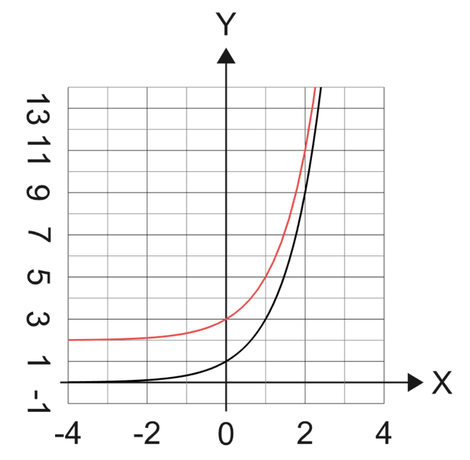
x	$f(x)$	g
-2	$\frac{1}{4}$	$(-1, \frac{1}{4})$
-1	$\frac{1}{2}$	$(0, \frac{1}{2})$
0	1	$(1, 1)$
1	2	$(2, 2)$
2	4	$(3, 4)$



Example

- On the same coordinate system, graph: $f(x) = 3^x$ and $g(x) = 3^x + 2$.

x	$f(x)$	$g(x)$
-2	$\frac{1}{9}$	$\frac{19}{9}$
-1	$\frac{1}{3}$	$\frac{7}{3}$
0	1	3
1	3	5
2	9	11



Solve Exponential Equations

- **One-to-One Property of Exponential Equations**: For $a > 0$ and $a \neq 1$,

$$\text{if } a^x = a^y, \text{ then } x = y.$$

- To solve an exponential equation follow the steps:
  - Write both sides of the equation with the same base, if possible;
  - Write a new equation by setting the exponents equal;
  - Solve the equation;
  - Check the solution.

Example

- Solve: $3^{3x-2} = 81$.

$$\begin{aligned} 3^{3x-2} &= 81 \\ 3^{3x-2} &= 3^4 \\ 3x-2 &= 4 \\ 3x &= 6 \\ x &= 2. \end{aligned}$$

Check the solution.

Example

- Recall that e is a constant (like π), with value
$$e \approx 2.718281827...$$
- Solve: $\frac{e^{x^2}}{e^x} = e^2$.
$$\begin{gathered}
\frac{e^{x^2}}{e^x} = e^2 \\
e^{x^2-x} = e^2 \\
x^2 - x = 2 \\
x^2 - x - 2 = 0 \\
(x-2)(x+1) = 0 \\
x = 2 \quad \text{or} \quad x = -1.
\end{gathered}$$

  Check the solutions.

Use Exponential Models in Applications

- **Compound Interest**: For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is:

$$A = P(1 + \tfrac{r}{n})^{nt} \quad \text{when compounded } n \text{ times a year}$$

$$A = Pe^{rt} \quad \text{when compounded continuously.}$$

Example

- Angela invested \$15,000 in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding? (a) compound quarterly (b) compound monthly (c) compound continuously.

(a) We use $A = P(1+\frac{r}{n})^{nt}$, with $P = 15000$, $r = 0.04$, $t = 10$ and $n = 4$.

$$A = P(1+\frac{r}{n})^{nt} = 15000(1+\frac{0.04}{4})^{4\cdot 10} = 15000\cdot 1.01^{40} \approx 22,332.96.$$

(b) We use $A = P(1+\frac{r}{n})^{nt}$, with $P = 15000$, $r = 0.04$, $t = 10$ and $n = 12$.

$$A = P(1+\frac{r}{n})^{nt} = 15000(1+\frac{0.04}{12})^{12\cdot 10} = 15000\cdot 1.0033^{120} \approx 22,362.49.$$

(c) We use $A = Pe^{rt}$, with $P = 15000$, $r = 0.04$, $t = 10$.

$$A = Pe^{rt} = 15000e^{0.04\cdot 10} = 15000e^{0.4} \approx 22,377.37.$$

Exponential Growth and Decay

- For an original amount, A_0, that grows or decays at a rate, r, for a certain time, t, the final amount, A, is:

$$A = A_0 e^{rt}.$$

- **Exponential growth** has a positive rate of growth;
- **Exponential decay** has a negative rate of growth.

Example

- Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

  We use $A = A_0 e^{rt}$, where $A_0 = 100$, $r = 0.1$ and $t = 24$.

$$A = A_0 e^{rt} = 100e^{0.1 \cdot 24} = 100e^{2.4} \approx 1,102.32.$$

  She will find approximately 1,102 viruses.

Subsection 3

Evaluate and Graph Logarithmic Functions

We Shall Learn and Practice

- Convert between exponential and logarithmic form.
- Evaluate logarithmic functions.
- Graph logarithmic functions.
- Solve logarithmic equations.
- Use logarithmic models in applications.

Convert Between Exponential and Logarithmic Form

- Suppose we want to find the inverse of the one-to-one exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

  We follow the usual work:

$$\begin{aligned} f(x) &= a^x \\ y &= a^x \\ x &= a^y \end{aligned}$$

  Solve for y? We have no way to solve for y!

- To deal with this we define the logarithm function with base a to be the inverse of the exponential function $f(x) = a^x$.
- The function $f(x) = \log_a x$ is the **logarithmic function with base** a, where $a > 0$, $x > 0$, and $a \neq 1$:

$$y = \log_a x \quad \text{is equivalent to} \quad x = a^y.$$

Example

- Convert to logarithmic form: (a) $3^2 = 9$ (b) $7^{\frac{1}{2}} = \sqrt{7}$ (c) $(\frac{1}{3})^x = \frac{1}{27}$.

(a)
$$3^2 = 9 \quad \text{equivalent to} \quad \log_3 9 = 2;$$

(b)
$$7^{\frac{1}{2}} = \sqrt{7} \quad \text{equivalent to} \quad \log_7 (\sqrt{7}) = \frac{1}{2};$$

(c)
$$\left(\frac{1}{3}\right)^x = \frac{1}{27} \quad \text{equivalent to} \quad \log_{\frac{1}{3}} \left(\frac{1}{27}\right) = x.$$

Example

- Convert to exponential form: (a) $3 = \log_4 64$ (b) $0 = \log_x 1$ (c) $-2 = \log_{10} \frac{1}{100}$.

(a)

$$3 = \log_4 64 \quad \text{equivalent to} \quad 4^3 = 64;$$

(b)

$$0 = \log_x 1 \quad \text{equivalent to} \quad x^0 = 1;$$

(c)

$$-2 = \log_{10} \frac{1}{100} \quad \text{equivalent to} \quad 10^{-2} = \frac{1}{100}.$$

Evaluate Logarithmic Functions

- To evaluate logarithmic equations convert the equation to its equivalent exponential equation.

Example

- Find the value of x: (a) $\log_x 64 = 2$ (b) $\log_5 x = 3$ (c) $\log_{\frac{1}{2}} \frac{1}{4} = x$.

(a)

$$\log_x 64 = 2 \quad \Leftrightarrow \quad x^2 = 64 \quad \Leftrightarrow \quad x = \pm\sqrt{64} \quad \Leftrightarrow \quad x = \pm 8.$$

Since the base must be positive, $x = 8$.

(b)

$$\log_5 x = 3 \quad \Leftrightarrow \quad 5^3 = x \quad \Leftrightarrow \quad x = 125.$$

(c)

$$\log_{\frac{1}{2}} \frac{1}{4} = x \ \Leftrightarrow \ \left(\frac{1}{2}\right)^x = \frac{1}{4} \ \Leftrightarrow \ \left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^2 \ \Leftrightarrow \ x = 2.$$

Example

- Find the exact value of each logarithm without using a calculator: (a) $\log_{12} 144$ (b) $\log_4 2$ (c) $\log_2 \frac{1}{32}$.

(a) Note that $y = \log_{12} 144$ is equivalent to $12^y = 144$.

So we are looking for the power of 12 that gives 144.

We see that $y = \log_{12} 144 = 2$.

(b) We have $y = \log_4 2$ equivalent to $4^y = 2$.

So we are looking for the power of 4 that gives 2.

Thus, $\log_4 2 = \frac{1}{2}$.

(c) We have $y = \log_2 \frac{1}{32}$ equivalent to $2^y = \frac{1}{32}$.

We are looking for the power of 2 that gives $\frac{1}{32}$.

Thus, $\log_2 \frac{1}{32} = -5$.

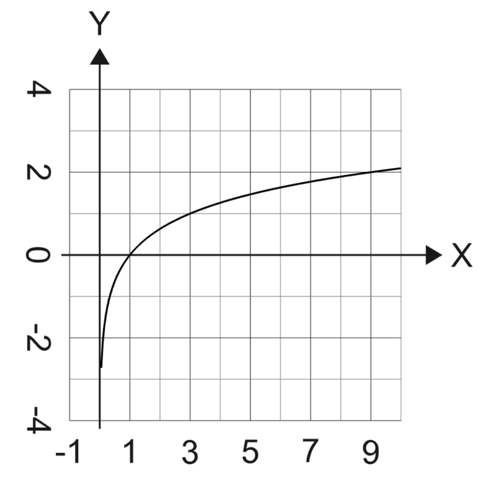
Graph Logarithmic Functions

- To graph a logarithmic function $y = \log_a x$, it is easiest to convert the equation to its exponential form, $x = a^y$.
- Generally, when we look for ordered pairs for the graph of a function, we usually choose an x-value and then determine its corresponding y-value.
- In this case it is easier to choose y-values and then determine its corresponding x-value.

Example

- Graph: $y = \log_3 x$.

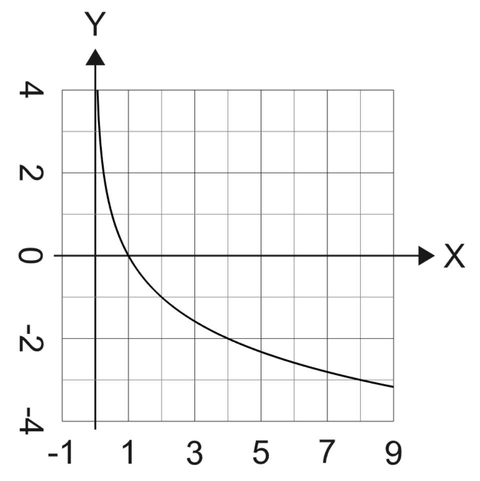
$x = 3^y$	$y = \log_3 x$
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2



Example

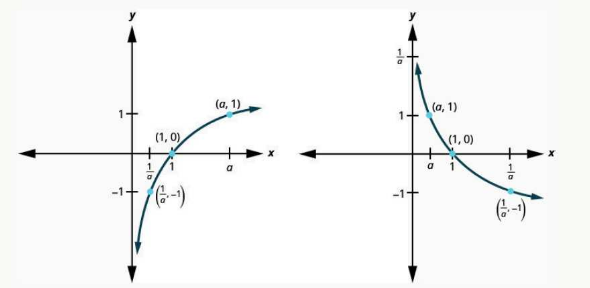
- Graph: $y = \log_{\frac{1}{2}} x$.

$x = (\frac{1}{2})^y$	$y = \log_{\frac{1}{2}} x$
4	-2
2	-1
1	0
$\frac{1}{2}$	1
$\frac{1}{4}$	2



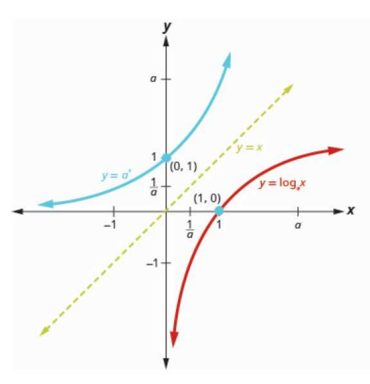
Properties of the Graph of $y = \log_a x$

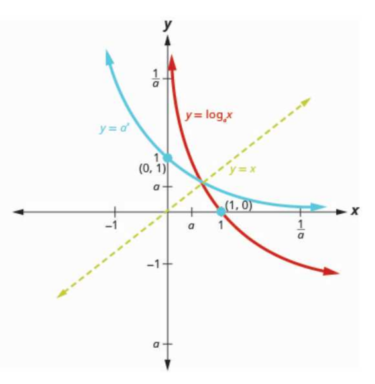
when $a > 1$		when $0 < a < 1$	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	x-intercept	$(1, 0)$
y-intercept	none	y-intercept	None
Contains	$(a, 1)$, $\left(\frac{1}{a}, -1\right)$	Contains	$(a, 1)$, $\left(\frac{1}{a}, -1\right)$
Asymptote	y-axis	Asymptote	y-axis
Basic shape	increasing	Basic shape	Decreasing



Graphs of Exponential and Logarithmic Functions

- We saw that the logarithmic function $f^{-1}(x) = \log_a x$ is the inverse of the exponential function $f(x) = a^x$.
- We know that graphs of inverse functions are mirror images of each other through $y = x$.





Solve Logarithmic Equations

- **Natural Logarithmic Function**: The function $f(x) = \ln x$ is the **natural logarithmic function** with base e, where $x > 0$.

$$y = \ln x \quad \text{is equivalent to} \quad x = e^y.$$

- **Common Logarithmic Function**: The function $f(x) = \log x$ is the **common logarithmic function** with base 10, where $x > 0$.

$$y = \log x \quad \text{is equivalent to} \quad x = 10^y.$$

Example

- Solve: (a) $\log_a 121 = 2$ (b) $\ln x = 7$.

(a)

$$\log_a 121 = 2 \quad \Leftrightarrow \quad a^2 = 121 \quad \Leftrightarrow \quad a = \sqrt{121} \quad \Leftrightarrow \quad a = 11.$$

(b)

$$\ln x = 7 \quad \Leftrightarrow \quad e^7 = x \quad \Leftrightarrow \quad x = e^7.$$

Example

- Solve: (a) $\log_2(5x-1)=6$ (b) $\ln e^{3x}=6$.

(a)

$$\log_2(5x-1)=6 \quad \Leftrightarrow \quad 2^6=5x-1 \quad \Leftrightarrow \quad 64=5x-1$$

$$\Leftrightarrow \quad 5x=65 \quad \Leftrightarrow \quad x=13.$$

(b)

$$\ln e^{3x}=6 \quad \Leftrightarrow \quad e^6=e^{3x} \quad \Leftrightarrow \quad 6=3x \quad \Leftrightarrow \quad x=2.$$

Use Logarithmic Models in Applications

- **Decibel Level of Sound**: The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is
$$D = 10\log\left(\frac{I}{10^{-12}}\right).$$
- What is the decibel level of one of the new quiet dishwashers with intensity 10^{-7} watts per square inch?

  We get for $I = 10^{-7}$,
$$\begin{aligned} D &= 10\log\left(\tfrac{I}{10^{-12}}\right) = 10\log\left(\tfrac{10^{-7}}{10^{-12}}\right) \\ &= 10\log 10^5 = 10\cdot 5 = 50 \text{ decibels.} \end{aligned}$$

Earthquake Intensity

- The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.
- In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

  We compute the two intensities:

$$\begin{aligned} 8.2 = \log I \quad &\Leftrightarrow \quad 10^{8.2} = I; \\ 5.1 = \log I' \quad &\Leftrightarrow \quad 10^{5.1} = I'. \end{aligned}$$

  Thus, we get

$$\frac{I}{I'} = \frac{10^{8.2}}{10^{5.1}} = 10^{3.1} \approx 1000.$$

  Therefore, Chile's quake was 1000 times more intense that the one in Los Angeles.

Subsection 4

Use the Properties of Logarithms

We Shall Learn and Practice

- Use the properties of logarithms.
- Use the Change of Base Formula

Properties of Logarithms

- If $M > 0$, $a > 0$, $a \neq 1$ and p is any real number then,

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse	$a^{\log_a x} = x$	$e^{\ln x} = x$
	$\log_a a^x = x$	$\ln e^x = x$
Product	$\log_a (M \cdot N) = \log_a M + log_a N$	$\ln (M \cdot N) = \ln M + \ln N$
Quotient	$\log_a \frac{M}{N} = log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

Example

- Evaluate using the properties of logarithms: (a) $\log_{13} 1$ (b) $\log_9 9$.

(a)

$$\log_{13} 1 = 0;$$

(b)

$$\log_9 9 = 1.$$

Example

- Evaluate using the properties of logarithms: (a) $5^{\log_5 15}$ (b) $\log_7 7^4$.

(a)

$$5^{\log_5 15} = 15;$$

(b)

$$\log_7 7^4 = 4.$$

Example

- Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible. (a) $\log_3 3x$ (b) $\log_2 8xy$

(a)

$$\log_3 3x = \log_3 3 + \log_3 x = 1 + \log_3 x;$$

(b)

$$\log_2 8xy = \log_2 8 + \log_2 x + \log_2 y = 3 + \log_2 x + \log_2 y.$$

Example

- Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible. (a) $\log_4 \frac{3}{4}$ (b) $\log \frac{x}{1000}$

(a)

$$\log_4 \frac{3}{4} = \log_4 3 - \log_4 4 = \log_4 3 - 1;$$

(b)

$$\log \frac{x}{1000} = \log x - \log 1000 = \log x - 3.$$

Example

- Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible. (a) $\log_7 5^4$ (b) $\log x^{100}$

(a)

$$\log_7 5^4 = 4 \log_7 5;$$

(b)

$$\log x^{100} = 100 \log x.$$

Example

- Use the Properties of Logarithms to expand the logarithm $\log_2(5x^4y^2)$. Simplify, if possible.

$$\begin{aligned} \log_2(5x^4y^2) &= \log_2 5 + \log_2 x^4 + \log_2 y^2 \\ &= \log_2 5 + 4\log_2 x + 2\log_2 y. \end{aligned}$$

Example

- Use the Properties of Logarithms to expand the logarithm $\log_4 \sqrt[5]{\frac{x^4}{2y^3z^2}}$. Simplify, if possible.

$$\begin{aligned}
\log_4 \sqrt[5]{\frac{x^4}{2y^3z^2}} &= \log_4 \left(\frac{x^4}{2y^3z^2}\right)^{\frac{1}{5}} \\
&= \frac{1}{5}\log_4 \frac{x^4}{2y^3z^2} \\
&= \frac{1}{5}(\log_4 (x^4) - \log_4 (2y^3z^2)) \\
&= \frac{1}{5}(\log_4 (x^4) - (\log_4 2 + \log_4 (y^3) + \log_4 (z^2))) \\
&= \frac{1}{5}(\log_4 (x^4) - \log_4 2 - \log_4 (y^3) - \log_4 (z^2)) \\
&= \frac{1}{5}(4\log_4 x - \frac{1}{2} - 3\log_4 y - 2\log_4 z).
\end{aligned}$$

Example

- Use the Properties of Logarithms to condense the logarithm $\log_2 5 + \log_2 x - \log_2 y$. Simplify, if possible.

$$\begin{aligned} &\log_2 5 + \log_2 x - \log_2 y \\ &= \log_2 (5x) - \log_2 y \\ &= \log_2 \frac{5x}{y}. \end{aligned}$$

Example

- Use the Properties of Logarithms to condense the logarithm $3\log_2 x + 2\log_2(x-1)$. Simplify, if possible.

$$\begin{aligned} &3\log_2 x + 2\log_2(x-1) \\ &= \log_2(x^3) + \log_2((x-1)^2) \\ &= \log_2(x^3(x-1)^2). \end{aligned}$$

Change of Base Formula

- **Change-of-Base Formula**: For any logarithmic bases a, b and $M > 0$,

$$\underset{\text{new base } b}{\log_a M = \frac{\log_b M}{\log_b a}} \quad \underset{\text{new base } 10}{\log_a M = \frac{\log M}{\log a}} \quad \underset{\text{new base } e}{\log_a M = \frac{\ln M}{\ln a}}$$

Example

- Rounding to three decimal places, approximate $\log_3 42$.
  We use natural logarithms.
  $$\log_3 42 = \frac{\ln 42}{\ln 3} \approx 3.402.$$

Subsection 5

Solve Exponential and Logarithmic Equations

We Shall Learn and Practice

- Solve logarithmic equations using the properties of logarithms.
- Solve exponential equations using logarithms.
- Use exponential models in applications.

Solve Logarithmic Equations Using the Properties

- **One-to-One Property of Logarithmic Equations**: For $M > 0$, $N > 0$ and $a > 0$, with $a \neq 1$, any real number:

$$\text{If } \log_a M = \log_a N, \text{ then } M = N.$$

Example

- Solve: $2\log_3 x = \log_3 36$.

$$\begin{aligned} 2\log_3 x &= \log_3 36 \\ \log_3 (x^2) &= \log_3 36 \\ x^2 &= 36 \\ x &= \pm\sqrt{36} \\ x &= \pm 6. \end{aligned}$$

Only $x = 6$ is admissible.

Example

- Solve: $\log_2 x + \log_2 (x-2) = 3$.

$$\begin{aligned} \log_2 x + \log_2 (x-2) &= 3 \\ \log_2 (x(x-2)) &= 3 \\ x(x-2) &= 2^3 \\ x^2 - 2x &= 8 \\ x^2 - 2x - 8 &= 0 \\ (x-4)(x+2) &= 0 \\ x = -2 \quad \text{or} \quad x &= 4. \end{aligned}$$

Only $x = 4$ is admissible.

Example

- Solve: $\log(x+2) - \log(4x+3) = -\log x$.

$$\begin{gathered}
\log(x+2) - \log(4x+3) = -\log x \\
\log\frac{x+2}{4x+3} = \log(x^{-1}) \\
\frac{x+2}{4x+3} = \frac{1}{x} \\
x(4x+3)\frac{x+2}{4x+3} = x(4x+3)\frac{1}{x} \\
x(x+2) = 4x+3 \\
x^2 + 2x - 4x - 3 = 0 \\
x^2 - 2x - 3 = 0 \\
(x-3)(x+1) = 0 \\
x = -1 \quad \text{or} \quad x = 3.
\end{gathered}$$

Only $x = 3$ is admissible.

Solve Exponential Equations Using Logarithms

- We solved some equations by writing both sides of the equation with the same base and then setting the exponents equal.
- It is not always possible or convenient to write the expressions with the same base.
- In that case we often take the common logarithm or natural logarithm of both sides once the exponential is isolated.

Example

- Solve $7^x = 43$. Find the exact answer and then approximate it to three decimal places.

$$\begin{aligned} 7^x &= 43 \\ \ln 7^x &= \ln 43 \\ x \ln 7 &= \ln 43 \\ x &= \frac{\ln 43}{\ln 7} \\ x &\approx 1.933. \end{aligned}$$

Example

- Solve $2e^{x-2} = 18$. Find the exact answer and then approximate it to three decimal places.

$$\begin{aligned} 2e^{x-2} &= 18 \\ e^{x-2} &= 9 \\ x - 2 &= \ln 9 \\ x &= 2 + \ln 9 \\ x &\approx 4.197. \end{aligned}$$

Use Exponential Models in Applications

- **Compound Interest**: For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is:

$$A = P(1 + \frac{r}{n})^{nt} \quad \text{when compounded } n \text{ times a year}$$

$$A = Pe^{rt} \quad \text{when compounded continuously.}$$

Example

- Hector invests \$10,000 at age 21. He hopes the investments will be worth \$150,000 when he turns 50. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal?

  We use $A = Pe^{rt}$, with $P = 10000$, $t = 29$ and $A = 150000$.

  We set up the equation and solve for r:

$$\begin{aligned} A &= Pe^{rt} \\ 150000 &= 10000e^{29r} \\ 15 &= e^{29r} \\ 29r &= \ln 15 \\ r &= \tfrac{\ln 15}{29} \\ r &\approx 0.093. \end{aligned}$$

  Hence, he will need rate of growth 9.3%.

Exponential Growth and Decay

- For an original amount, A_0, that grows or decays at a rate, r, for a certain time, t, the final amount, A, is:

$$A = A_0 e^{rt}.$$

- **Exponential growth** has a positive rate of growth;
- **Exponential decay** has a negative rate of growth.

Example

- Researchers recorded that a certain bacteria population grew from 100 to 500 in 6 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?
  First, we compute the rate of growth r. ($A_0 = 100$, $A = 500$, $t = 6$)

$$\begin{aligned} A &= A_0 e^{rt} \\ 500 &= 100 e^{6r} \\ 5 &= e^{6r} \\ 6r &= \ln 5 \\ r &= \tfrac{\ln 5}{6} \\ r &\approx 0.268. \end{aligned}$$

  Now we get ($A_0 = 100$, $r = 0.268$, $t = 24$)

$$A = 100 e^{0.268 \cdot 24} \approx 62141.$$

  So there will be around 62,141 bacteria 24 hours from the start of the experiment.

Example

- The half-life of magnesium-27 is 9.45 minutes. How much of a 10-mg sample will be left in 6 minutes?

  First, we compute the rate of decay. $(A_0 = 10,\ A = 5,\ t = 9.45)$

$$\begin{aligned} A &= A_0 e^{rt} \\ 5 &= 10e^{9.45r} \\ \tfrac{1}{2} &= e^{9.45r} \\ 9.45r &= \ln \tfrac{1}{2} \\ r &= \frac{\ln \frac{1}{2}}{9.45} \\ r &\approx -0.073. \end{aligned}$$

  Now we get $(A_0 = 10,\ r = -0.073,\ t = 6)$

$$A = 10e^{-0.073 \cdot 6} \approx 6.45.$$

  So there will be around 6.45 mg left in 6 minutes.