Intermediate Algebra

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Subsection 1

Review of Factoring

We Shall Review and Practice

- Factor completely a polynomial expression.
- Solve polynomial equations.

General Strategy

- Factor out the Greatest Common Factor (GCF).
- Then detect the number of terms:
  - For a binomial (2 terms):
    - Sum of squares does not factor;
    - Difference of squares $a^2 - b^2 = (a+b)(a-b)$;
    - Sum of cubes $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$;
    - Difference of cubes $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$;
  - For a trinomial (3 terms):
    - Square of a sum $a^2 + 2ab + b^2 = (a+b)^2$;
    - Square of a difference $a^2 - 2ab + b^2 = (a-b)^2$;
    - Apply the ac-Method;
  - For a polynomial of more than 3 terms:
    - Use Grouping.

Example

- Factor completely $5x^2 - 45y^2$.

$$\begin{aligned} 5x^2 - 45y^2 &= 5(x^2 - 9y^2) \\ &= 5(x^2 - (3y)^2) \\ &= 5(x + 3y)(x - 3y). \end{aligned}$$

Example

- Factor completely $2x^3 + 16$.

$$\begin{aligned} 2x^3 + 16 &= 2(x^3 + 8) \\ &= 2(x^3 + 2^3) \\ &= 2(x + 2)(x^2 - 2x + 4). \end{aligned}$$

Example

- Factor completely $3x^3 - 36x^2 + 108x$.

$$\begin{aligned} 3x^3 - 36x^2 + 108x &= 3x(x^2 - 12x + 36) \\ &= 3x(x^2 - 2 \cdot x \cdot 6 + 6^2) \\ &= 3x(x-6)^2. \end{aligned}$$

Example

- Factor completely $9x^2 - 12x + 4$.

$$\begin{aligned} 9x^2 - 12x + 4 &= (3x)^2 - 2 \cdot (3x) \cdot 2 + 2^2 \\ &= (3x - 2)^2. \end{aligned}$$

Example

- Factor completely $8x^3 + 25x^2 + 3x$.

$$\begin{array}{rcl} 8x^3 + 25x^2 + 3x & = & x(8x^2 + 25x + 3) \\ & \stackrel{\text{ac}}{=} & x(8x^2 + 24x + x + 3) \\ & \stackrel{\text{group}}{=} & x[8x(x+3) + (x+3)] \\ & = & x(8x+1)(x+3). \end{array}$$

Example

- Factor completely $3x^3 - 21x^2 + 30x$.

$$\begin{aligned} 3x^3 - 21x^2 + 30x &= 3x(x^2 - 7x + 10) \\ &\overset{\text{ac (a=1)}}{=} 3x(x-5)(x-2). \end{aligned}$$

Example

- Factor completely $xy - 8y + 7x - 56$.

$$\begin{aligned} xy - 8y + 7x - 56 &= y(x-8) + 7(x-8) \\ &= (y+7)(x-8). \end{aligned}$$

Solving Polynomial Equations

- To solve a polynomial equation follow the steps:
  - Set one side equal to zero;
  - Factor completely the non-zero side;
  - Set each factor equal to zero;
  - Solve the resulting (linear or quadratic) equations.

Example

- Solve the equation $x^3 + 9x^2 + 20x = 0$.

$$
\begin{gathered}
x^3 + 9x^2 + 20x = 0 \\
x(x^2 + 9x + 20) = 0 \\
x(x+4)(x+5) = 0 \\
x = 0 \quad \text{or} \quad x + 4 = 0 \quad \text{or} \quad x + 5 = 0 \\
x = 0 \quad \text{or} \quad x = -4 \quad \text{or} \quad x = -5.
\end{gathered}
$$

Example

- Solve the equation $2x^3 + 2x^2 = 12x$.

$$2x^3 + 2x^2 = 12x$$
$$2x^3 + 2x^2 - 12x = 0$$
$$2x(x^2 + x - 6) = 0$$
$$2x(x-2)(x+3) = 0$$
$$2 = 0 \quad \text{or} \quad x = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 0 \quad \text{or} \quad x = 2 \quad \text{or} \quad x = -3.$$

Example

- Solve the equation $(3x-2)(x+4)=12x$.

$$\begin{gathered}
(3x-2)(x+4)=12x \\
(3x-2)(x+4)-12x=0 \\
3x^2+12x-2x-8-12x=0 \\
3x^2-2x-8=0 \\
3x^2-6x+4x-8=0 \\
3x(x-2)+4(x-2)=0 \\
(3x+4)(x-2)=0 \\
3x+4=0 \quad \text{or} \quad x-2=0 \\
x=-\frac{4}{3} \quad \text{or} \quad x=2.
\end{gathered}$$

Subsection 2

Multiply and Divide Rational Expressions

We Shall Learn and Practice

- Determine the values for which a rational expression is undefined.
- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Multiply and divide rational functions.

Values for Which a Rational Expression is Undefined

- Given a rational expression $\frac{p}{q}$, where p and q are polynomials, to find the values for which it is undefined:
  - Set the denominator equal to zero, $q = 0$;
  - Solve the polynomial equation $q = 0$.
- It follows that $\frac{p}{q}$ is then defined at all values *except* those found above.

Example

- Determine the value for which the rational expression $\frac{5y^2}{7x}$ is undefined.

  Set $7x = 0$.

  Solve $7x = 0$.

$$\begin{aligned} 7x &= 0 \\ x &= 0. \end{aligned}$$

  Thus, $\frac{5y^2}{7x}$ is undefined for $x = 0$.

Example

- Determine the value for which the rational expression $\frac{5n-1}{7n+3}$ is undefined.

  Set $7n+3=0$.

  Solve $7n+3=0$.

$$\begin{aligned} 7n+3 &= 0 \\ 7n &= -3 \\ n &= -\frac{3}{7}. \end{aligned}$$

  Thus, $\frac{5n-1}{7n+3}$ is undefined for $n=-\frac{3}{7}$.

Example

- Determine the value for which the rational expression $\frac{a+1}{a^2+9a+20}$ is undefined.

  Set $a^2+9a+20=0$.

  Solve $a^2+9a+20=0$.

$$\begin{aligned} a^2+9a+20&=0\\ (a+4)(a+5)&=0\\ a+4=0 \quad \text{or} \quad a+5&=0\\ a=-4 \quad \text{or} \quad a&=-5 \end{aligned}$$

  Thus, $\frac{a+1}{a^2+9a+20}$ is undefined for $a=-4$ and $a=-5$.

Simplify Rational Expressions

- Follow the steps:
  - Write the numerator and denominator in factored form;
  - Remove the common factors.

Caution:

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$ is legal because c is a common factor;

$$\frac{a+c}{b+c} = \frac{a}{b}$$ is illegal because c is not a common factor.

Example

- Simplify the rational expression $\frac{x^2-5x+6}{x^2+3x-10}$.

$$\frac{x^2-5x+6}{x^2+3x-10} = \frac{(x-2)(x-3)}{(x-2)(x+5)} = \frac{x-3}{x+5}.$$

Example

- Simplify the rational expression $\frac{3x^2-18xy+27y^2}{2x^2-18y^2}$.

$$\begin{aligned}\frac{3x^2-18xy+27y^2}{2x^2-18y^2} &= \frac{3(x^2-6xy+9y^2)}{2(x^2-9y^2)}\\ &= \frac{3(x^2-2\cdot x\cdot 3y+(3y)^2)}{2(x^2-(3y)^2)}\\ &= \frac{3(x-3y)^2}{2(x-3y)(x+3y)}\\ &= \frac{3(x-3y)}{2(x+3y)}.\end{aligned}$$

Example

- Simplify the rational expression $\frac{x^2-x-30}{36-x^2}$.

$$\begin{aligned}\frac{x^2-x-30}{36-x^2} &= \frac{(x-6)(x+5)}{(6-x)(6+x)} \\ &= \frac{(x-6)(x+5)}{-(x-6)(x+6)} \\ &= -\frac{x+5}{x+6}.\end{aligned}$$

Multiply Rational Expressions

- Follow the steps:
  - Factor each numerator and denominator completely;
  - Multiply straight across;
  - Simplify by canceling common factors.

Example

- Simplify $\frac{3x}{x^2-2x-15} \cdot \frac{x^2-9}{6x}$.

$$\begin{aligned}
\frac{3x}{x^2-2x-15} \cdot \frac{x^2-9}{6x} &= \frac{3x}{(x-5)(x+3)} \cdot \frac{(x+3)(x-3)}{6x} \\
&= \frac{3x(x+3)(x-3)}{6x(x-5)(x+3)} \\
&= \frac{x-3}{2(x-5)}.
\end{aligned}$$

Example

- Simplify $\frac{3x^2+7x+2}{x^2-4} \cdot \frac{x^2-4x+4}{3x^2-20x-7}$.
  Before the main work, we do some factoring:

$$3x^2+7x+2 = 3x^2+6x+x+2 = 3x(x+2)+(x+2)$$
$$= (3x+1)(x+2);$$
$$x^2-4x+4 = x^2-2\cdot x\cdot 2+2^2 = (x-2)^2;$$
$$3x^2-20x-7 = 3x^2-21x+x-7 = 3x(x-7)+(x-7)$$
$$= (3x+1)(x-7).$$

  Now we do the main work:

$$\begin{aligned}\frac{3x^2+7x+2}{x^2-4} \cdot \frac{x^2-4x+4}{3x^2-20x-7} &= \frac{(3x+1)(x+2)}{(x+2)(x-2)} \cdot \frac{(x-2)^2}{(3x+1)(x-7)} \\ &= \frac{(3x+1)(x+2)(x-2)^2}{(x+2)(x-2)(3x+1)(x-7)} \\ &= \frac{x-2}{x-7}.\end{aligned}$$

Divide Rational Expressions

- Follow the steps:
    - Rewrite the division as the product of the first rational expression and the reciprocal of the second;
    - Factor each numerator and denominator completely;
    - Multiply straight across;
    - Simplify by canceling common factors.

Example

- Simplify $\frac{x^3-27}{5x^2-15x+45} \div \frac{x^2-9}{15}$.

$$\begin{aligned} & \frac{x^3-27}{5x^2-15x+45} \div \frac{x^2-9}{15} \\ &= \frac{x^3-27}{5x^2-15x+45} \cdot \frac{15}{x^2-9} \\ &= \frac{(x-3)(x^2+3x+9)}{5(x^2-3x+9)} \cdot \frac{15}{(x+3)(x-3)} \\ &= \frac{15(x-3)(x^2+3x+9)}{5(x^2-3x+9)(x+3)(x-3)} \\ &= \frac{3(x^2+3x+9)}{(x^2-3x+9)(x+3)}. \end{aligned}$$

Example

- Simplify $\dfrac{\frac{5x^2-3x-2}{7x-21}}{\frac{5x^2+12x+4}{x^2-x-6}}$.

  We do some factoring first:

$$\begin{aligned}5x^2-3x-2 &= 5x^2-5x+2x-2 = 5x(x-1)+2(x-1)\\ &= (5x+2)(x-1);\\ 5x^2+12x+4 &= 5x^2+10x+2x+4 = 5x(x+2)+2(x+2)\\ &= (5x+2)(x+2).\end{aligned}$$

  Now the main work:

$$\begin{aligned}\frac{\frac{5x^2-3x-2}{7x-21}}{\frac{5x^2+12x+4}{x^2-x-6}} &= \frac{5x^2-3x-2}{7x-21}\cdot\frac{x^2-x-6}{5x^2+12x+4}\\ &= \frac{(5x+2)(x-1)}{7(x-3)}\cdot\frac{(x-3)(x+2)}{(5x+2)(x+2)}\\ &= \frac{(5x+2)(x-1)(x-3)(x+2)}{7(x-3)(5x+2)(x+2)} = \frac{x-1}{7}.\end{aligned}$$

Example

- Simplify $\frac{4x+4}{3x-15} \div \frac{x^2-4x-32}{x^2-3x-10} \cdot \frac{6x-48}{12x-36}$.

$$\begin{aligned}
&\frac{4x+4}{3x-15} \div \frac{x^2-4x-32}{x^2-3x-10} \cdot \frac{6x-48}{12x-36} \\
&= \frac{4x+4}{3x-15} \cdot \frac{x^2-3x-10}{x^2-4x-32} \cdot \frac{6x-48}{12x-36} \\
&= \frac{4(x+1)}{3(x-5)} \cdot \frac{(x-5)(x+2)}{(x-8)(x+4)} \cdot \frac{6(x-8)}{12(x-3)} \\
&= \frac{24(x+1)(x-5)(x+2)(x-8)}{36(x-5)(x-8)(x+4)(x-3)} \\
&= \frac{2(x+1)(x+2)}{3(x+4)(x-3)}.
\end{aligned}$$

Determine the Domain of a Rational Function

- A **rational function** is a function determined by a rational expression.
- The **domain** of a rational function consists of all real numbers except those that would cause division by zero.
- To determine the domain follow the steps:
  - Set the denominator equal to zero;
  - Solve the equation;
  - The domain consists of all real numbers except the values found as solutions.

Example

- Find the domain of $R(x) = \frac{7x^2-21x}{5x^2+10x-75}$.

  Set $5x^2 + 10x - 75 = 0$.

  Solve $5x^2 + 10x - 75 = 0$.

$$\begin{gathered} 5x^2 + 10x - 75 = 0 \\ 5(x^2 + 2x - 15) = 0 \\ 5(x-3)(x+5) = 0 \\ 5 = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{or} \quad x + 5 = 0 \\ x = 3 \quad \text{or} \quad x = -5. \end{gathered}$$

  Thus, the domain of $R(x)$ is the set of real numbers except $x = 3$ and $x = -5$.

  Sometimes this set is written $\mathbb{R} - \{-5, 3\}$.

Multiply and Divide Rational Functions

- To multiply rational functions, we multiply the resulting rational expressions on the right side of the equation using the same techniques we used to multiply rational expressions.
- To divide rational functions, we divide the resulting rational expressions on the right side of the equation using the same techniques we used to divide rational expressions.

Example

- Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \frac{3x-9}{x^2-10x+21}$ and $g(x) = \frac{2x^2-98}{3x+21}$.

$$\begin{aligned} R(x) &= f(x) \cdot g(x) \\ &= \frac{3x-9}{x^2-10x+21} \cdot \frac{2x^2-98}{3x+21} \\ &= \frac{3(x-3)}{(x-7)(x-3)} \cdot \frac{2(x^2-49)}{3(x+7)} \\ &= \frac{3(x-3)}{(x-7)(x-3)} \cdot \frac{2(x+7)(x-7)}{3(x+7)} \\ &= \frac{6(x-3)(x+7)(x-7)}{3(x-7)(x-3)(x+7)} \\ &= 2. \end{aligned}$$

Example

- Find $R(x) = \frac{f(x)}{g(x)}$, where $f(x) = \frac{5x^2}{x^2-3x}$ and $g(x) = \frac{10x^2+20x}{x^2+9x+14}$.

$$\begin{aligned} R(x) &= \frac{f(x)}{g(x)} = \frac{\frac{5x^2}{x^2-3x}}{\frac{10x^2+20x}{x^2+9x+14}} \\ &= \frac{5x^2}{x^2-3x} \cdot \frac{x^2+9x+14}{10x^2+20x} \\ &= \frac{5x^2}{x(x-3)} \cdot \frac{(x+2)(x+7)}{10x(x+2)} \\ &= \frac{5x^2(x+2)(x+7)}{10x^2(x-3)(x+2)} \\ &= \frac{x+7}{2(x-3)}. \end{aligned}$$

Subsection 3

Add and Subtract Rational Expressions

We Shall Learn and Practice

- Add and subtract rational expressions with a common denominator.
- Add and subtract rational expressions whose denominators are opposites.
- Find the least common denominator of rational expressions.
- Add and subtract rational expressions with unlike denominators.
- Add and subtract rational functions.

Add and Subtract with a Common Denominator

- To add or subtract rational expressions with a common denominator, follow the steps:
  - Add or subtract the numerators and place the result over the common denominator;
$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r},\quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$
  - Simplify the resulting rational expression.
    Recall this involves:
    - Factoring numerator and denominator completely;
    - Canceling common factors.

Example

- Simplify $\frac{x^2}{x+3} - \frac{2x+15}{x+3}$.

$$\begin{aligned} \frac{x^2}{x+3} - \frac{2x+15}{x+3} &= \frac{x^2 - (2x+15)}{x+3} \\ &= \frac{x^2 - 2x - 15}{x+3} \\ &= \frac{(x-5)(x+3)}{x+3} \\ &= x - 5. \end{aligned}$$

Example

- Simplify $\frac{4x^2-3x+30}{x^2-15x+56} - \frac{3x^2+10x-10}{x^2-15x+56}$.

$$\begin{aligned}
&\frac{4x^2-3x+30}{x^2-15x+56} - \frac{3x^2+10x-10}{x^2-15x+56} \\
&= \frac{4x^2-3x+30-(3x^2+10x-10)}{x^2-15x+56} \\
&= \frac{4x^2-3x+30-3x^2-10x+10}{x^2-15x+56} \\
&= \frac{x^2-13x+40}{x^2-15x+56} \\
&= \frac{(x-5)(x-8)}{(x-7)(x-8)} \\
&= \frac{x-5}{x-7}.
\end{aligned}$$

Add and Subtract with Opposite Denominators

- When the denominators of two rational expressions are opposites follow the steps:
  - Multiply one of the fractions by $\frac{-1}{-1}$;
  - Add or subtract with common denominator;
  - Simplify.

Example

- Simplify $\frac{x^2-7x}{x^2-9} - \frac{3x-21}{9-x^2}$.

$$\begin{aligned}
\frac{x^2-7x}{x^2-9} - \frac{3x-21}{9-x^2} &= \frac{x^2-7x}{x^2-9} - \frac{-(3x-21)}{-(9-x^2)} \\
&= \frac{x^2-7x}{x^2-9} - \frac{-(3x-21)}{x^2-9} \\
&= \frac{x^2-7x+3x-21}{x^2-9} \\
&= \frac{x^2-4x-21}{x^2-9} \\
&= \frac{(x-7)(x+3)}{(x-3)(x+3)} \\
&= \frac{x-7}{x-3}.
\end{aligned}$$

Example

- Simplify $\frac{4x^2+12x-1}{x^2-4} - \frac{x^2-3x-1}{4-x^2}$.

$$\begin{aligned}
&\frac{4x^2+12x-1}{x^2-4} - \frac{x^2-3x-1}{4-x^2} \\
&= \frac{4x^2+12x-1}{x^2-4} - \frac{-(x^2-3x-1)}{-(4-x^2)} \\
&= \frac{4x^2+12x-1}{x^2-4} - \frac{-(x^2-3x-1)}{x^2-4} \\
&= \frac{4x^2+12x-1+x^2-3x-1}{x^2-4} \\
&= \frac{5x^2+9x-2}{x^2-4} \\
&= \frac{(5x-1)(x+2)}{(x-2)(x+2)} \qquad (5x^2+9x-2 = 5x^2+10x-x-2 = 5x(x+2)-(x+2)) \\
&= \frac{5x-1}{x-2}.
\end{aligned}$$

Find the Least Common Denominator

- To find the Least Common Denominator (LCD) follow the steps:
  - Factor each denominator completely;
  - List the factors of each denominator; Match factors vertically when possible;
  - Bring down the columns by including all factors, but do not include common factors twice;
  - Write the LCD as the product of the factors.

Example

- (a) Find the LCD for the expressions $\frac{2}{x^2-x-12}$, $\frac{5}{x^2-9}$. (b) Rewrite them as equivalent rational expressions with the lowest common denominator.

(a)

$$\begin{array}{ccccc} \frac{\frac{2}{x^2-x-12}}{\frac{2}{(x-4)(x+3)}} & & & & \frac{\frac{5}{x^2-9}}{\frac{5}{(x-3)(x+3)}} \\ \to & (x-4) & & (x+3) & \\ & & (x-3) & (x+3) & \leftarrow \\ \hline \text{LCD:} & (x-4) & (x-3) & (x+3) & \end{array}$$

(b)

$$\begin{array}{rcl} \frac{2}{x^2-x-12} & = & \frac{2}{(x-4)(x+3)} = \frac{2(x-3)}{(x-4)(x-3)(x+3)} = \frac{2x-6}{(x-4)(x-3)(x+3)} \\ \frac{5}{x^2-9} & = & \frac{5}{(x-3)(x+3)} = \frac{5(x-4)}{(x-4)(x-3)(x+3)} = \frac{5x-20}{(x-4)(x-3)(x+3)}. \end{array}$$

Add and Subtract with Unlike Denominators

- To add or subtract with unline denominators follow the steps:
  - Determine if the rational expressions have a common denominator;
  - If no, rewrite each rational expression with the LCD;
  - Add or subtract using common denominators;
  - Simplify, if possible.

Example

- Simplify $\frac{2}{x-7}+\frac{9}{x+4}$.

$$\begin{aligned}\frac{2}{x-7}+\frac{9}{x+4} &= \frac{2(x+4)}{(x-7)(x+4)}+\frac{9(x-7)}{(x-7)(x+4)}\\ &= \frac{2x+8}{(x-7)(x+4)}+\frac{9x-63}{(x-7)(x+4)}\\ &= \frac{2x+8+9x-63}{(x-7)(x+4)}\\ &= \frac{11x-55}{(x-7)(x+4)}\\ &= \frac{11(x-5)}{(x-7)(x+4)}.\end{aligned}$$

Example

- Simplify $\frac{1}{x^2-x-2}+\frac{5x}{x^2+3x+2}$.

$$\begin{aligned}
&\frac{1}{x^2-x-2}+\frac{5x}{x^2+3x+2} \\
&= \frac{1}{(x-2)(x+1)}+\frac{5x}{(x+1)(x+2)} \\
&= \frac{x+2}{(x-2)(x+1)(x+2)}+\frac{5x(x-2)}{(x-2)(x+1)(x+2)} \\
&= \frac{x+2+5x^2-10x}{(x-2)(x+1)(x+2)} \\
&= \frac{5x^2-9x+2}{(x-2)(x+1)(x+2)}.
\end{aligned}$$

Example

- Simplify $\frac{3}{x+3} - \frac{6x}{x^2-9}$.

$$\begin{aligned}
\frac{3}{x+3} - \frac{6x}{x^2-9} &= \frac{3}{x+3} - \frac{6x}{(x-3)(x+3)} \\
&= \frac{3(x-3)}{(x-3)(x+3)} - \frac{6x}{(x-3)(x+3)} \\
&= \frac{3x-9-6x}{(x-3)(x+3)} \\
&= \frac{-3x-9}{(x-3)(x+3)} \\
&= \frac{-3(x+3)}{(x-3)(x+3)} \\
&= \frac{-3}{x-3}.
\end{aligned}$$

Example

- Simplify $\frac{3x-1}{x^2-5x-6} - \frac{2}{6-x}$.

$$\begin{aligned}
\frac{3x-1}{x^2-5x-6} - \frac{2}{6-x} &= \frac{3x-1}{(x-6)(x+1)} - \frac{-2}{-(6-x)} \\
&= \frac{3x-1}{(x-6)(x+1)} - \frac{-2}{x-6} \\
&= \frac{3x-1}{(x-6)(x+1)} - \frac{-2(x+1)}{(x-6)(x+1)} \\
&= \frac{3x-1+2(x+1)}{(x-6)(x+1)} \\
&= \frac{3x-1+2x+2}{(x-6)(x+1)} \\
&= \frac{5x+1}{(x-6)(x+1)}.
\end{aligned}$$

Example

- Simplify $\frac{3}{x^2-4x-5} - \frac{2}{x^2-6x+5}$.

$$\begin{aligned}
&\frac{3}{x^2-4x-5} - \frac{2}{x^2-6x+5} \\
&= \frac{3}{(x-5)(x+1)} - \frac{2}{(x-5)(x-1)} \\
&= \frac{3(x-1)}{(x-5)(x-1)(x+1)} - \frac{2(x+1)}{(x-5)(x-1)(x+1)} \\
&= \frac{3(x-1)-2(x+1)}{(x-5)(x-1)(x+1)} \\
&= \frac{3x-3-2x-2}{(x-5)(x-1)(x+1)} \\
&= \frac{x-5}{(x-5)(x-1)(x+1)} \\
&= \frac{1}{(x-1)(x+1)}.
\end{aligned}$$

Add and Subtract Rational Functions

- To add or subtract rational functions, we simply add or subtract the rational expression that define the rational functions.

Example

- Find $R(x) = f(x) - g(x)$ where $f(x) = \frac{x+1}{x+3}$ and $g(x) = \frac{x+17}{x^2-x-12}$.

$$\begin{aligned}
R(x) &= \frac{x+1}{x+3} - \frac{x+17}{x^2-x-12} \\
&= \frac{x+1}{x+3} - \frac{x+17}{(x-4)(x+3)} \\
&= \frac{(x+1)(x-4)}{(x-4)(x+3)} - \frac{x+17}{(x-4)(x+3)} \\
&= \frac{(x+1)(x-4)-(x+17)}{(x-4)(x+3)} \\
&= \frac{x^2-4x+x-4-x-17}{(x-4)(x+3)} \\
&= \frac{x^2-4x-21}{(x-4)(x+3)} \\
&= \frac{(x-7)(x+3)}{(x-4)(x+3)} = \frac{x-7}{x-4}.
\end{aligned}$$

Subsection 4

Simplify Complex Rational Expressions

We Shall Learn and Practice

- Simplify a complex rational expression by writing it as division.
- Simplify a complex rational expression by using the LCD.

Complex Rational Expressions

- A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}, \quad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}, \quad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}.$$

Simplify by Writing as Division

- To simplify a complex rational expression by writing it as a division follow the steps:
  - Simplify the numerator and denominator;
  - Rewrite the complex rational expression as a division.
  - Divide the expressions involved.

Example

- Simplify $\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$.

$$\begin{aligned}\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}} &= \frac{1}{x^2-7x+12}\cdot\frac{x-4}{2}\\ &= \frac{1}{(x-4)(x-3)}\cdot\frac{x-4}{2}\\ &= \frac{x-4}{2(x-4)(x-3)}\\ &= \frac{1}{2(x-3)}.\end{aligned}$$

Example

- Simplify $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}$.

$$\begin{aligned}\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}} &= \frac{\frac{y}{xy}+\frac{x}{xy}}{\frac{y^2}{x^2y^2}-\frac{x^2}{x^2y^2}} = \frac{\frac{y+x}{xy}}{\frac{y^2-x^2}{x^2y^2}} \\ &= \frac{\frac{y+x}{xy}}{\frac{(y-x)(y+x)}{x^2y^2}} = \frac{y+x}{xy}\cdot\frac{x^2y^2}{(y-x)(y+x)} \\ &= \frac{(y+x)x^2y^2}{xy(y-x)(y+x)} = \frac{xy}{y-x}.\end{aligned}$$

Example

- Simplify $\dfrac{1-\frac{3}{x+4}}{\frac{1}{x+4}+\frac{x}{3}}$.

$$
\begin{aligned}
\frac{1-\frac{3}{x+4}}{\frac{1}{x+4}+\frac{x}{3}} &= \frac{\frac{x+4}{x+4}-\frac{3}{x+4}}{\frac{3}{3(x+4)}+\frac{x(x+4)}{3(x+4)}} = \frac{\frac{x+4-3}{x+4}}{\frac{3+x(x+4)}{3(x+4)}} \\
&= \frac{\frac{x+1}{x+4}}{\frac{x^2+4x+3}{3(x+4)}} = \frac{\frac{x+1}{x+4}}{\frac{(x+1)(x+3)}{3(x+4)}} \\
&= \frac{x+1}{x+4}\cdot\frac{3(x+4)}{(x+1)(x+3)} \\
&= \frac{3(x+4)(x+1)}{(x+1)(x+3)(x+4)} \\
&= \frac{3}{x+3}.
\end{aligned}
$$

Simplify by Using the LCD

- To simplify a complex rational expression by using the LCD follow the steps:
  - Find the LCD of all fractions in the complex rational expression;
  - Multiply the numerator and denominator by the LCD;
  - Simplify the expression.

Example

- Simplify $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

$$\begin{aligned}\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} &= \frac{xy(\frac{1}{x}+\frac{1}{y})}{xy(\frac{x}{y}-\frac{y}{x})} \\ &= \frac{y+x}{x^2-y^2} \\ &= \frac{y+x}{(x-y)(x+y)} \\ &= \frac{1}{x-y}.\end{aligned}$$

Example

- Simplify $\dfrac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}$.

$$\begin{aligned}
\frac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}} &= \frac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{(x-7)(x+7)}} \\
&= \frac{(x-7)(x+7)(\frac{2}{x-7}-\frac{1}{x+7})}{(x-7)(x+7)(\frac{6}{x+7}-\frac{1}{(x-7)(x+7)})} \\
&= \frac{2(x+7)-(x-7)}{6(x-7)-1} \\
&= \frac{2x+14-x+7}{6x-42-1} \\
&= \frac{x+21}{6x-43}.
\end{aligned}$$

Example

- Simplify $\dfrac{\frac{4x}{x+5}+\frac{2}{x+6}}{\frac{3x}{x^2+11x+30}}$.

$$\begin{aligned}\frac{\frac{4x}{x+5}+\frac{2}{x+6}}{\frac{3x}{x^2+11x+30}} &= \frac{\frac{4x}{x+5}+\frac{2}{x+6}}{\frac{3x}{(x+5)(x+6)}} \\ &= \frac{(x+5)(x+6)(\frac{4x}{x+5}+\frac{2}{x+6})}{(x+5)(x+6)\frac{3x}{(x+5)(x+6)}} \\ &= \frac{4x(x+6)+2(x+5)}{3x} \\ &= \frac{4x^2+24x+2x+10}{3x} \\ &= \frac{4x^2+26x+10}{3x} \\ &= \frac{2(2x^2+13x+5)}{3x}.\end{aligned}$$

Example

- Simplify $\dfrac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$.

$$\begin{aligned}\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}} &= \frac{(x-1)(x+1)(1+\frac{1}{x-1})}{(x-1)(x+1)\frac{3}{x+1}} \\ &= \frac{(x-1)(x+1)+(x+1)}{3(x-1)} \\ &= \frac{x^2-1+x+1}{3(x-1)} \\ &= \frac{x^2+x}{3(x-1)} \\ &= \frac{x(x+1)}{3(x-1)}.\end{aligned}$$

Subsection 5

Solve Rational Equations

We Shall Learn and Practice

- Solve rational equations.
- Use rational functions.
- Solve a rational equation for a specific variable.

Solve Rational Equations

- First, note those values that would make denominators zero!
- Then follow the steps:
  - Find the LCD of all denominators in the equation;
  - Multiply both sides of the equation by the LCD (this clears the fractions);
  - Solve the resulting equation;
  - Admit only those solutions that were not forbidden initially.
- Check that correct solutions were obtained using the equation.

Example

- Solve $1 - \frac{2}{x} = \frac{15}{x^2}$.

$$\begin{aligned} 1 - \frac{2}{x} &= \frac{15}{x^2}, \quad x \neq 0 \\ x^2\left(1 - \frac{2}{x}\right) &= x^2 \frac{15}{x^2} \\ x^2 - 2x &= 15 \\ x^2 - 2x - 15 &= 0 \\ (x-5)(x+3) &= 0 \\ x - 5 = 0 \quad &\text{or} \quad x + 3 = 0 \\ x = 5 \quad &\text{or} \quad x = -3. \end{aligned}$$

Both solutions admissible (none equals 0!).

Example

- Solve $\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$.

$$\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$$
$$\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{(x-1)(x+1)}, \quad x\neq -1,1$$
$$(x-1)(x+1)(\frac{2}{x+1}+\frac{1}{x-1})=(x-1)(x+1)\frac{1}{(x-1)(x+1)}$$
$$2(x-1)+(x+1)=1$$
$$2x-2+x+1=1$$
$$3x-1=1$$
$$3x=2$$
$$x=\frac{2}{3}$$

The solution is admissible (does not equal -1 or 1).

Example

- Solve $\frac{x+13}{x^2-7x+10} = \frac{6}{x-5} - \frac{4}{x-2}$.

$$\frac{x+13}{x^2-7x+10} = \frac{6}{x-5} - \frac{4}{x-2}$$
$$\frac{x+13}{(x-5)(x-2)} = \frac{6}{x-5} - \frac{4}{x-2}, \quad x \neq 2, 5$$
$$(x-5)(x-2)\frac{x+13}{(x-5)(x-2)} = (x-5)(x-2)(\frac{6}{x-5} - \frac{4}{x-2})$$
$$x+13 = 6(x-2) - 4(x-5)$$
$$x+13 = 6x - 12 - 4x + 20$$
$$x+13 = 2x+8$$
$$x = 5$$

Not an admissible solution. (2 and 5 were forbidden!)

So, this equation has no solutions.

Example

- Solve $\frac{x}{x+4} = \frac{32}{x^2-16} + 5$.

$$\frac{x}{x+4} = \frac{32}{x^2-16} + 5$$
$$\frac{x}{x+4} = \frac{32}{(x-4)(x+4)} + 5, \quad x \neq -4, 4$$
$$(x-4)(x+4)\frac{x}{x+4} = (x-4)(x+4)(\frac{32}{(x-4)(x+4)} + 5)$$
$$x(x-4) = 32 + 5(x-4)(x+4)$$
$$x^2 - 4x = 32 + 5(x^2-16)$$
$$x^2 - 4x = 32 + 5x^2 - 80$$
$$4x^2 + 4x - 48 = 0$$
$$4(x^2 + x - 12) = 0$$
$$4(x-3)(x+4) = 0$$
$$x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$
$$x = 3 \quad \text{or} \quad x = -4$$

Only $x = 3$ is an admissible solution!

Example

- Solve $\frac{x}{5x-10} - \frac{5}{3x+6} = \frac{2x^2-19x+54}{15x^2-60}$.

$$\frac{x}{5x-10} - \frac{5}{3x+6} = \frac{2x^2-19x+54}{15x^2-60}$$
$$\frac{x}{5(x-2)} - \frac{5}{3(x+2)} = \frac{2x^2-19x+54}{15(x-2)(x+2)}, \quad x \neq -2, 2$$
$$15(x-2)(x+2)\left(\frac{x}{5(x-2)} - \frac{5}{3(x+2)}\right) = 15(x-2)(x+2)\frac{2x^2-19x+54}{15(x-2)(x+2)}$$
$$3x(x+2) - 25(x-2) = 2x^2-19x+54$$
$$3x^2+6x-25x+50 = 2x^2-19x+54$$
$$x^2-4 = 0$$
$$(x-2)(x+2) = 0$$
$$x = -2 \quad \text{or} \quad x = 2$$

None of the two solutions is admissible.

Use Rational Functions

- For rational function, $f(x) = \frac{8-x}{x^2-7x+12}$. (a) find the domain of the function. (b) Solve $f(x) = 3$ (c) Find the points on the graph at this function value.

(a)

$$\begin{gathered} x^2 - 7x + 12 = 0 \\ (x-4)(x-3) = 0 \\ x = 4 \quad \text{or} \quad x = 3 \end{gathered}$$

The domain consists of all real numbers except $x = 3$ and $x = 4$.

(b)

$$\begin{gathered} \frac{8-x}{x^2-7x+12} = 3 \qquad \frac{8-x}{(x-4)(x-3)} = 3, \quad x \neq 3, 4 \\ (x-4)(x-3)\frac{8-x}{(x-4)(x-3)} = 3(x-4)(x-3) \\ 8 - x = 3x^2 - 21x + 36 \qquad 3x^2 - 20x + 28 = 0 \\ 3x^2 - 6x - 14x + 28 = 0 \qquad 3x(x-2) - 14(x-2) = 0 \\ (3x-14)(x-2) = 0 \qquad x = \frac{14}{3} \quad \text{or} \quad x = 2. \end{gathered}$$

(c) The points are $(2, 3)$ and $(\frac{14}{3}, 3)$.

Solve a Rational Equation for a Specific Variable

- Follow the steps:
  - Note any value of the variable that would make any denominator zero;
  - Clear the fractions by multiplying both sides of the equation by the LCD;
  - Simplify;
  - Isolate the terms with the variable to solve for;
  - Solve for the variable by factoring it out and dividing.

Example

- Solve: $m = \frac{y-5}{x-4}$ for y.

$$m = \frac{y-5}{x-4}, \quad x \neq 4$$
$$(x-4)m = (x-4)\frac{y-5}{x-4}$$
$$xm - 4m = y - 5$$
$$xm - 4m + 5 = y.$$

Thus, $y = xm - 4m + 5$.

Example

- Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

$$\begin{aligned}
\frac{1}{a}+\frac{1}{b} &= c, \quad a,b\neq 0\\
ab\left(\frac{1}{a}+\frac{1}{b}\right) &= abc\\
b+a &= abc\\
b &= abc-a\\
b &= a(bc-1)\\
\frac{b}{bc-1} &= a.
\end{aligned}$$

Thus, $a=\dfrac{b}{bc-1}$.

Subsection 6

Solve Applications with Rational Equations

We Shall Learn and Practice

- Solve proportions.
- Solve similar figure applications.
- Solve uniform motion applications.
- Solve work applications.
- Solve direct variation problems.
- Solve inverse variation problems.

Solve Proportions

- A proportion is an equation of the form
$$\frac{a}{b} = \frac{c}{d}, \quad b \neq 0, d \neq 0.$$
- A proportion is an equation with rational expressions.
  So to solve a proportion we follow the steps:
  - Multiply both sides of the equation by the LCD;
  - Clear the fractions;
  - Solve the resulting equation.

Example

- Solve the proportion: $\frac{x}{x+55} = \frac{3}{8}$.

$$\frac{x}{x+55} = \frac{3}{8}, \quad x \neq -55$$
$$8(x+55)\frac{x}{x+55} = 8(x+55)\frac{3}{8}$$
$$8x = 3(x+55)$$
$$8x = 3x + 165$$
$$5x = 165$$
$$x = 33.$$

Example

- Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Agatha, who weighs 65 pounds?

  Suppose she prescribes x ml for Agatha.

  Then we get the proportion $\frac{5}{25} = \frac{x}{65}$.

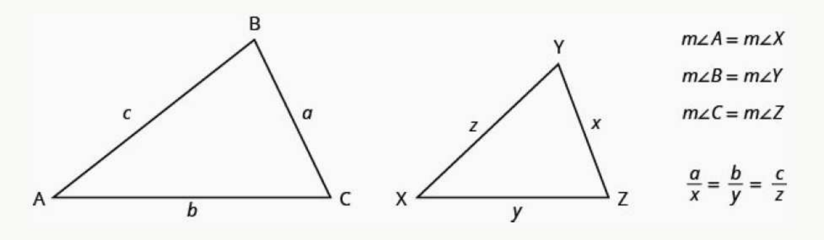
  We solve for x.

$$\frac{1}{5} = \frac{x}{65}$$
$$65 \cdot \frac{1}{5} = 65 \cdot \frac{x}{65}$$
$$13 = x.$$

  Thus, the prescription for Agatha is 13 ml.

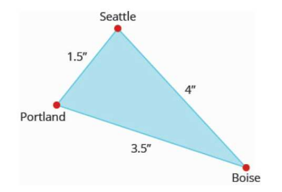
Solve Similar Figure Applications

- Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.



Example

- On the map, Seattle, Portland, and Boise form a triangle. The distance between the cities is measured in inches. The actual distance from Seattle to Boise is 400 miles. Find the actual distance from Seattle to Portland.



  Let x be the actual distance from Seattle to Portland.

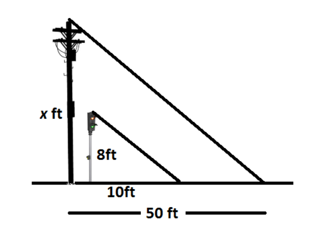
  Then we get the proportion

$$\frac{x}{1.5} = \frac{400}{4}$$
$$\frac{x}{1.5} = 100$$
$$1.5\frac{x}{1.5} = 1.5 \cdot 100$$
$$x = 150.$$

  Thus the actual distance from Seattle to Portland is 150 miles.

Example

- A telephone pole casts a shadow that is 50 feet long. Nearby, an 8 foot tall traffic sign casts a shadow that is 10 feet long. How tall is the telephone pole?



Suppose the pole is x feet tall.

We get the proportion

$$\frac{x}{8} = \frac{50}{10}$$
$$\frac{x}{8} = 5$$
$$8 \cdot \frac{x}{8} = 8 \cdot 5$$
$$x = 40.$$

Thus the telephone pole is 40 feet tall.

Solve Uniform Motion Applications

- The formula

$$\begin{aligned} D &= rt \\ \text{(Distance)} &= \text{(Velocity)} \times \text{(Time)} \end{aligned}$$

assumes we know r and t and use them to find D.

- If we know D and r and need to find t, we would solve the equation for t:

$$t = \frac{D}{r}.$$

Example

- Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link's biking speed?

  Suppose Link's biking speed is x.

	D	r	t
Head Drive	20	$x-3$	t
Tail Drive	30	$x+3$	t

  Since times are the same, from $t=\frac{D}{r}$ we get: $\frac{20}{x-3}=\frac{30}{x+3}$.

$$\begin{aligned}(x-3)(x+3)\tfrac{20}{x-3} &= (x-3)(x+3)\tfrac{30}{x+3}\\ 20(x+3) &= 30(x-3)\\ 20x+60 &= 30x-90\\ 150 &= 10x\\ 15 &= x.\end{aligned}$$

  Hence, Link's biking speed is 15 mph.

Example

- Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis' speed going uphill and his speed going downhill?

  Suppose Dennis' speed going uphill was x.

	D	r	t
Uphill	20	x	t
Downhill	20	$x+5$	$6-t$

  From $t = \frac{D}{r}$ we get $t = \frac{20}{x}$ and $6 - t = \frac{20}{x+5}$.
  Combining the two gives

$$6 - \frac{20}{x} = \frac{20}{x+5}.$$

Example (Cont'd)

- We now solve

$$\begin{aligned}
6-\frac{20}{x}&=\frac{20}{x+5}\\
x(x+5)(6-\frac{20}{x})&=x(x+5)\frac{20}{x+5}\\
6x(x+5)-20(x+5)&=20x\\
6x^2+30x-20x-100&=20x\\
6x^2-10x-100&=0\\
2(3x^2-5x-50)&=0\\
2(3x^2-15x+10x-50)&=0\\
2[3x(x-5)+10(x-5)]&=0\\
2(3x+10)(x-5)&=0\\
x=-\tfrac{10}{3} \quad \text{or} \quad x&=5.
\end{aligned}$$

Hence, Dennis' speed going uphill was 5mph and his speed going downhill was 10mph.

Solve Work Applications

- Follow the steps:
  - Tabulate the data as follows:

	Hours to Complete Job	Part of Job Completed/Hour
Worker A	3	$\frac{1}{3}$
Worker B	5	$\frac{1}{5}$
Together	t	$\frac{1}{t}$

  - Set up the equation

$$\begin{array}{ccccc} \text{Work Completed} & & \text{Work Complete by} & & \text{Work Completed by} \\ \text{Together/Hour} & = & \text{Worker A/Hour} & + & \text{Worker B/Hour} \\ \dfrac{1}{t} & = & \dfrac{1}{3} & + & \dfrac{1}{5} \end{array}$$

  - Solve for t.

Example

- One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course?

	Hours/Job	Jobs/Hour
Gardener A	4	$\frac{1}{4}$
Gardener B	6	$\frac{1}{6}$
Together	t	$\frac{1}{t}$

$$\begin{aligned} \frac{1}{t} &= \frac{1}{4} + \frac{1}{6} \\ 12t\frac{1}{t} &= 12t\left(\frac{1}{4} + \frac{1}{6}\right) \\ 12 &= 3t + 2t \\ 12 &= 5t \\ t &= \frac{12}{5}. \end{aligned}$$

So it would take them 2 hours and 24 minutes working together.

Example

- Alice can paint a room in 6 hours. If Kristina helps her it takes them 4 hours to paint the room. How long would it take Kristina to paint the room by herself?

	Hours/Job	Jobs/Hour
Alice	6	$\frac{1}{6}$
Kristina	t	$\frac{1}{t}$
Together	4	$\frac{1}{4}$

$$\frac{1}{4} = \frac{1}{6} + \frac{1}{t}$$
$$12t\frac{1}{4} = 12t(\frac{1}{6} + \frac{1}{t})$$
$$3t = 2t + 12$$
$$t = 12.$$

So it would take Kristina 12 hours to paint the room by herself.

Solve Direct Variation Problems

- For any two variables x and y, y **varies directly with** x if

$$y = kx, \quad k \neq 0.$$

  The constant k is called the **constant of variation**.
- To solve direct variation problems follow the steps:
  - Write the formula for direct variation;
  - Substitute the given values for the variables;
  - Solve for the constant of variation;
  - Write the equation that relates x and y using the constant of variation.

Example

- The number of calories, c, burned varies directly with the amount of time, t, spent exercising. Arnold burned 312 calories in 65 minutes exercising. (a) Write the equation that relates c and t. (b) How many calories would he burn if he exercises for 90 minutes?

(a) Let k be the constant of variation.

$$c = kt$$
$$312 = k \cdot 65$$
$$k = \frac{312}{65} = \frac{24}{5} \text{ cal/min.}$$

Thus, we have $c = \frac{24}{5}t$.

(b) Now, for a $t = 90$ minute exercise, we get

$$c = \frac{24}{5} \cdot 90 = 432 \text{ cal.}$$

Solve Inverse Variation Problems

- For any two variables x and y, y **varies inversely with** x if

$$y = \frac{k}{x}, \quad k \neq 0.$$

  The constant k is called the **constant of variation**.
- To solve inverse variation problems follow the steps:
  - Write the formula for inverse variation;
  - Substitute the given values for the variables;
  - Solve for the constant of variation;
  - Write the equation that relates x and y using the constant of variation.

Example

- The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees Celsius. (a) Write the equation of variation. (b) How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

(a) Let t be the time in hours, T the air temperature in degrees Celsius and k be the constant of variation.

$$t = \frac{k}{T}$$
$$2 = \frac{k}{65}$$
$$k = 2 \cdot 65 = 130^{\circ}\mathrm{C} \cdot \mathrm{hr}.$$

Thus, we have $t = \frac{130}{T}$.

(b) Now, for $T = 78^{\circ}\mathrm{C}$, we get

$$t = \frac{130}{78} = \frac{5}{3}.$$

Thus, it would take 1 hour and 40 minutes for the ice to melt.

Subsection 7

Solve Rational Inequalities

We Shall Learn and Practice

- Solve rational inequalities.
- Solve an inequality with rational functions.

Solve Rational Inequalities

- To solve a rational inequality follow the steps:
  - Write the inequality as one quotient on the left and zero on the right;
  - Determine the critical points - the points where the rational expression will be zero or undefined;
  - Use the critical points to divide the number line into intervals;
  - Test a value in each interval to determine the sign of each factor of the numerator and denominator in each interval and then determine the sign of the quotient;
  - Determine the intervals where the inequality is correct;
  - Write the solution in interval notation.

Example

- Solve and write the solution in interval notation: $\frac{x-5}{x+3} \geq 0$.

$$\frac{x-5}{x+3} \geq 0, \quad x \neq -3$$

Find the critical points:

- $x-5=0 \Rightarrow x=5$;
- $x+3=0 \Rightarrow x=-3$.

Create the sign table:

	$(-\infty,-3)$	$(-3,5)$	$(5,+\infty)$
$x-5$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$
$\frac{x-5}{x+3}$	$+$	$-$	$+$

Since we want $\frac{x-5}{x+3} \geq 0$, we need x in $(-\infty,-3) \cup [5,+\infty)$.

Example

- Solve and write the solution in interval notation: $\frac{3x}{x-3} < 1$.

$$\frac{3x}{x-3} < 1 \Rightarrow \frac{3x}{x-3} - 1 < 0 \Rightarrow \frac{3x}{x-3} - \frac{x-3}{x-3} < 0$$
$$\Rightarrow \frac{3x-(x-3)}{x-3} < 0 \Rightarrow \frac{3x-x+3}{x-3} < 0 \Rightarrow \frac{2x+3}{x-3} < 0.$$

Critical points: Numerator: $x = -\frac{3}{2}$; Denominator: $x = 3$.

Create the sign table:

	$(-\infty, -\frac{3}{2})$	$(-\frac{3}{2}, 3)$	$(3, +\infty)$
$2x+3$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
$\frac{2x+3}{x-3}$	$+$	$-$	$+$

Since we want $\frac{2x+3}{x-3} < 0$, we need x in $(-\frac{3}{2}, 3)$.

Example

- Solve and write the solution in interval notation: $\frac{1}{x^2+2x-8} > 0$.

$$\frac{1}{x^2+2x-8} > 0 \Rightarrow \frac{1}{(x-2)(x+4)} > 0, \quad x \neq -4, 2.$$

Critical points: Numerator: None; Denominator: $x=-4$ and $x=2$

Create the sign table:

	$(-\infty,-4)$	$(-4,2)$	$(2,+\infty)$
$x-2$	$-$	$-$	$+$
$x+4$	$-$	$+$	$+$
$\frac{1}{(x-2)(x+4)}$	$+$	$-$	$+$

Since we want $\frac{1}{(x-2)(x+4)} > 0$, we need x in $(-\infty,-4) \cup (2,+\infty)$.

Example

- Solve and write the solution in interval notation: $\frac{1}{2}+\frac{4}{x^2}<\frac{3}{x}$.

$$\frac{1}{2}+\frac{4}{x^2}<\frac{3}{x} \Rightarrow \frac{1}{2}+\frac{4}{x^2}-\frac{3}{x}<0 \Rightarrow \frac{x^2}{2x^2}+\frac{8}{2x^2}-\frac{6x}{2x^2}<0$$
$$\Rightarrow \frac{x^2+8-6x}{2x^2}<0 \Rightarrow \frac{x^2-6x+8}{2x^2}<0 \Rightarrow \frac{(x-4)(x-2)}{2x^2}<0.$$

Critical points: Numerator: $x=2$ and $x=4$; Denominator: $x=0$.

Create the sign table:

	$(-\infty,0)$	$(0,2)$	$(2,4)$	$(4,+\infty)$
$x-4$	$-$	$-$	$-$	$+$
$x-2$	$-$	$-$	$+$	$+$
$\frac{(x-4)(x-2)}{2x^2}$	$+$	$+$	$-$	$+$

Since we want $\frac{(x-4)(x-2)}{2x^2}<0$, we need x in $(2,4)$.

Solve an Inequality with Rational Functions

- Given the function $R(x) = \frac{x-2}{x+4}$, find the values of x that make the function less than or equal to 0.

$$\frac{x-2}{x+4} \le 0, \quad x \neq -4$$

Find the critical points:
  - $x - 2 = 0 \ \Rightarrow \ x = 2;$
  - $x + 4 = 0 \ \Rightarrow \ x = -4.$

Create the sign table:

	$(-\infty, -4)$	$(-4, 2)$	$(2, +\infty)$
$x-2$	$-$	$-$	$+$
$x+4$	$-$	$+$	$+$
$\frac{x-2}{x+4}$	$+$	$-$	$+$

Since we want $\frac{x-2}{x+4} \le 0$, we need x in $(-4, 2]$.

Example

- The function $C(x) = 20x + 6000$ represents the cost to produce x, number of items. Find (a) the average cost function, $c(x)$. (b) how many items should be produced so that the average cost is less than \$60?

(a) $c(x) = \frac{C(x)}{x} = \frac{20x+6000}{x}$.

(b) $c(x) < 60 \Rightarrow \frac{20x+6000}{x} < 60 \Rightarrow \frac{20x+6000}{x} - 60 < 0 \Rightarrow \frac{20x+6000}{x} - \frac{60x}{x} < 0 \Rightarrow \frac{20x+6000-60x}{x} < 0 \Rightarrow \frac{-40x+6000}{x} < 0 \Rightarrow \frac{-40(x-150)}{x} < 0$.

Critical points: Numerator: $x = 150$; Denominator: $x = 0$.

Create the sign table:

	$(-\infty, 0)$	$(0, 150)$	$(150, +\infty)$
$x - 150$	$-$	$-$	$+$
x	$-$	$+$	$+$
$\frac{-40(x-150)}{x}$	$-$	$+$	$-$

Since we want $\frac{-40(x-150)}{x} < 0$ (and $x \geq 0$!!), we need x in $(150, +\infty)$.