Intermediate Algebra

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LSSU Math 102

1. Roots and Radicals
   - Simplify Expressions with Roots
   - Simplify Radical Expressions
   - Simplify Rational Exponents
   - Add, Subtract, and Multiply Radical Expressions
   - Divide Radical Expressions
   - Solve Radical Equations
   - Use Radicals in Functions
   - Use the Complex Number System

Subsection 1

Simplify Expressions with Roots

We Shall Learn and Practice

- Simplify expressions with roots.
- Estimate and approximate roots.
- Simplify variable expressions with roots.

Simplify Expressions with Roots

- If $n^2 = m$, then m is the **square** of n.
- If $n^2 = m$, then n is a **square root** of m.
- Note that $3^2 = (-3)^2 = 9$.

  So both -3 and 3 are square roots of 9.
- What if we only wanted the positive square root of a positive number?
- We use a radical sign, and write,

  $$\sqrt{m},$$

  which denotes the **positive square root** of m.
- The positive square root is also called the **principal square root**.

Example

- Simplify: (a) $-\sqrt{64}$ (b) $\sqrt{225}$.

(a) $-\sqrt{64} = -8$ (because $8^2 = 64$)

(b) $\sqrt{225} = 15$ (because $15^2 = 225$)

Example

- Simplify: (a) $\sqrt{-169}$ (b) $-\sqrt{81}$.

(a) $\sqrt{-169}$ is not a real number;

(b) $-\sqrt{81} = -9$.

n-th Roots

- If $b^n = a$, then b is an ***n*-th root** of a.
- The **principal *n*-th root** of a is written $\sqrt[n]{a}$.
- n is called the **index** of the radical.
- Properties of $\sqrt[n]{a}$:
  - When n is an even number and
    - $a \geq 0$, then $\sqrt[n]{a}$ is a real number;
    - $a < 0$, then $\sqrt[n]{a}$ is not a real number.
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

Example

- Simplify: (a) $\sqrt[3]{27}$ (b) $\sqrt[4]{256}$ (c) $\sqrt[5]{243}$

(a) $\sqrt[3]{27} = 3$; $\quad (3^3 = 27)$

(b) $\sqrt[4]{256} = 4$; $\quad (4^4 = 256)$

(c) $\sqrt[5]{243} = 3$. $\quad (3^5 = 243)$

Example

- Simplify: (a) $\sqrt[3]{-27}$ (b) $\sqrt[4]{-256}$ (c) $\sqrt[5]{-32}$.

(a) $\sqrt[3]{-27} = -3$; $((-3)^3 = -27)$

(b) $\sqrt[4]{-256}$ is not a real number; $(?^4 = -256)$

(c) $\sqrt[5]{-32} = -2$. $((-2)^5 = -32)$

Estimate and Approximate Roots

- Idea of Estimation:

Number	Square Root
4	2
9	3
16	4
25	5

Number	Cube Root
8	2
27	3
64	4
125	5

- Suppose we want to estimate $\sqrt{11}$.
  - Locate 11: $9 < 11 < 16$;
  - So $3 < \sqrt{11} < 4$.
- Suppose we want to estimate $\sqrt[3]{91}$.
  - Locate 91: $64 < 91 < 125$;
  - So $4 < \sqrt[3]{91} < 5$.

Example

- Estimate each root between two consecutive whole numbers: (a) $\sqrt{38}$ (b) $\sqrt[3]{93}$

  Here are the tables again

Number	Square Root
16	4
25	5
36	6
49	7

Number	Cube Root
8	2
27	3
64	4
125	5

(a) For $\sqrt{38}$:
- Locate 38: $36 < 38 < 49$;
- So $6 < \sqrt{38} < 7$.

(b) For $\sqrt[3]{93}$:
- Locate 93: $64 < 93 < 125$;
- So $4 < \sqrt[3]{93} < 5$.

Simplify Variable Expressions with Roots

- Simplifying Odd and Even Roots:

  For any integer $n \geq 2$,
  - when the index n is odd $\sqrt[n]{a^n} = a$;
  - when the index n is even $\sqrt[n]{a^n} = |a|$.

  We must use the absolute value signs when we take an even root of an expression with a variable in the radical.
- To understand why look at the following:

$$\sqrt{(-3)^2} = \sqrt{9} = 3.$$

  So writing
  - $\sqrt{(-3)^2} = (-3)$ is wrong;
  - $\sqrt{(-3)^2} = |-3|$ is right!

Example

- Simplify: (a) $\sqrt{b^2}$ (b) $\sqrt[3]{w^3}$ (c) $\sqrt[4]{m^4}$ (d) $\sqrt[5]{q^5}$.

(a) $\sqrt{b^2} = |b|$;

(b) $\sqrt[3]{w^3} = w$;

(c) $\sqrt[4]{m^4} = |m|$;

(d) $\sqrt[5]{q^5} = q$.

Example

- Simplify: (a) $\sqrt{y^{18}}$ (b) $\sqrt{z^{12}}$.

(a) $\sqrt{y^{18}} = \sqrt{(y^9)^2} = |y^9|$;

(b) $\sqrt{z^{12}} = \sqrt{(z^6)^2} = |z^6| = z^6$.

Example

- Simplify: (a) $\sqrt[4]{u^{12}}$ (b) $\sqrt[3]{v^{15}}$.

(a) $\sqrt[4]{u^{12}} = \sqrt[4]{(u^3)^4} = |u^3|$;

(b) $\sqrt[3]{v^{15}} = \sqrt[3]{(v^5)^3} = v^5$.

Example

- Simplify: (a) $\sqrt{64x^2}$ (b) $-\sqrt{100p^2}$.

(a) $\sqrt{64x^2} = 8|x|$;

(b) $-\sqrt{100p^2} = -10|p|$.

Example

- Simplify: (a) $\sqrt[3]{27x^{27}}$ (b) $\sqrt[4]{81q^{28}}$.

(a) $\sqrt[3]{27x^{27}} = \sqrt[3]{27(x^9)^3} = 3x^9$;

(b) $\sqrt[4]{81q^{28}} = \sqrt[4]{81(q^7)^4} = 3|q^7|$.

Example

- Simplify: (a) $\sqrt{100a^2b^2}$ (b) $\sqrt{144p^{12}q^{20}}$ (c) $\sqrt[3]{8x^{30}y^{12}}$.

(a) $\sqrt{100a^2b^2} = 10|a||b|$;

(b) $\sqrt{144p^{12}q^{20}} = \sqrt{144(p^6)^2(q^{10})^2} = 12|p^6||q^{10}| = 12p^6q^{10}$;

(c) $\sqrt[3]{8x^{30}y^{12}} = \sqrt[3]{8(x^{10})^3(y^4)^3} = 2x^{10}y^4$.

Subsection 2

Simplify Radical Expressions

We Shall Learn and Practice

- Use the Product Property to simplify radical expressions.
- Use the Quotient Property to simplify radical expressions.

Use the Product Property to Simplify Radicals

- Follow the steps:
  - Find the largest factor in the radicand that is a perfect power of the index;
  - Rewrite the radicand as a product of two factors, using that factor;
  - Use the product rule
    $$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$
    to rewrite the radical as the product of two radicals;
  - Simplify the root of the perfect power.

Example

- Simplify: $\sqrt{48}$.

$$\begin{aligned}\sqrt{48} &= \sqrt{16 \cdot 3} \\ &= \sqrt{16}\sqrt{3} \\ &= 4\sqrt{3}.\end{aligned}$$

Example

- Simplify: (a) $\sqrt{288}$ (b) $\sqrt[3]{81}$ (c) $\sqrt[4]{64}$.

(a)

$$\sqrt{288} = \sqrt{144 \cdot 2} = \sqrt{144}\sqrt{2} = 12\sqrt{2};$$

(b)

$$\sqrt[3]{81} = \sqrt[3]{27 \cdot 3} = \sqrt[3]{27}\sqrt[3]{3} = 3\sqrt[3]{3};$$

(c)

$$\sqrt[4]{64} = \sqrt[4]{16 \cdot 4} = \sqrt[4]{16}\sqrt[4]{4} = 2\sqrt[4]{4}.$$

Example

- Simplify: (a) $\sqrt{b^5}$ (b) $\sqrt[4]{y^6}$ (c) $\sqrt[3]{z^5}$.

(a)

$$\sqrt{b^5} = \sqrt{b^4 b} = \sqrt{(b^2)^2}\sqrt{b} = |b^2|\sqrt{b} = b^2\sqrt{b};$$

(b)

$$\sqrt[4]{y^6} = \sqrt[4]{y^4 y^2} = \sqrt[4]{y^4}\sqrt[4]{y^2} = |y|\sqrt[4]{y^2};$$

(c)

$$\sqrt[3]{z^5} = \sqrt[3]{z^3 z^2} = \sqrt[3]{z^3}\sqrt[3]{z^2} = z\sqrt[3]{z^2}.$$

Example

- Simplify: (a) $\sqrt{32y^5}$ (b) $\sqrt[3]{54p^{10}}$ (c) $\sqrt[4]{64q^{10}}$.

(a)

$$\sqrt{32y^5} = \sqrt{16y^4 \cdot 2y} = \sqrt{16(y^2)^2}\sqrt{2y} = 4|y^2|\sqrt{2y} = 4y^2\sqrt{2y};$$

(b)

$$\sqrt[3]{54p^{10}} = \sqrt[3]{27p^9 \cdot 2p} = \sqrt[3]{27(p^3)^3}\sqrt[3]{2p} = 3p^3\sqrt[3]{2p};$$

(c)

$$\sqrt[4]{64q^{10}} = \sqrt[4]{16q^8 \cdot 4q^2} = \sqrt[4]{16(q^2)^4}\sqrt[4]{4q^2} = 2|q^2|\sqrt[4]{4q^2} = 2q^2\sqrt[4]{4q^2}.$$

Example

- Simplify: (a) $\sqrt{98a^7b^5}$ (b) $\sqrt[3]{56x^5y^4}$ (c) $\sqrt[4]{32x^5y^8}$.

(a)

$$\begin{aligned}\sqrt{98a^7b^5} &= \sqrt{49a^6b^4 \cdot 2ab} = \sqrt{49(a^3)^2(b^2)^2}\sqrt{2ab} \\ &= 7|a^3||b^2|\sqrt{2ab} = 7|a^3|b^2\sqrt{2ab};\end{aligned}$$

(b)

$$\sqrt[3]{56x^5y^4} = \sqrt[3]{8x^3y^3 \cdot 7x^2y} = \sqrt[3]{8x^3y^3}\sqrt[3]{7x^2y} = 2xy\sqrt[3]{7x^2y};$$

(c)

$$\begin{aligned}\sqrt[4]{32x^5y^8} &= \sqrt[4]{16x^4y^8 \cdot 2x} = \sqrt[4]{16x^4(y^2)^4}\sqrt[4]{2x} \\ &= 2|x||y^2|\sqrt[4]{2x} = 2|x|y^2\sqrt[4]{2x}.\end{aligned}$$

Example

- Simplify: (a) $5+\sqrt{75}$ (b) $\frac{10-\sqrt{75}}{5}$

(a)

$$5+\sqrt{75}=5+\sqrt{25\cdot 3}=5+\sqrt{25}\sqrt{3}=5+5\sqrt{3};$$

(b)

$$\begin{aligned}\frac{10-\sqrt{75}}{5} &= \frac{10-\sqrt{25\cdot 3}}{5}=\frac{10-\sqrt{25}\sqrt{3}}{5} \\ &= \frac{10-5\sqrt{3}}{5}=\frac{5(2-\sqrt{3})}{5} \\ &= 2-\sqrt{3}.\end{aligned}$$

Use the Quotient Property to Simplify Radicals

- Follow the steps:
  - Simplify the fraction in the radicand, if possible;
  - Use the Quotient Property

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

    to rewrite the radical as the quotient of two radicals;
  - Simplify the radicals in the numerator and the denominator.

Example

- Simplify: (a) $\sqrt{\frac{75}{48}}$ (b) $\sqrt[3]{\frac{54}{250}}$ (c) $\sqrt[4]{\frac{32}{162}}$.

(a)

$$\sqrt{\frac{75}{48}} = \sqrt{\frac{3 \cdot 25}{3 \cdot 16}} = \sqrt{\frac{25}{16}} = \frac{\sqrt{25}}{\sqrt{16}} = \frac{5}{4};$$

(b)

$$\sqrt[3]{\frac{54}{250}} = \sqrt[3]{\frac{2 \cdot 27}{2 \cdot 125}} = \sqrt[3]{\frac{27}{125}} = \frac{\sqrt[3]{27}}{\sqrt[3]{125}} = \frac{3}{5};$$

(c)

$$\sqrt[4]{\frac{32}{162}} = \sqrt[4]{\frac{2 \cdot 16}{2 \cdot 81}} = \sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{2}{3}.$$

Example

- Simplify: (a) $\sqrt{\frac{a^8}{a^6}}$ (b) $\sqrt[4]{\frac{x^7}{x^3}}$ (c) $\sqrt[4]{\frac{y^{17}}{y^5}}$.

(a)

$$\sqrt{\frac{a^8}{a^6}} = \sqrt{a^2} = |a|;$$

(b)

$$\sqrt[4]{\frac{x^7}{x^3}} = \sqrt[4]{x^4} = |x|;$$

(c)

$$\sqrt[4]{\frac{y^{17}}{y^5}} = \sqrt[4]{y^{12}} = \sqrt[4]{(y^3)^4} = |y^3|.$$

Example

- Simplify: $\sqrt{\frac{24p^3}{49}}$.

$$\begin{aligned}\sqrt{\frac{24p^3}{49}} &= \frac{\sqrt{24p^3}}{\sqrt{49}} \\ &= \frac{\sqrt{4p^2 \cdot 6p}}{7} \\ &= \frac{\sqrt{4p^2}\sqrt{6p}}{7} \\ &= \frac{2|p|\sqrt{6p}}{7}.\end{aligned}$$

Example

- Simplify: (a) $\sqrt{\frac{80m^3}{n^6}}$ (b) $\sqrt[3]{\frac{108c^{10}}{d^6}}$ (c) $\sqrt[4]{\frac{80x^{10}}{y^4}}$.

(a)

$$\begin{aligned}\sqrt{\frac{80m^3}{n^6}} &= \frac{\sqrt{80m^3}}{\sqrt{n^6}} = \frac{\sqrt{16m^2 \cdot 5m}}{\sqrt{n^6}} \\ &= \frac{\sqrt{16m^2}\sqrt{5m}}{\sqrt{(n^3)^2}} = \frac{4|m|\sqrt{5m}}{|n^3|};\end{aligned}$$

(b)

$$\begin{aligned}\sqrt[3]{\frac{108c^{10}}{d^6}} &= \frac{\sqrt[3]{108c^{10}}}{\sqrt[3]{d^6}} = \frac{\sqrt[3]{27c^9 \cdot 4c}}{\sqrt[3]{d^6}} \\ &= \frac{\sqrt[3]{27(c^3)^3}\sqrt[3]{4c}}{\sqrt[3]{(d^2)^3}} = \frac{3c^3\sqrt[3]{4c}}{d^2};\end{aligned}$$

(c)

$$\begin{aligned}\sqrt[4]{\frac{80x^{10}}{y^4}} &= \frac{\sqrt[4]{80x^{10}}}{\sqrt[4]{y^4}} = \frac{\sqrt[4]{16x^8 \cdot 5x^2}}{\sqrt[4]{y^4}} = \frac{\sqrt[4]{16(x^2)^4}\sqrt[4]{5x^2}}{\sqrt[4]{y^4}} \\ &= \frac{2|x^2|\sqrt[4]{5x^2}}{|y|} = \frac{2x^2\sqrt[4]{5x^2}}{|y|}.\end{aligned}$$

Example

- Simplify: (a) $\sqrt{\frac{50x^5y^3}{72x^4y}}$ (b) $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$ (c) $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$.

(a)

$$\begin{aligned}\sqrt{\frac{50x^5y^3}{72x^4y}} &= \sqrt{\frac{25xy^2}{36}} = \frac{\sqrt{25xy^2}}{\sqrt{36}} \\ &= \frac{\sqrt{25y^2}\sqrt{x}}{\sqrt{36}} = \frac{5|y|\sqrt{x}}{6};\end{aligned}$$

(b)

$$\begin{aligned}\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}} &= \sqrt[3]{\frac{8x^3y^5}{27}} = \frac{\sqrt[3]{8x^3y^3y^2}}{\sqrt[3]{27}} \\ &= \frac{\sqrt[3]{8x^3y^3}\,\sqrt[3]{y^2}}{\sqrt[3]{27}} = \frac{2xy\,\sqrt[3]{y^2}}{3};\end{aligned}$$

(c)

$$\begin{aligned}\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}} &= \sqrt[4]{\frac{a^5b^4}{16}} = \frac{\sqrt[4]{a^4b^4a}}{\sqrt[4]{16}} \\ &= \frac{\sqrt[4]{a^4b^4}\,\sqrt[4]{a}}{\sqrt[4]{16}} = \frac{|a||b|\,\sqrt[4]{a}}{2}.\end{aligned}$$

Example

- Simplify: (a) $\frac{\sqrt{98z^5}}{\sqrt{2z}}$ (b) $\frac{\sqrt[3]{-500}}{\sqrt[3]{2}}$ (c) $\frac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$.

(a)

$$\frac{\sqrt{98z^5}}{\sqrt{2z}} = \sqrt{\frac{98z^5}{2z}} = \sqrt{49z^4} = \sqrt{49(z^2)^2} = 7|z^2| = 7z^2;$$

(b)

$$\frac{\sqrt[3]{-500}}{\sqrt[3]{2}} = \sqrt[3]{\frac{-500}{2}} = \sqrt[3]{-250} = \sqrt[3]{-125\cdot 2} = \sqrt[3]{-125}\sqrt[3]{2} = -5\sqrt[3]{2};$$

(c)

$$\frac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}} = \sqrt[4]{\frac{486m^{11}}{3m^5}} = \sqrt[4]{162m^6} = \sqrt[4]{81m^4\cdot 2m^2} = \sqrt[4]{81m^4}\sqrt[4]{2m^2} = 3|m|\sqrt[4]{2m^2}.$$

Subsection 3

Simplify Rational Exponents

We Shall Learn and Practice

- Simplify expressions with $a^{\frac{1}{n}}$;
- Simplify expressions with $a^{\frac{m}{n}}$;
- Use the properties of exponents to simplify expressions with rational exponents.

Simplify Expressions With $a^{\frac{1}{n}}$

- If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

- If $\sqrt[n]{a}$ is a non zero real number and $n \geq 2$, then

$$a^{-\frac{1}{n}} = \frac{1}{a^{\frac{1}{n}}} = \frac{1}{\sqrt[n]{a}}.$$

Example

- Write as a radical expression: (a) $t^{\frac{1}{2}}$ (b) $m^{\frac{1}{3}}$ (c) $r^{\frac{1}{4}}$

(a)

$$t^{\frac{1}{2}} = \sqrt{t};$$

(b)

$$m^{\frac{1}{3}} = \sqrt[3]{m};$$

(c)

$$r^{\frac{1}{4}} = \sqrt[4]{r}.$$

Example

- Write with a rational exponent: (a) $\sqrt{10m}$ (b) $\sqrt[5]{3n}$ (c) $3\sqrt[4]{6y}$

(a)

$$\sqrt{10m} = (10m)^{\frac{1}{2}};$$

(b)

$$\sqrt[5]{3n} = (3n)^{\frac{1}{5}};$$

(c)

$$3\sqrt[4]{6y} = 3(6y)^{\frac{1}{4}}.$$

Example

- Simplify: (a) $36^{\frac{1}{2}}$ (b) $8^{\frac{1}{3}}$ (c) $16^{\frac{1}{4}}$

(a)

$$36^{\frac{1}{2}} = \sqrt{36} = 6;$$

(b)

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2;$$

(c)

$$16^{\frac{1}{4}} = \sqrt[4]{16} = 2.$$

Example

- Simplify: (a) $(-64)^{-\frac{1}{2}}$ (b) $-64^{\frac{1}{2}}$ (c) $(64)^{-\frac{1}{2}}$

(a)

$$(-64)^{-\frac{1}{2}} = \frac{1}{(-64)^{\frac{1}{2}}} = \frac{1}{\sqrt{-64}} \quad \text{Not a real number;}$$

(b)

$$-64^{\frac{1}{2}} = -\sqrt{64} = -8;$$

(c)

$$(64)^{-\frac{1}{2}} = \frac{1}{64^{\frac{1}{2}}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

Simplify Expressions With $a^{\frac{m}{n}}$

- For any positive integers m and n,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

- The m-th power of the n-th root of a is equal to the n-th root of the m-th power of a.

Example

- Write with a rational exponent: (a) $\sqrt{x^5}$ (b) $(\sqrt[4]{3y})^3$ (c) $\sqrt{(\frac{2m}{3n})^5}$.

(a)

$$\sqrt{x^5} = x^{\frac{5}{2}};$$

(b)

$$(\sqrt[4]{3y})^3 = (3y)^{\frac{3}{4}};$$

(c)

$$\sqrt{\left(\frac{2m}{3n}\right)^5} = \left(\frac{2m}{3n}\right)^{\frac{5}{2}}.$$

Example

- Simplify: (a) $27^{\frac{2}{3}}$ (b) $81^{-\frac{3}{2}}$ (c) $16^{-\frac{3}{4}}$

(a)

$$27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9;$$

(b)

$$81^{-\frac{3}{2}} = \frac{1}{81^{\frac{3}{2}}} = \frac{1}{(\sqrt{81})^3} = \frac{1}{9^3} = \frac{1}{729};$$

(c)

$$16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}} = \frac{1}{(\sqrt[4]{16})^3} = \frac{1}{2^3} = \frac{1}{8}.$$

Example

- Simplify: (a) $-16^{\frac{3}{2}}$ (b) $-16^{-\frac{3}{2}}$ (c) $(-16)^{-\frac{3}{2}}$

(a)

$$-16^{\frac{3}{2}} = -(\sqrt{16})^3 = -4^3 = -64;$$

(b)

$$-16^{-\frac{3}{2}} = -\frac{1}{16^{\frac{3}{2}}} = -\frac{1}{(\sqrt{16})^3} = -\frac{1}{4^3} = -\frac{1}{64};$$

(c)

$$(-16)^{-\frac{3}{2}} = \frac{1}{(-16)^{\frac{3}{2}}} = \frac{1}{(\sqrt{-16})^3} \quad \text{Not a real number.}$$

Simplify Expressions With Rational Exponents

- If a and b are real numbers and m and n are rational numbers, then:

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$
Zero Exponent Definition	$a^0 = 1, \quad a \neq 0$
Quotient to a Power Property	$(\frac{a}{b})^m = \frac{a^m}{b^m}, \quad b \neq 0$
Negative Exponent Property	$a^{-n} = \frac{1}{a^n}, \quad a \neq 0$

Example

- Simplify: (a) $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$ (b) $(x^6)^{\frac{4}{3}}$ (c) $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$

(a)

$$x^{\frac{1}{6}} \cdot x^{\frac{4}{3}} = x^{\frac{1}{6}+\frac{4}{3}} = x^{\frac{1}{6}+\frac{8}{6}} = x^{\frac{9}{6}} = x^{\frac{3}{2}};$$

(b)

$$(x^6)^{\frac{4}{3}} = x^{6\cdot\frac{4}{3}} = x^8;$$

(c)

$$\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}} = x^{\frac{2}{3}-\frac{5}{3}} = x^{-\frac{3}{3}} = x^{-1}.$$

Example

- Simplify: (a) $(32x^{\frac{1}{3}})^{\frac{3}{5}}$ (b) $(x^{\frac{3}{4}}y^{\frac{1}{2}})^{\frac{2}{3}}$

(a)

$$(32x^{\frac{1}{3}})^{\frac{3}{5}} = 32^{\frac{3}{5}}(x^{\frac{1}{3}})^{\frac{3}{5}} = (\sqrt[5]{32})^3 x^{\frac{1}{3}\cdot\frac{3}{5}} = 2^3 x^{\frac{1}{5}} = 8x^{\frac{1}{5}};$$

(b)

$$(x^{\frac{3}{4}}y^{\frac{1}{2}})^{\frac{2}{3}} = (x^{\frac{3}{4}})^{\frac{2}{3}}(y^{\frac{1}{2}})^{\frac{2}{3}} = x^{\frac{3}{4}\cdot\frac{2}{3}}y^{\frac{1}{2}\cdot\frac{2}{3}} = x^{\frac{1}{2}}y^{\frac{1}{3}}.$$

Example

- Simplify: (a) $\dfrac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$ (b) $\left(\dfrac{25m^{\frac{1}{6}}n^{\frac{11}{6}}}{m^{\frac{2}{3}}n^{-\frac{1}{6}}}\right)^{\frac{1}{2}}$

(a)

$$\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}} = \frac{m^{\frac{2}{3}-\frac{1}{3}}}{m^{-\frac{5}{3}}} = \frac{m^{\frac{1}{3}}}{m^{-\frac{5}{3}}} = m^{\frac{1}{3}-(-\frac{5}{3})} = m^{\frac{6}{3}} = m^2;$$

(b)

$$\begin{aligned}\left(\frac{25m^{\frac{1}{6}}n^{\frac{11}{6}}}{m^{\frac{2}{3}}n^{-\frac{1}{6}}}\right)^{\frac{1}{2}} &= (25m^{\frac{1}{6}-\frac{2}{3}}n^{\frac{11}{6}-(-\frac{1}{6})})^{\frac{1}{2}} = (25m^{-\frac{3}{6}}n^{\frac{12}{6}})^{\frac{1}{2}} \\ &= 25^{\frac{1}{2}}(m^{-\frac{1}{2}})^{\frac{1}{2}}(n^2)^{\frac{1}{2}} = \sqrt{25}m^{-\frac{1}{2}\cdot\frac{1}{2}}n^{2\cdot\frac{1}{2}} \\ &= 5m^{-\frac{1}{4}}n.\end{aligned}$$

Subsection 4

Add, Subtract, and Multiply Radical Expressions

We Shall Learn and Practice

- Add and subtract radical expressions.
- Multiply radical expressions.
- Use polynomial multiplication to multiply radical expressions.

Add and Subtract Radical Expressions

- Adding radical expressions with the same index and the same radicand is just like adding like terms.
- We call radicals with the same index and the same radicand **like radicals**.

Example

- Simplify: (a) $8\sqrt{2} - 9\sqrt{2}$ (b) $4\sqrt[3]{x} + 7\sqrt[3]{x}$ (c) $3\sqrt[4]{x} - 5\sqrt[4]{y}$.

(a)

$$8\sqrt{2} - 9\sqrt{2} = -\sqrt{2};$$

(b)

$$4\sqrt[3]{x} + 7\sqrt[3]{x} = 11\sqrt[3]{x};$$

(c)

$3\sqrt[4]{x} - 5\sqrt[4]{y}$ Not simplifiable; not like radicals.

Example

- Simplify: (a) $\sqrt{7x} - 7\sqrt{7x} + 4\sqrt{7x}$ (b) $4\sqrt[4]{5xy} + 2\sqrt[4]{5xy} - 7\sqrt[4]{5xy}$.

(a)

$$\sqrt{7x} - 7\sqrt{7x} + 4\sqrt{7x} = -6\sqrt{7x} + 4\sqrt{7x} = -2\sqrt{7x};$$

(b)

$$4\sqrt[4]{5xy} + 2\sqrt[4]{5xy} - 7\sqrt[4]{5xy} = 6\sqrt[4]{5xy} - 7\sqrt[4]{5xy} = -\sqrt[4]{5xy}.$$

Example

- Simplify: (a) $\sqrt{18}+6\sqrt{2}$ (b) $6\sqrt[3]{16}-2\sqrt[3]{250}$ (c) $\frac{2}{3}\sqrt[3]{81}-\frac{1}{2}\sqrt[3]{24}$.

(a)

$$\sqrt{18}+6\sqrt{2}=\sqrt{9\cdot 2}+6\sqrt{2}=\sqrt{9}\sqrt{2}+6\sqrt{2}$$
$$=3\sqrt{2}+6\sqrt{2}=9\sqrt{2};$$

(b)

$$6\sqrt[3]{16}-2\sqrt[3]{250}=6\sqrt[3]{8\cdot 2}-2\sqrt[3]{125\cdot 2}=6\sqrt[3]{8}\sqrt[3]{2}-2\sqrt[3]{125}\sqrt[3]{2}$$
$$=6\cdot 2\sqrt[3]{2}-2\cdot 5\sqrt[3]{2}=12\sqrt[3]{2}-10\sqrt[3]{2}=2\sqrt[3]{2};$$

(c)

$$\frac{2}{3}\sqrt[3]{81}-\frac{1}{2}\sqrt[3]{24}=\frac{2}{3}\sqrt[3]{27\cdot 3}-\frac{1}{2}\sqrt[3]{8\cdot 3}=\frac{2}{3}\sqrt[3]{27}\sqrt[3]{3}-\frac{1}{2}\sqrt[3]{8}\sqrt[3]{3}$$
$$=\frac{2}{3}\cdot 3\sqrt[3]{3}-\frac{1}{2}\cdot 2\sqrt[3]{3}=2\sqrt[3]{3}-\sqrt[3]{3}=\sqrt[3]{3}.$$

Example

- Simplify: (a) $\sqrt{32m^7} - \sqrt{50m^7}$ (b) $\sqrt[3]{135x^7} - \sqrt[3]{40x^7}$.

(a)

$$\begin{aligned}\sqrt{32m^7} - \sqrt{50m^7} &= \sqrt{16m^6 \cdot 2m} - \sqrt{25m^6 \cdot 2m} \\ &= \sqrt{16(m^3)^2}\sqrt{2m} - \sqrt{25(m^3)^2}\sqrt{2m} \\ &= 4|m^3|\sqrt{2m} - 5|m^3|\sqrt{2m} = -|m^3|\sqrt{2m};\end{aligned}$$

(b)

$$\begin{aligned}\sqrt[3]{135x^7} - \sqrt[3]{40x^7} &= \sqrt[3]{27x^6 \cdot 5x} - \sqrt[3]{8x^6 \cdot 5x} \\ &= \sqrt[3]{27(x^2)^3}\sqrt[3]{5x} - \sqrt[3]{8(x^2)^3}\sqrt[3]{5x} \\ &= 3x^2\sqrt[3]{5x} - 2x^2\sqrt[3]{5x} = x^2\sqrt[3]{5x}.\end{aligned}$$

Multiply Radical Expressions

- For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

- When we multiply two radicals they must have the same index.
- Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.
- Multiplying radicals with coefficients is much like multiplying variables with coefficients.

  To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables.

Example

- Simplify: (a) $(3\sqrt{2})(2\sqrt{30})$ (b) $(2\sqrt[3]{18})(-3\sqrt[3]{6})$.

(a)

$$\begin{aligned}(3\sqrt{2})(2\sqrt{30}) &= 3\cdot 2\sqrt{2}\sqrt{30} = 6\sqrt{2\cdot 30}\\ &= 6\sqrt{60} = 6\sqrt{4\cdot 15} = 6\sqrt{4}\sqrt{15}\\ &= 6\cdot 2\sqrt{15} = 12\sqrt{15};\end{aligned}$$

(b)

$$\begin{aligned}(2\sqrt[3]{18})(-3\sqrt[3]{6}) &= 2\cdot(-3)\sqrt[3]{18}\sqrt[3]{6} = -6\sqrt[3]{18\cdot 6}\\ &= -6\sqrt[3]{108} = -6\sqrt[3]{27\cdot 4} = -6\sqrt[3]{27}\sqrt[3]{4}\\ &= -6\cdot 3\sqrt[3]{4} = -18\sqrt[3]{4}.\end{aligned}$$

Example

- Simplify: (a) $(6\sqrt{6x^2})(8\sqrt{30x^4})$ (b) $(-4\sqrt[4]{12y^3})(-\sqrt[4]{8y^3})$.

(a)

$$\begin{aligned}(6\sqrt{6x^2})(8\sqrt{30x^4}) &= 6\cdot 8\sqrt{6x^2}\sqrt{30x^4} = 48\sqrt{6x^2\cdot 30x^4}\\ &= 48\sqrt{180x^6} = 48\sqrt{36x^6\cdot 5} = 48\sqrt{36(x^3)^2}\sqrt{5}\\ &= 48\cdot 6|x^3|\sqrt{5} = 288|x^3|\sqrt{5};\end{aligned}$$

(b)

$$\begin{aligned}(-4\sqrt[4]{12y^3})(-\sqrt[4]{8y^3}) &= 4\sqrt[4]{12y^3\cdot 8y^3} = 4\sqrt[4]{96y^6}\\ &= 4\sqrt[4]{16y^4\cdot 6y^2} = 4\sqrt[4]{16y^4}\sqrt[4]{6y^2} = 4\cdot 2|y|\sqrt[4]{6y^2}\\ &= 8|y|\sqrt[4]{6y^2}.\end{aligned}$$

Polynomial Multiplication for Radical Expressions

- We use the Distributive Property to multiply expressions with radicals:
  - First distribute;
  - Then simplify the radicals when possible.

Example

- Simplify: (a) $\sqrt{6}(1+3\sqrt{6})$ (b) $\sqrt[3]{4}(-2-\sqrt[3]{6})$.

(a)

$$\begin{aligned}\sqrt{6}(1+3\sqrt{6}) &= \sqrt{6}+3\sqrt{6}\sqrt{6} = \sqrt{6}+3\sqrt{6^2} \\ &= \sqrt{6}+3\cdot 6 = \sqrt{6}+18;\end{aligned}$$

(b)

$$\begin{aligned}\sqrt[3]{4}(-2-\sqrt[3]{6}) &= -2\sqrt[3]{4}-\sqrt[3]{4\cdot 6} = -2\sqrt[3]{4}-\sqrt[3]{24} \\ &= -2\sqrt[3]{4}-\sqrt[3]{8\cdot 3} = -2\sqrt[3]{4}-\sqrt[3]{8}\sqrt[3]{3} = -2\sqrt[3]{4}-2\sqrt[3]{3};\end{aligned}$$

Example

- Simplify: (a) $(6-3\sqrt{7})(3+4\sqrt{7})$ (b) $(\sqrt[3]{x}-2)(\sqrt[3]{x}-3)$.

(a)

$$(6-3\sqrt{7})(3+4\sqrt{7}) = 18+24\sqrt{7}-9\sqrt{7}-12\sqrt{7}\sqrt{7}$$
$$= 18+15\sqrt{7}-12\cdot 7 = 18+15\sqrt{7}-84 = -66+15\sqrt{7};$$

(b)

$$(\sqrt[3]{x}-2)(\sqrt[3]{x}-3) = \sqrt[3]{x}\sqrt[3]{x}-3\sqrt[3]{x}-2\sqrt[3]{x}+6$$
$$= \sqrt[3]{x^2}-5\sqrt[3]{x}+6.$$

Example

- Simplify: $(5\sqrt{3}-\sqrt{7})(\sqrt{3}+2\sqrt{7})$.

$$\begin{aligned}&(5\sqrt{3}-\sqrt{7})(\sqrt{3}+2\sqrt{7})\\&=5\sqrt{3}\sqrt{3}+10\sqrt{3}\sqrt{7}-\sqrt{3}\sqrt{7}-2\sqrt{7}\sqrt{7}\\&=5\cdot 3+10\sqrt{21}-\sqrt{21}-2\cdot 7\\&=15+9\sqrt{21}-14\\&=1+9\sqrt{21}.\end{aligned}$$

Example

- Simplify: (a) $(10+\sqrt{2})^2$ (b) $(9-2\sqrt{10})^2$.

(a)

$$(10+\sqrt{2})^2 = 10^2 + 2\cdot 10\cdot\sqrt{2} + (\sqrt{2})^2$$
$$= 100 + 20\sqrt{2} + 2 = 102 + 20\sqrt{2};$$

(b)

$$(9-2\sqrt{10})^2 = 9^2 - 2\cdot 9\cdot 2\sqrt{10} + (2\sqrt{10})^2$$
$$= 81 - 36\sqrt{10} + 4\cdot 10 = 81 - 36\sqrt{10} + 40$$
$$= 121 - 36\sqrt{10}.$$

Example

- Simplify: $(3-2\sqrt{5})(3+2\sqrt{5})$.

$$\begin{aligned}
&(3-2\sqrt{5})(3+2\sqrt{5}) \\
&= 9+6\sqrt{5}-6\sqrt{5}-4\sqrt{5}^2 \\
&= 9-4\cdot 5 \\
&= 9-20 \\
&= -11.
\end{aligned}$$

Subsection 5

Divide Radical Expressions

We Shall Learn and Practice

- Divide radical expressions.
- Rationalize a one term denominator.
- Rationalize a two term denominator.

Divide Radical Expressions

- **Quotient Property** If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, where $n \geq 2$ is any integer, then

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

- We will use the Quotient Property when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index.
- When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

Example

- Simplify: (a) $\frac{\sqrt{50s^3}}{\sqrt{128s}}$ (b) $\frac{\sqrt[3]{56a}}{\sqrt[3]{7a^4}}$.

(a)

$$\frac{\sqrt{50s^3}}{\sqrt{128s}} = \sqrt{\frac{50s^3}{128s}} = \sqrt{\frac{25s^2}{64}} = \frac{5|s|}{8}.$$

(b)

$$\frac{\sqrt[3]{56a}}{\sqrt[3]{7a^4}} = \sqrt[3]{\frac{56a}{7a^4}} = \sqrt[3]{\frac{8}{a^3}} = \frac{2}{a}.$$

Example

- Simplify: (a) $\frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$ (b) $\frac{\sqrt[3]{128x^2y^{-1}}}{\sqrt[3]{2x^{-1}y^2}}$.

(a)

$$\frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}} = \sqrt{\frac{162x^{10}y^2}{2x^6y^6}} = \sqrt{\frac{81x^4}{y^4}} = \frac{9|x^2|}{|y^2|} = \frac{9x^2}{y^2}.$$

(b)

$$\frac{\sqrt[3]{128x^2y^{-1}}}{\sqrt[3]{2x^{-1}y^2}} = \sqrt[3]{\frac{128x^2y^{-1}}{2x^{-1}y^2}} = \sqrt[3]{\frac{64x^3}{y^3}} = \frac{4x}{y}.$$

Example

- Simplify $\frac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$.

$$\frac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}} = \sqrt{\frac{64x^4y^5}{2xy^3}} = \sqrt{32x^3y^2}$$
$$= \sqrt{16x^2y^2 \cdot 2x} = \sqrt{16x^2y^2}\sqrt{2x}$$
$$= 4|x||y|\sqrt{2x}.$$

Rationalize a One Term Denominator

- A radical expression is considered simplified if there are:
  - no factors in the radicand that have perfect powers of the index;
  - no fractions in the radicand;
  - no radicals in the denominator of a fraction.
- **Rationalizing the denominator** is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is radical-free.

Example

- Simplify: (a) $\frac{5}{\sqrt{3}}$ (b) $\sqrt{\frac{3}{32}}$ (c) $\frac{2}{\sqrt{2x}}$.

(a)

$$\frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{(\sqrt{3})^2} = \frac{5\sqrt{3}}{3};$$

(b)

$$\sqrt{\frac{3}{32}} = \frac{\sqrt{3}}{\sqrt{32}} = \frac{\sqrt{3}}{\sqrt{16}\sqrt{2}} = \frac{\sqrt{3}}{4\sqrt{2}} = \frac{\sqrt{3}\sqrt{2}}{4(\sqrt{2})^2} = \frac{\sqrt{6}}{4 \cdot 2} = \frac{\sqrt{6}}{8};$$

(c)

$$\frac{2}{\sqrt{2x}} = \frac{2\sqrt{2x}}{(\sqrt{2x})^2} = \frac{2\sqrt{2x}}{2x} = \frac{\sqrt{2x}}{x}.$$

Example

- Simplify: (a) $\frac{1}{\sqrt[3]{7}}$ (b) $\sqrt[3]{\frac{5}{12}}$ (c) $\frac{5}{\sqrt[3]{9y}}$.

(a)

$$\frac{1}{\sqrt[3]{7}} = \frac{(\sqrt[3]{7})^2}{(\sqrt[3]{7})^3} = \frac{\sqrt[3]{49}}{7};$$

(b)

$$\sqrt[3]{\frac{5}{12}} = \frac{\sqrt[3]{5}}{\sqrt[3]{12}} = \frac{\sqrt[3]{5}}{\sqrt[3]{4}\sqrt[3]{3}} = \frac{\sqrt[3]{5}\sqrt[3]{2}\sqrt[3]{9}}{\sqrt[3]{4}\sqrt[3]{2}\sqrt[3]{3}\sqrt[3]{9}} = \frac{\sqrt[3]{90}}{\sqrt[3]{8}\sqrt[3]{27}} = \frac{\sqrt[3]{90}}{6};$$

(c)

$$\frac{5}{\sqrt[3]{9y}} = \frac{5\sqrt[3]{3y^2}}{\sqrt[3]{9y}\sqrt[3]{3y^2}} = \frac{5\sqrt[3]{3y^2}}{\sqrt[3]{27y^3}} = \frac{5\sqrt[3]{3y^2}}{3y}.$$

Example

- Simplify: (a) $\frac{1}{\sqrt[4]{3}}$ (b) $\sqrt[4]{\frac{3}{64}}$ (c) $\frac{3}{\sqrt[4]{125x}}$.

(a)

$$\frac{1}{\sqrt[4]{3}} = \frac{(\sqrt[4]{3})^3}{(\sqrt[4]{3})^4} = \frac{\sqrt[4]{27}}{3};$$

(b)

$$\sqrt[4]{\frac{3}{64}} = \frac{\sqrt[4]{3}}{\sqrt[4]{64}} = \frac{\sqrt[4]{3}\sqrt[4]{4}}{\sqrt[4]{64}\sqrt[4]{4}} = \frac{\sqrt[4]{12}}{\sqrt[4]{256}} = \frac{\sqrt[4]{12}}{4};$$

(c)

$$\frac{3}{\sqrt[4]{125x}} = \frac{3\sqrt[4]{5x^3}}{\sqrt[4]{125x}\sqrt[4]{5x^3}} = \frac{3\sqrt[4]{5x^3}}{\sqrt[4]{625x^4}} = \frac{3\sqrt[4]{5x^3}}{5|x|}.$$

Rationalize a Two Term Denominator

- When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates Pattern to rationalize the denominator.

$$(a-b)(a+b)=a^2-b^2$$

- In this way, e.g.,

$$(3-\sqrt{2})(3+\sqrt{2})=3^2-(\sqrt{2})^2=9-2=7.$$

Example

- Simplify: $\frac{3}{1-\sqrt{5}}$.

$$\frac{3}{1-\sqrt{5}} = \frac{3(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})} = \frac{3(1+\sqrt{5})}{1^2-(\sqrt{5})^2}$$
$$= \frac{3(1+\sqrt{5})}{1-5} = -\frac{3(1+\sqrt{5})}{4}.$$

Example

- Simplify: $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

$$\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}} = \frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{(\sqrt{x}+\sqrt{2})(\sqrt{x}-\sqrt{2})}$$
$$= \frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{(\sqrt{x})^2-(\sqrt{2})^2} = \frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{x-2}.$$

Example

- Simplify: $\frac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

$$\frac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}} = \frac{(\sqrt{p}+\sqrt{2})(\sqrt{p}+\sqrt{2})}{(\sqrt{p}-\sqrt{2})(\sqrt{p}+\sqrt{2})}$$
$$= \frac{(\sqrt{p}+\sqrt{2})^2}{\sqrt{p}^2-\sqrt{2}^2} = \frac{(\sqrt{p}+\sqrt{2})^2}{p-2}.$$

Subsection 6

Solve Radical Equations

We Shall Learn and Practice

- Solve radical equations.
- Solve radical equations with two radicals.
- Use radicals in applications.

Solve Radical Equations

- Follow the steps:
  - Isolate the radical on one side of the equation;
  - Raise both sides of the equation to the power of the index;
  - Solve the new equation;
  - Check the answer in the original equation.

Example

- Solve: $\sqrt{3m+2}-5=0$.

$$\begin{aligned} \sqrt{3m+2}-5 &= 0 \\ \sqrt{3m+2} &= 5 \\ (\sqrt{3m+2})^2 &= 5^2 \\ 3m+2 &= 25 \\ 3m &= 23 \\ m &= \frac{23}{3}. \end{aligned}$$

Check: $\sqrt{3\cdot\frac{23}{3}+2}-5=\sqrt{23+2}-5=\sqrt{25}-5=5-5=0$.

So $x=\frac{23}{3}$ is an admissible solution.

Example

- Solve: $\sqrt{2r-3}+5=0$.

$$\begin{gathered}\sqrt{2r-3}+5=0\\ \sqrt{2r-3}=-5\end{gathered}$$

Since a square root is equal, by definition, to a positive number, this equation has no solutions.

Example

- Solve: $\sqrt{x-2}+2=x$.

$$\begin{aligned} \sqrt{x-2}+2 &= x \\ \sqrt{x-2} &= x-2 \\ (\sqrt{x-2})^2 &= (x-2)^2 \\ x-2 &= x^2-4x+4 \\ x^2-5x+6 &= 0 \\ (x-3)(x-2) &= 0 \\ x=3 \quad \text{or} \quad x &= 2. \end{aligned}$$

Check!

Both $x=2$ and $x=3$ are admissible solutions.

Example

- Solve: $\sqrt[3]{4x-3}+8=5$.

$$\begin{aligned}\sqrt[3]{4x-3}+8&=5\\ \sqrt[3]{4x-3}&=-3\\ (\sqrt[3]{4x-3})^3&=(-3)^3\\ 4x-3&=-27\\ 4x&=-24\\ x&=-6.\end{aligned}$$

Check!

$x=-6$ is an admissible solution.

Example

- Solve: $(9x+9)^{\frac{1}{4}} - 2 = 1$.

$$
\begin{aligned}
(9x+9)^{\frac{1}{4}} - 2 &= 1 \\
\sqrt[4]{9x+9} - 2 &= 1 \\
\sqrt[4]{9x+9} &= 3 \\
(\sqrt[4]{9x+9})^4 &= 3^4 \\
9x+9 &= 81 \\
9x &= 72 \\
x &= 8.
\end{aligned}
$$

Check!

$x = 8$ is an admissible solution.

Example

- Solve: $\sqrt{m+9} - m + 3 = 0$.

$$\begin{gathered}
\sqrt{m+9} - m + 3 = 0 \\
\sqrt{m+9} = m - 3 \\
(\sqrt{m+9})^2 = (m-3)^2 \\
m + 9 = m^2 - 6m + 9 \\
m^2 - 7m = 0 \\
m(m-7) = 0 \\
m = 0 \quad \text{or} \quad m = 7.
\end{gathered}$$

Check!

Only $m = 7$ is an admissible solution.

Example

- Solve: $2\sqrt{4a+4}-16=16$.

$$\begin{aligned} 2\sqrt{4a+4}-16 &= 16 \\ 2\sqrt{4a+4} &= 32 \\ \sqrt{4a+4} &= 16 \\ (\sqrt{4a+4})^2 &= 16^2 \\ 4a+4 &= 256 \\ 4a &= 252 \\ a &= 63. \end{aligned}$$

Check!

$a=63$ is an admissible solution.

Solve Radical Equations with Two Radicals

- Follow the steps:
  - Isolate one of the radicals on one side of the equation;
  - Raise both sides of the equation to the power of the index;
  - If there are more radicals repeat the preceding steps;
  - If there are no more radicals solve the new equation;
  - Check the answer in the original equation.

Example

- Solve: $\sqrt[3]{5x-4} = \sqrt[3]{2x+5}$.

$$\begin{aligned} \sqrt[3]{5x-4} &= \sqrt[3]{2x+5} \\ (\sqrt[3]{5x-4})^3 &= (\sqrt[3]{2x+5})^3 \\ 5x-4 &= 2x+5 \\ 3x &= 9 \\ x &= 3. \end{aligned}$$

Check!

$x = 3$ is an admissible solution.

Example

- Solve: $3 - \sqrt{x} = \sqrt{x-3}$.

$$\begin{aligned} 3 - \sqrt{x} &= \sqrt{x-3} \\ (3-\sqrt{x})^2 &= (\sqrt{x-3})^2 \\ 9 - 2 \cdot 3 \cdot \sqrt{x} + (\sqrt{x})^2 &= x - 3 \\ 9 - 6\sqrt{x} + x &= x - 3 \\ 12 &= 6\sqrt{x} \\ 2 &= \sqrt{x} \\ 2^2 &= (\sqrt{x})^2 \\ 4 &= x \end{aligned}$$

Check!

$x = 4$ is an admissible solution.

Example

- Solve: $\sqrt{x-1}+2=\sqrt{2x+6}$.

$$\begin{aligned}
\sqrt{x-1}+2 &= \sqrt{2x+6} \\
(\sqrt{x-1}+2)^2 &= (\sqrt{2x+6})^2 \\
(\sqrt{x-1})^2+2\cdot\sqrt{x-1}\cdot 2+2^2 &= 2x+6 \\
x-1+4\sqrt{x-1}+4 &= 2x+6 \\
4\sqrt{x-1} &= x+3 \\
(4\sqrt{x-1})^2 &= (x+3)^2 \\
16(x-1) &= x^2+6x+9 \\
16x-16 &= x^2+6x+9 \\
x^2-10x+25 &= 0 \\
(x-5)^2 &= 0 \\
x &= 5.
\end{aligned}$$

Check!

$x=5$ is an admissible solution.

Use Radicals in Applications

- Follow the steps:
  - Read the problem carefully.
    When appropriate, draw a figure with the given information.
  - Identify what we are looking for.
  - Name what we are looking for by choosing a variable to represent it.
  - Translate into an equation by writing the appropriate formula or model for the situation.
    Substitute in the given information.
  - Solve the equation using good algebra techniques.
  - Check the answer in the problem and make sure it makes sense.
  - Answer the question with a complete sentence.

Example

- On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- A helicopter dropped a rescue package from a height of $1,296$ feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

  We are looking for the time t it took for the package to reach the ground.

  We have $t = \frac{\sqrt{h}}{4}$ and $h = 1296$ feet.

$$t = \frac{\sqrt{1296}}{4} = \frac{36}{4} = 9.$$

  Thus, the package reached the ground in $t = 9$ seconds.

Example

- If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.
- An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

  We are looking for the speed s of the car before the brakes were applied.

  We have $s = \sqrt{24d}$ and $d = 76$ feet.

$$s = \sqrt{24 \cdot 76} = \sqrt{1824} \approx 42.7.$$

  The speed of the car before the brakes were applied was $s = 42.7$ mph.

Subsection 7

Use Radicals in Functions

We Shall Learn and Practice

- Evaluate a radical function.
- Find the domain of a radical function.
- Graph radical functions.

Evaluate a Radical Function

- A **radical function** is a function that is defined by a radical expression.
- To evaluate a radical function, we find the value of $f(x)$ for a given value of x just as we did in our previous work with functions.

Example

- For the function $f(x) = \sqrt{3x-2}$, find (a) $f(6)$ (b) $f(0)$.

(a)

$$f(6) = \sqrt{3 \cdot 6 - 2} = \sqrt{18-2} = \sqrt{16} = 4;$$

(b)

$$f(0) = \sqrt{3 \cdot 0 - 2} = \sqrt{-2} \quad \text{Not a real number.}$$

Example

- For the function $g(x) = \sqrt[3]{3x-4}$, find (a) $g(4)$ (b) $g(1)$.

(a)

$$g(4) = \sqrt[3]{3 \cdot 4 - 4} = \sqrt[3]{12-4} = \sqrt[3]{8} = 2;$$

(b)

$$g(1) = \sqrt[3]{3 \cdot 1 - 4} = \sqrt[3]{-1} = -1.$$

Example

- For the function $f(x) = \sqrt[4]{3x+4}$, find (a) $f(4)$ (b) $f(-1)$.

(a)
$$f(4) = \sqrt[4]{3 \cdot 4 + 4} = \sqrt[4]{12+4} = \sqrt[4]{16} = 2;$$

(b)
$$f(-1) = \sqrt[4]{3 \cdot (-1) + 4} = \sqrt[4]{-3+4} = \sqrt[4]{1} = 1.$$

Find the Domain of a Radical Function

- When the index of the radical is even, the radicand must be greater than or equal to zero.
- When the index of the radical is odd, the radicand can be any real number.

Example

- Find the domain of the function, $f(x) = \sqrt{6x-5}$. Write the domain in interval notation.

  Since the index is even, we must have

$$\begin{gathered} 6x - 5 \geq 0 \\ 6x \geq 5 \\ x \geq \frac{5}{6}. \end{gathered}$$

  Hence the domain of f is the interval $[\frac{5}{6}, +\infty)$.

Example

- Find the domain of the function, $f(x) = \sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

  Since the index is even, we must have
  $$\frac{4}{x+3} \geq 0.$$
  Since the numerator is positive, for the fraction to be positive, the denominator must also be positive. (But, notice, it is not allowed to be zero!)
  $$x + 3 > 0.$$
  Thus, $x > -3$.

  Hence the domain of f is the interval $(-3, +\infty)$.

Example

- Find the domain of the function, $f(x) = \sqrt[3]{3x^2 - 1}$. Write the domain in interval notation.

  Since the index is odd, no restrictions are necessary.

  Thus x can be any real number.

  In interval notation the domain is $(-\infty, +\infty)$.

Graph Radical Functions

- Before we graph any radical function, we first find the domain of the function.
- We choose x-values in the domain, substitute them in and create a chart.
- Then plot the points.
- The **range** of the function is the set of the y-values that the function assumes.

Example

- For the function $f(x) = \sqrt{x+2}$, (a) find the domain (b) graph the function (c) use the graph to determine the range.

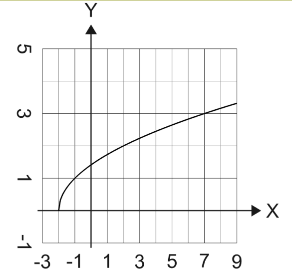
(a) Since the index is even, we must have

$$x + 2 \geq 0$$
$$x \geq -2.$$

Thus the domain is the interval $[-2, +\infty)$.

(b)

x	y
-2	0
-1	1
2	2
7	3



(c) The range of f is the interval $[0, +\infty)$.

Example

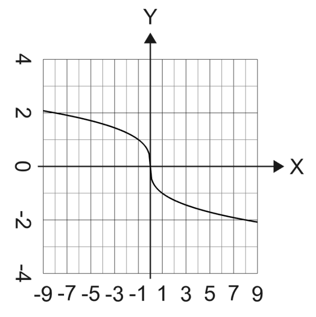
- For the function $f(x) = -\sqrt[3]{x}$, (a) find the domain (b) graph the function (c) use the graph to determine the range.

(a) Since the index is odd no restrictions are necessary.

Thus the domain is the interval $(-\infty, +\infty)$.

(b)

x	y
-8	2
-1	1
0	0
1	-1
8	-2



(c) The range of f is the interval $(-\infty, +\infty)$.

Subsection 8

Use the Complex Number System

We Shall Learn and Practice

- Evaluate the square root of a negative number.
- Add and subtract complex numbers.
- Multiply complex numbers.
- Divide complex numbers.
- Simplify powers of i.

Evaluate the Square Root of a Negative Number

- The **imaginary unit** i is the number whose square is -1.

$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}.$$

- If b is a positive real number, then

$$\sqrt{-b} = \sqrt{b}\sqrt{-1} = \sqrt{b}i.$$

Example

- Write each expression in terms of i and simplify if possible: (a) $\sqrt{-81}$ (b) $\sqrt{-5}$ (c) $\sqrt{-18}$.

(a)

$$\sqrt{-81} = \sqrt{81}i = 9i;$$

(b)

$$\sqrt{-5} = \sqrt{5}i;$$

(c)

$$\sqrt{-18} = \sqrt{18}i = \sqrt{9 \cdot 2}i = \sqrt{9}\sqrt{2}i = 3\sqrt{2}i.$$

Add and Subtract Complex Numbers

- A **complex number** is of the form $a + bi$, where a and b are real numbers.
  - If $b = 0$, then $a + bi$ becomes $a + 0 \cdot i = a$, and is a real number;
  - If $a = 0$, then $a + bi$ becomes $0 + bi = bi$, and is called an **imaginary number**.
- Adding and subtracting complex numbers is much like adding or subtracting like terms.
  - We add or subtract the real parts;
  - We add or subtract the imaginary parts.

  Our final result should be in standard form $a + bi$.

Example

- Add: $\sqrt{-8}+\sqrt{-32}$.

$$\sqrt{-8}+\sqrt{-32}=\sqrt{8}i+\sqrt{32}i=\sqrt{4\cdot 2}i+\sqrt{16\cdot 2}i$$
$$=\sqrt{4}\sqrt{2}i+\sqrt{16}\sqrt{2}i=2\sqrt{2}i+4\sqrt{2}i=6\sqrt{2}i.$$

Example

- Simplify: (a) $(2+7i)+(4-2i)$ (b) $(8-4i)-(2-i)$.

(a)

$$(2+7i)+(4-2i)=2+7i+4-2i=6+5i;$$

(b)

$$(8-4i)-(2-i)=8-4i-2+i=6-3i.$$

Multiply Complex Numbers

- Multiplying complex numbers is also much like multiplying expressions with coefficients and variables.
- Always keep in mind that

$$i^2 = -1.$$

Example

- Multiply: $4i(5-3i)$.

$$4i(5-3i) = 20i - 12i^2 = 20i - 12(-1) = 12 + 20i.$$

Example

- Multiply: $(5-3i)(-1-2i)$.

$$\begin{aligned}(5-3i)(-1-2i) &= -5-10i+3i+6i^2 \\ &= -5-10i+3i-6 = -11-7i.\end{aligned}$$

Example

- Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

$$\begin{aligned}(-2-5i)^2 &= (-2)^2 - 2\cdot(-2)\cdot(5i)+(5i)^2\\ &= 4+20i+25i^2 = 4+20i-25\\ &= -21+20i.\end{aligned}$$

Example

- Multiply: $\sqrt{-49} \cdot \sqrt{-4}$.

$$\sqrt{-49} \cdot \sqrt{-4} = \sqrt{49}i \cdot \sqrt{4}i = 7i \cdot 2i = 14i^2 = -14.$$

Example

- Multiply: $(4-\sqrt{-12})(3-\sqrt{-48})$.

$$\begin{aligned}
&(4-\sqrt{-12})(3-\sqrt{-48}) = (4-\sqrt{12}i)(3-\sqrt{48}i) \\
&= (4-2\sqrt{3}i)(3-4\sqrt{3}i) = 12-16\sqrt{3}i-6\sqrt{3}i+8(\sqrt{3})^2i^2 \\
&= 12-22\sqrt{3}i+8\cdot 3\cdot(-1) = 12-22\sqrt{3}i-24 \\
&= -12-22\sqrt{3}i.
\end{aligned}$$

Complex Conjugate Pairs

- A complex conjugate pair is of the form

$$a + bi, \quad a - bi.$$

- Multiply: $(4 - 3i) \cdot (4 + 3i)$.

$$(4 - 3i) \cdot (4 + 3i) = 16 + 12i - 12i - 9i^2 = 16 + 9 = 25.$$

- If a and b are real numbers, then

$$(a - bi)(a + bi) = a^2 + b^2.$$

Example

- Multiply using the Product of Complex Conjugates Pattern: $(3-10i)(3+10i)$.

$$(3-10i)(3+10i) = 3^2 + 10^2 = 9 + 100 = 109.$$

Divide Complex Numbers

- Follow the steps:
  - Write both the numerator and denominator in standard form;
  - Multiply the numerator and the denominator by the complex conjugate of the denominator;
  - Simplify and write the result in standard form.

Example

- Divide: $\frac{2+5i}{5-2i}$.

$$\frac{2+5i}{5-2i} = \frac{(2+5i)(5+2i)}{(5-2i)(5+2i)} = \frac{10+4i+25i-10}{5^2+2^2} = \frac{29i}{29} = i.$$

Example

- Divide, writing the answer in standard form: $\frac{4}{1-4i}$.

$$\frac{4}{1-4i} = \frac{4(1+4i)}{(1-4i)(1+4i)} = \frac{4+16i}{1^2+4^2} = \frac{4+16i}{17} = \frac{4}{17} + \frac{16}{17}i.$$

Example

- Divide: $\frac{3+3i}{2i}$.

$$\frac{3+3i}{2i} = \frac{-2i(3+3i)}{2i\cdot(-2i)} = \frac{-6i-6i^2}{2^2} = \frac{6-6i}{4}$$
$$= \frac{2(3-3i)}{4} = \frac{3-3i}{2} = \frac{3}{2} - \frac{3}{2}i.$$

Simplify Powers of i

- Notice the pattern:

$$\begin{array}{llll} i^1 = i & i^2 = -1 & i^3 = -i & i^4 = 1 \\ i^5 = i & i^6 = -1 & i^7 = -i & i^8 = 1 \end{array}$$

- To compute i^{103}:
  - Divide 103 by 4: Quotient is 25 and Remainder is 3.
  - Write

$$i^{103} = i^{4\cdot 25+3} = i^{4\cdot 25} i^3 = (i^4)^{25} i^3 = 1^{25}(-i) = -i.$$

- Thus $i^n = i^r$, where r is the remainder upon dividing n by 4.

Example

- Simplify: (a) i^{75} (b) i^{90}.

(a)

$$i^{75} = i^{4 \cdot 18+3} = i^{4 \cdot 18} i^3 = (i^4)^{18} i^3 = 1 \cdot (-i) = -i;$$

(b)

$$i^{90} = i^{4 \cdot 22+2} = (i^4)^{22} i^2 = 1 \cdot (-1) = -1.$$