Intermediate Algebra

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 102

Subsection 1

Solve Quadratic Equations Using the Square Root Property

We Shall Learn and Practice

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property.
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property.

Quadratic Equations of the Form $ax^2 = k$

- **The Square Root Property:** If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k}.$$

  This is sometimes abbreviated by writing $x = \pm\sqrt{k}$.
- To solve an equation of the form $ax^2 = k$ follow the steps:
  - Isolate the quadratic term and make its coefficient 1;
  - Use the Square Root Property;
  - Simplify the radical;
  - Check the solutions.

Example

- Solve: $x^2 - 48 = 0$.

$$\begin{gathered} x^2 - 48 = 0 \\ x^2 = 48 \\ x = \pm\sqrt{48} \\ x = \pm\sqrt{16 \cdot 3} \\ x = \pm 4\sqrt{3}. \end{gathered}$$

Check the solutions!

Example

- Solve: $2x^2 = 98$.

$$\begin{aligned} 2x^2 &= 98 \\ x^2 &= 49 \\ x &= \pm\sqrt{49} \\ x &= \pm 7. \end{aligned}$$

Check the solutions!

Example

- Solve: $c^2 + 12 = 0$.

$$\begin{aligned} c^2 + 12 &= 0 \\ c^2 &= -12 \\ c &= \pm\sqrt{-12} \\ c &= \pm\sqrt{12}i \\ c &= \pm\sqrt{4}\sqrt{3}i \\ c &= \pm 2\sqrt{3}i. \end{aligned}$$

Check the solutions!

Example

- Solve: $\frac{1}{2}x^2 + 4 = 24$.

$$\begin{aligned} \tfrac{1}{2}x^2 + 4 &= 24 \\ \tfrac{1}{2}x^2 &= 20 \\ x^2 &= 40 \\ x &= \pm\sqrt{40} \\ x &= \pm\sqrt{4}\sqrt{10} \\ x &= \pm 2\sqrt{10}. \end{aligned}$$

Check the solutions!

Example

- Solve: $5r^2 - 2 = 34$.

$$\begin{gathered} 5r^2 - 2 = 34 \\ 5r^2 = 36 \\ r^2 = \frac{36}{5} \\ r = \pm\sqrt{\frac{36}{5}} \\ r = \pm\frac{\sqrt{36}}{\sqrt{5}} \\ r = \pm\frac{6}{\sqrt{5}} \\ r = \pm\frac{6\sqrt{5}}{(\sqrt{5})^2} \\ r = \pm\frac{6\sqrt{5}}{5}. \end{gathered}$$

Check the solutions!

Quadratic Equations of the Form $a(x-h)^2 = k$

- To solve a quadratic equation of the form $a(x-h)^2 = k$, follow the steps:
  - Isolate the term that has the variable squared;
    In this case, a binomial is being squared.
  - Once the binomial is isolated, by dividing each side by the coefficient a, we make the coefficient of $(x-h)^2$ equal to 1;
  - Then we use the Square Root Property on $(x-h)^2$.

Example

- Solve: $3(a-3)^2 = 54$.

$$
\begin{aligned}
3(a-3)^2 &= 54 \\
(a-3)^2 &= 18 \\
a-3 &= \pm\sqrt{18} \\
a-3 &= \pm\sqrt{9}\sqrt{2} \\
a-3 &= \pm 3\sqrt{2} \\
a &= 3 \pm 3\sqrt{2}.
\end{aligned}
$$

Check the solutions!

Example

- Solve: $(x-\frac{1}{2})^2 = \frac{5}{4}$.

$$\begin{aligned}
(x-\tfrac{1}{2})^2 &= \tfrac{5}{4} \\
x-\tfrac{1}{2} &= \pm\sqrt{\tfrac{5}{4}} \\
x-\tfrac{1}{2} &= \pm\tfrac{\sqrt{5}}{\sqrt{4}} \\
x-\tfrac{1}{2} &= \pm\tfrac{\sqrt{5}}{2} \\
x &= \tfrac{1}{2} \pm \tfrac{\sqrt{5}}{2} \\
x &= \tfrac{1\pm\sqrt{5}}{2}.
\end{aligned}$$

Check the solutions!

Example

- Solve: $5(a-5)^2+4=104$.

$$\begin{aligned} 5(a-5)^2+4 &= 104 \\ 5(a-5)^2 &= 100 \\ (a-5)^2 &= 20 \\ a-5 &= \pm\sqrt{20} \\ a-5 &= \pm\sqrt{4}\sqrt{5} \\ a-5 &= \pm 2\sqrt{5} \\ a &= 5 \pm 2\sqrt{5}. \end{aligned}$$

Check the solutions!

Example

- Solve: $(3r+4)^2 = -8$.

$$\begin{aligned}(3r+4)^2 &= -8 \\ 3r+4 &= \pm\sqrt{-8} \\ 3r+4 &= \pm\sqrt{8}i \\ 3r+4 &= \pm\sqrt{4}\sqrt{2}i \\ 3r+4 &= \pm 2\sqrt{2}i \\ 3r &= -4 \pm 2\sqrt{2}i \\ r &= -\frac{4}{3} \pm \frac{2\sqrt{2}}{3}i.\end{aligned}$$

Check the solutions!

Example

- Solve: $9m^2 - 12m + 4 = 25$.

$$\begin{gathered}
9m^2 - 12m + 4 = 25 \\
(3m-2)^2 = 25 \\
3m - 2 = \pm\sqrt{25} \\
3m - 2 = \pm 5 \\
3m = 2 \pm 5 \\
m = \tfrac{2 \pm 5}{3} \\
m = -1 \quad \text{or} \quad m = \tfrac{7}{3}.
\end{gathered}$$

Check the solutions!

Subsection 2

Solve Quadratic Equations by Completing the Square

We Shall Learn and Practice

- Complete the square of a binomial expression.
- Solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.
- Solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.

Complete the Square of a Binomial Expression

- This technique is based on

$$(a+b)^2 = a^2 + 2\cdot a\cdot b + b^2;$$
$$(a-b)^2 = a^2 - 2\cdot a\cdot b + b^2.$$

- To complete a square of $x^2 + bx$ follow the steps:
  - Identify b, the coefficient of x;
  - Find $(\frac{1}{2}b)^2$, the number to complete the square;
  - Add the $(\frac{1}{2}b)^2$ to $x^2 + bx$;
  - Factor the perfect square trinomial, writing it as a binomial squared.

Example

- Complete the square to make a perfect square trinomial. Then write the result as a binomial squared. (a) $a^2 - 20a$ (b) $m^2 - 5m$ (c) $p^2 + \frac{1}{4}p$

(a)

$$a^2 - 20a \qquad (\tfrac{1}{2}b)^2 = (\tfrac{1}{2}20)^2 = 10^2 = 100;$$
$$a^2 - 20a + 100 = a^2 - 2 \cdot a \cdot 10 + (10)^2 = (a - 10)^2;$$

(b)

$$m^2 - 5m \qquad (\tfrac{1}{2}b)^2 = (\tfrac{1}{2}5)^2 = (\tfrac{5}{2})^2 = \tfrac{25}{4};$$
$$m^2 - 5m + \tfrac{25}{4} = m^2 - 2 \cdot m \cdot \tfrac{5}{2} + (\tfrac{5}{2})^2 = (m - \tfrac{5}{2})^2;$$

(c)

$$p^2 + \tfrac{1}{4}p \qquad (\tfrac{1}{2}b)^2 = (\tfrac{1}{2}\tfrac{1}{4})^2 = (\tfrac{1}{8})^2 = \tfrac{1}{64};$$
$$p^2 + \tfrac{1}{4}p + \tfrac{1}{64} = p^2 + 2 \cdot p \cdot \tfrac{1}{8} + (\tfrac{1}{8})^2 = (p + \tfrac{1}{8})^2.$$

Quadratic Equations of the Form $x^2 + bx + c = 0$

- To solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square follow the steps:
  - Isolate the variable terms on one side and the constant terms on the other;
  - Find $(\frac{1}{2}b)^2$, the number to complete the square, and add it to both sides;
  - Factor the perfect square trinomial as a binomial square;
  - Use the Square Root Property;
  - Simplify the radical and then solve the two resulting equations;
  - Check the solutions.

Example

- Solve by completing the square: $x^2 + 4x = 5$.

$$\begin{aligned} x^2 + 4x &= 5 \\ x^2 + 4x + 4 &= 5 + 4 \\ (x+2)^2 &= 9 \\ x + 2 &= \pm\sqrt{9} \\ x &= -2 \pm 3 \\ x = -5 \quad \text{or} &\quad x = 1. \end{aligned}$$

Check the solutions!

Example

- Solve by completing the square: $y^2 - 10y = -35$.

$$\begin{aligned} y^2 - 10y &= -35 \\ y^2 - 10y + 25 &= -35 + 25 \\ (y-5)^2 &= -10 \\ y - 5 &= \pm\sqrt{-10} \\ y - 5 &= \pm\sqrt{10}i \\ y &= 5 \pm \sqrt{10}i. \end{aligned}$$

Check the solutions!

Example

- Solve by completing the square: $x^2 - 16x = -16$.

$$\begin{aligned} x^2 - 16x &= -16 \\ x^2 - 16x + 64 &= -16 + 64 \\ (x-8)^2 &= 48 \\ x - 8 &= \pm\sqrt{48} \\ x - 8 &= \pm\sqrt{16}\sqrt{3} \\ x - 8 &= \pm 4\sqrt{3} \\ x &= 8 \pm 4\sqrt{3}. \end{aligned}$$

Check the solutions!

Example

- Solve by completing the square: $a^2 + 4a + 9 = 30$.

$$\begin{gathered}
a^2 + 4a + 9 = 30 \\
a^2 + 4a = 21 \\
a^2 + 4a + 4 = 21 + 4 \\
(a+2)^2 = 25 \\
a + 2 = \pm\sqrt{25} \\
a + 2 = \pm 5 \\
a = -2 \pm 5 \\
a = -7 \quad \text{or} \quad a = 3.
\end{gathered}$$

Check the solutions!

Example

- Solve by completing the square: $p^2 = 5p + 9$.

$$\begin{gathered} p^2 = 5p + 9 \\ p^2 - 5p = 9 \\ p^2 - 5p + \frac{25}{4} = 9 + \frac{25}{4} \\ (p - \frac{5}{2})^2 = \frac{61}{4} \\ p - \frac{5}{2} = \pm \sqrt{\frac{61}{4}} \\ p - \frac{5}{2} = \pm \frac{\sqrt{61}}{2} \\ p = \frac{5}{2} \pm \frac{\sqrt{61}}{2} \\ p = \frac{5 \pm \sqrt{61}}{2}. \end{gathered}$$

Check the solutions!

Example

- Solve by completing the square: $(c-2)(c+8)=11$.

$$\begin{aligned}(c-2)(c+8)&=11\\ c^2+6c-16&=11\\ c^2+6c&=27\\ c^2+6c+9&=27+9\\ (c+3)^2&=36\\ c+3&=\pm\sqrt{36}\\ c+3&=\pm 6\\ c&=-3\pm 6\\ c=-9 \quad \text{or} \quad c&=3.\end{aligned}$$

Check the solutions!

Quadratic Equations of the Form $ax^2 + bx + c = 0$

- The process of completing the square works best when the coefficient of x^2 is 1.
- If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Example

- Solve by completing the square: $2m^2 + 16m + 14 = 0$.

$$\begin{gathered}
2m^2 + 16m + 14 = 0 \\
m^2 + 8m + 7 = 0 \\
m^2 + 8m = -7 \\
m^2 + 8m + 16 = -7 + 16 \\
(m+4)^2 = 9 \\
m + 4 = \pm\sqrt{9} \\
m + 4 = \pm 3 \\
m = -4 \pm 3 \\
m = -7 \quad \text{or} \quad m = -1.
\end{gathered}$$

Check the solutions!

Example

- Solve by completing the square: $3r^2 - 2r = 21$.

$$\begin{gathered} 3r^2 - 2r = 21 \\ r^2 - \frac{2}{3}r = 7 \\ r^2 - \frac{2}{3}r + \frac{1}{9} = 7 + \frac{1}{9} \\ \left(r - \frac{1}{3}\right)^2 = \frac{64}{9} \\ r - \frac{1}{3} = \pm\sqrt{\frac{64}{9}} \\ r - \frac{1}{3} = \pm\frac{8}{3} \\ r = \frac{1}{3} \pm \frac{8}{3} \\ r = -\frac{7}{3} \quad \text{or} \quad r = 3. \end{gathered}$$

Check the solutions!

Subsection 3

Solve Quadratic Equations Using the Quadratic Formula

We Shall Learn and Practice

- Solve quadratic equations using the Quadratic Formula.
- Use the Discriminant to predict the number and type of solutions of a quadratic equation.
- Identify the most appropriate method to use to solve a quadratic equation.

Solve Quadratic Equations Using the Quadratic Formula

- **Quadratic Formula**: The solutions to a quadratic equation of the form
$$ax^2 + bx + c = 0, \text{ where } a \neq 0,$$
are given by the formula:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
- To use the Quadratic Formula follow the steps:
  - Write the quadratic equation in standard form to identify the values of a, b and c;
  - Write the quadratic formula and substitute in the values of a, b and c;
  - Simply the fraction and solve for x;
  - Check the solution(s).

Example

- Solve by using the Quadratic Formula: $3y^2 - 5y + 2 = 0$.

  We have $a = 3$, $b = -5$ and $c = 2$.

  We set up the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3}$$
$$= \frac{5 \pm \sqrt{1}}{6} = \frac{5 \pm 1}{6} = \begin{cases} \frac{2}{3} \\ 1 \end{cases}$$

  Check the solutions!

Example

- Solve by using the Quadratic Formula: $a^2 - 2a = 15$.

  Rewrite in standard form $a^2 - 2a - 15 = 0$.

  We have $a = 1$, $b = -2$ and $c = -15$.

  We set up the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-15)}}{2 \cdot 1}$$
$$= \frac{2 \pm \sqrt{64}}{2} = \frac{2 \pm 8}{2} = \begin{cases} -3 \\ 5 \end{cases}$$

  Check the solutions!

Example

- Solve by using the Quadratic Formula: $3m^2 + 12m + 7 = 0$.

  We have $a = 3$, $b = 12$ and $c = 7$.

  We set up the quadratic formula:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 3 \cdot 7}}{2 \cdot 3} \\
&= \frac{-12 \pm \sqrt{60}}{6} = \frac{-12 \pm \sqrt{4}\sqrt{15}}{6} \\
&= \frac{-12 \pm 2\sqrt{15}}{6} = \frac{2(-6 \pm \sqrt{15})}{6} = \frac{-6 \pm \sqrt{15}}{3}.
\end{aligned}
$$

  Check the solutions!

Example

- Solve by using the Quadratic Formula: $4a^2 - 2a + 8 = 0$.

  We may simplify first: $2a^2 - a + 4 = 0$.

  We have $a = 2,\ b = -1$ and $c = 4$.

  We set up the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot 4}}{2 \cdot 2}$$
$$= \frac{1 \pm \sqrt{-31}}{4} = \frac{1 \pm \sqrt{31}i}{4}.$$

  Check the solutions!

Example

- Solve by using the Quadratic Formula: $x(x+2)-5=0$.

  Rewrite in standard form: $x^2+2x-5=0$.

  We have $a=1,\ b=2$ and $c=\ -5$.

  We set up the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{2^2-4\cdot 1\cdot(-5)}}{2\cdot 1} \\ &= \frac{-2 \pm \sqrt{24}}{2} = \frac{-2 \pm \sqrt{4}\sqrt{6}}{2} = \frac{-2 \pm 2\sqrt{6}}{2} \\ &= \frac{2(-1 \pm \sqrt{6})}{2} = -1 \pm \sqrt{6}. \end{aligned}$$

  Check the solutions!

Example

- Solve by using the Quadratic Formula: $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$.

  Rewrite in standard form: $\frac{1}{4}c^2 - \frac{1}{3}c - \frac{1}{12} = 0$.

  Multiply by the LCD to clear denominators:

$$12(\tfrac{1}{4}c^2 - \tfrac{1}{3}c - \tfrac{1}{12}) = 12 \cdot 0$$
$$3c^2 - 4c - 1 = 0.$$

  We have $a = 3$, $b = -4$ and $c = -1$.

  We set up the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 3 \cdot (-1)}}{2 \cdot 3}$$
$$= \frac{4 \pm \sqrt{28}}{6} = \frac{4 \pm \sqrt{4}\sqrt{7}}{6} = \frac{4 \pm 2\sqrt{7}}{6} = \frac{2 \pm \sqrt{7}}{3}.$$

  Check the solutions!

Discriminant to Predict the Number and Type of Solutions

- In the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

the quantity

$$D = b^2 - 4ac$$

is called the **discriminant**.

- To use the discriminant, $b^2 - 4ac$, to determine the number and type of solutions of a quadratic equation, follow the steps:
  - If $b^2 - 4ac > 0$, the equation has 2 real solutions;
  - If $b^2 - 4ac = 0$, the equation has 1 real solution;
  - If $b^2 - 4ac < 0$, the equation has 2 complex solutions.

Example

- Determine the number and type of solutions to each quadratic equation. (a) $8m^2 - 3m + 6 = 0$ (b) $5z^2 + 6z - 2 = 0$ (c) $9w^2 + 24w + 16 = 0$.

(a) Compute

$$b^2 - 4ac = (-3)^2 - 4 \cdot 8 \cdot 6 = 9 - 192 = -183 < 0.$$

Thus, $8m^2 - 3m + 6 = 0$ has two complex solutions;

(b) Compute

$$b^2 - 4ac = 6^2 - 4 \cdot 5 \cdot (-2) = 36 + 40 = 76 > 0.$$

Thus, $5z^2 + 6z - 2 = 0$ has two real solutions;

(c) Compute

$$b^2 - 4ac = 24^2 - 4 \cdot 9 \cdot 16 = 576 - 576 = 0.$$

Thus, $9w^2 + 24w + 16 = 0$ has one real solution.

Identify the Most Appropriate Solution Method

- To identify the most appropriate method to solve a quadratic equation:
  - We try Factoring first.
    If the quadratic factors easily, this method is very quick;
  - Try the Square Root Property next.
    If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property;
  - Use the Quadratic Formula.
    Any other quadratic equation is best solved by using the Quadratic Formula.

Example

- Identify the most appropriate method to use to solve each quadratic equation. (a) $x^2 + 6x + 8 = 0$ (b) $(n-3)^2 = 16$ (c) $5p^2 - 6p = 9$.

(a) Since factoring $x^2 + 6x + 8$ is easy, the factoring method works best.

(b) Since the equation has form $(x-h)^2 = k$, the Square Root Property works best.

(c) Rewrite $5p^2 - 6p - 9 = 0$.

It seems like factoring is either difficult or impossible. So the quadratic formula may be the best method.

Subsection 4

Solve Quadratic Equations in Quadratic Form

We Shall Learn and Practice

- Solve equations in quadratic form.

Solve Equations in Quadratic Form

- To solve equations in quadratic form follow the steps:
  - Identify a substitution that will put the equation in quadratic form;
  - Rewrite the equation with the substitution to put it in quadratic form;
  - Solve the quadratic equation for u;
  - Substitute the original variable back into the results, using the substitution;
  - Solve for the original variable;
  - Check the solutions.

Example

- Solve: $x^4 - 6x^2 + 8 = 0$.
  Set $u = x^2$. Then $u^2 = (x^2)^2 = x^4$.
  Substitute: $u^2 - 6u + 8 = 0$.
  Solve for u:
  $$\begin{gathered} u^2 - 6u + 8 - 0 \\ (u-4)(u-2) = 0 \\ u = 4 \quad \text{or} \quad u = 2. \end{gathered}$$
  Go back to the substitution formula to get equations for x:
  $$\begin{gathered} x^2 = 2 \quad \text{or} \quad x^2 = 4 \\ x = \pm\sqrt{2} \quad \text{or} \quad x = \pm\sqrt{4} \\ x = \pm\sqrt{2} \quad \text{or} \quad x = \pm 2. \end{gathered}$$
  Check the solutions!

Example

- Solve: $(x-5)^2+6(x-5)+8=0$.
  Set $u=x-5$. Then $u^2=(x-5)^2$.
  Substitute: $u^2+6u+8=0$.
  Solve for u:
  $$\begin{gathered} u^2+6u+8-0 \\ (u+2)(u+4)=0 \\ u=-2 \quad \text{or} \quad u=-4. \end{gathered}$$
  Go back to the substitution formula to get equations for x:
  $$\begin{gathered} x-5=-2 \quad \text{or} \quad x-5=-4 \\ x=3 \quad \text{or} \quad x=1. \end{gathered}$$
  Check the solutions!

Example

- Solve: $x - 7\sqrt{x} + 12 = 0$.
  Set $u = \sqrt{x}$. Then $u^2 = (\sqrt{x})^2 = x$.
  Substitute: $u^2 - 7u + 12 = 0$.
  Solve for u:
  $$\begin{aligned} u^2 - 7u + 12 &= 0 \\ (u-3)(u-4) &= 0 \\ u = 3 \quad \text{or} \quad u &= 4. \end{aligned}$$
  Go back to the substitution formula to get equations for x:
  $$\begin{array}{ccc} \sqrt{x} = 3 & \text{or} & \sqrt{x} = 4 \\ x = 3^2 & \text{or} & x = 4^2 \\ x = 9 & \text{or} & x = 16. \end{array}$$
  Check the solutions!

Example

- Solve: $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} - 14 = 0$.
  Set $u = x^{\frac{1}{3}}$. Then $u^2 = (x^{\frac{1}{3}})^2 = x^{\frac{2}{3}}$.
  Substitute: $u^2 - 5u - 14 = 0$.
  Solve for u:
  $$\begin{gathered} u^2 - 5u - 14 = 0 \\ (u-7)(u+2) = 0 \\ u = 7 \quad \text{or} \quad u = -2. \end{gathered}$$
  Go back to the substitution formula to get equations for x:
  $$\begin{gathered} x^{\frac{1}{3}} = 7 \quad \text{or} \quad x^{\frac{1}{3}} = -2 \\ \sqrt[3]{x} = 7 \quad \text{or} \quad \sqrt[3]{x} = -2 \\ x = 7^3 \quad \text{or} \quad x = (-2)^3 \\ x = 343 \quad \text{or} \quad x = -8. \end{gathered}$$
  Check the solutions!

Example

- Solve: $8x^{-2} - 10x^{-1} + 3 = 0$.

  Set $u = x^{-1}$. Then $u^2 = (x^{-1})^2 = x^{-2}$.

  Substitute: $8u^2 - 10u + 3 = 0$.

  Solve for u:

$$\begin{aligned} 8u^2 - 10u + 3 &= 0 \\ 8u^2 - 4u - 6u + 3 &= 0 \\ 4u(2u - 1) - 3(2u - 1) &= 0 \\ (4u - 3)(2u - 1) &= 0 \\ u = \tfrac{3}{4} \quad \text{or} \quad u &= \tfrac{1}{2}. \end{aligned}$$

  Go back to the substitution formula to get equations for x:

$$\begin{aligned} x^{-1} = \tfrac{3}{4} \quad &\text{or} \quad x^{-1} = \tfrac{1}{2} \\ x = \tfrac{4}{3} \quad &\text{or} \quad x = 2. \end{aligned}$$

  Check the solutions!

Subsection 5

Solve Applications of Quadratic Equations

We Shall Learn and Practice

- Solve applications modeled by quadratic equations.

Solve Applications Modeled by Quadratic Equations

- Follow the steps:
  - Read the problem carefully;
  - Identify what we are looking for and choose a variable to represent that quantity;
  - Translate into an equation;
  - Solve the equation using algebra techniques;
  - Check the answer in the problem and make sure it makes sense;
  - Answer the question with a complete sentence.

Example

- The product of two consecutive odd integers is 99. Find the integers.

  Suppose n and $n+2$ are the two consecutive odd integers.

  Since their product is 99, we get $n(n+2) = 99$.

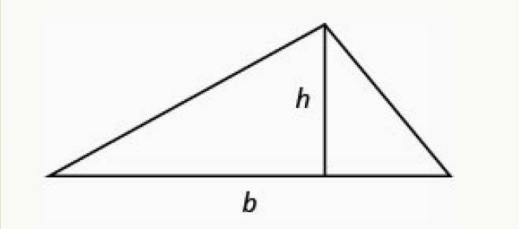
  We solve for n to find the integers.

$$\begin{gathered} n(n+2) = 99 \\ n^2 + 2n = 99 \\ n^2 + 2n - 99 = 0 \\ (n-9)(n+11) = 0 \\ n = 9 \quad \text{or} \quad n = -11. \end{gathered}$$

  Hence the two integers are either 9 and 11 or -11 and -9.

Area of a Triangle

- For a triangle with base, b, and height, h,



the area, A, is given by the formula

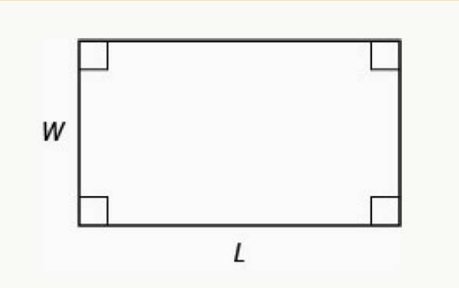
$$A = \frac{1}{2}bh.$$

Example

- Find the base and height of a triangle whose base is four inches more than six times its height and has an area of 456 square inches.
  Let b be the base and h be the height of the given triangle.
  The base is four inches more than six times the height. So we have the equation $b = 6h + 4$.
  Moreover, we know that the area is given by $A = \frac{1}{2}bh$.
  So we get the equation:
  $$\frac{1}{2}(6h+4)h = 456$$
  $$(3h+2)h = 456$$
  $$3h^2 + 2h - 456 = 0$$
  $$h = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3 \cdot (-456)}}{2 \cdot 3}$$
  $$= \frac{-2 \pm \sqrt{5476}}{6} = \frac{-2 \pm 74}{6} = \frac{72}{6} = 12.$$
  Hence, the triangle has height 12 inches and base $b = 76$ inches.

Area of a Rectangle

- For a rectangle with length, L, and width, W,



the area, A, is given by the formula

$$A = LW.$$

Example

- The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

  Let L be the width and W be the length of the garden.

  The length is four feet less than twice the width: $L = 2W - 4$.

  Moreover, the area is given by $A = LW$.

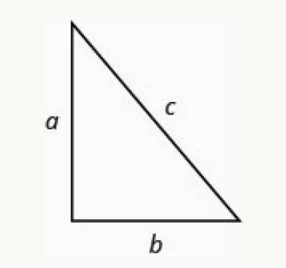
  So we get the equation:

$$\begin{gathered}(2W-4)W = 200\\ 2W^2 - 4W = 200\\ 2W^2 - 4W - 200 = 0\\ W^2 - 2W - 100 = 0\\ W = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-100)}}{2 \cdot 1} = \frac{2 \pm \sqrt{404}}{2} \approx 11.\end{gathered}$$

Hence, the garden has width ≈ 11 inches and length $L \approx 18$ inches.

Pythagorean Theorem

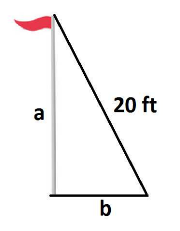
- **Pythagorean Theorem**: In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse,



$$a^2 + b^2 = c^2.$$

Example

- The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole.
  Let a be the length of the flag pole and b the length of the shadow.
  The height of the pole is three times the length of its shadow: $a = 3b$.
  Moreover, the hypothenuse of the right triangle is 20 ft.
  So, by the Pythagorean Theorem, we get the equation:



$$\begin{aligned}(3b)^2 + b^2 &= 20^2\\ 9b^2 + b^2 &= 400\\ 10b^2 &= 400\\ b^2 &= 40\\ b = \pm\sqrt{40} &= \pm 2\sqrt{10}.\end{aligned}$$

Hence, the shadow has length $2\sqrt{10}$ ft. and the pole $6\sqrt{10}$ ft.

Height of an Object Shot Upward

- The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula
$$h = -16t^2 + v_0 t.$$
- We can use this formula to find how many seconds it will take for the object to reach a specific height.

Example

- An arrow is shot from the ground into the air at an initial speed of 108 ft/s. Use the formula $h = -16t^2 + v_0 t$ to determine when the arrow will be 180 feet from the ground.

  Let t be the time when the arrow will be 180 feet from the ground.

  Then, substituting in $h = -16t^2 + v_0 t$, we get:

$$\begin{gathered} 180 = -16t^2 + 108t \\ 16t^2 - 108t + 180 = 0 \\ 4t^2 - 27t + 45 = 0 \\ 4t^2 - 12t - 15t + 45 = 0 \\ 4t(t-3) - 15(t-3) = 0 \\ (4t-15)(t-3) = 0 \\ t = \frac{15}{4} \quad \text{or} \quad t = 3. \end{gathered}$$

  Hence, the object will be at height 180 feet at time $t = 3$ secs and at time $t = 3.75$ secs

Uniform Motion

- We already encountered the formula

$$\text{Distance} = \text{Velocity} \times \text{Time}$$
$$D = rt.$$

- The formula $D = rt$ assumes we know r and t and use them to find D.
- If we know D and r and need to find t, we would solve the equation for t and get the formula $t = \frac{D}{r}$.

Example

- MaryAnne just returned from a visit with her grandchildren back east. The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?
  Suppose that the speed of the jet stream is s.
  Then the speed of the forward trip was $(500+s)$ mph and that of the return trip $(500-s)$ mph.
  Assume the forward trip took t hours and the return trip took t' hours.
  Using $t=\frac{D}{r}$, we get the two equations $t=\frac{2400}{500+s}$ and $t'=\frac{2400}{500-s}$.
  We also know that $t+t'=10$.
  Hence we get the eqution
  $$\frac{2400}{500+s}+\frac{2400}{500-s}=10.$$

Example (Cont'd)

- We solve the equation.

$$\begin{aligned}
\frac{2400}{500+s}+\frac{2400}{500-s} &= 10\\
(500+s)(500-s)\left(\frac{2400}{500+s}+\frac{2400}{500-s}\right) &= 10(500+s)(500-s)\\
2400(500-s)+2400(500+s) &= 10(500+s)(500-s)\\
240(500-s)+240(500+s) &= (500+s)(500-s)\\
120000-240s+120000+240s &= 250000-s^2\\
240000 &= 250000-s^2\\
s^2 &= 10000\\
s &= 100.
\end{aligned}$$

Thus the speed of the jet stream was 100 mph.

Work

- Recall the framework in which
  - a worker A needs a hours to complete a job;
  - a worker B needs b hours to complete the same job.
- To find the time t it would take them working together to complete the same job, we set up the equation

$$\frac{1}{t} = \frac{1}{a} + \frac{1}{b}.$$

Example

- Press #1 takes 6 hours more than Press #2 to print a magazine When both are running they can print the magazine in 4 hours. How long does it take for each press to print the job alone?
  Suppose it takes Press #1 t hours to print the job alone.
  Then it takes Press #2 $t-6$ hours to print the job alone.
  Since it takes them together 4 hours to do the job, we get

$$\begin{gathered}\frac{1}{t}+\frac{1}{t-6}=\frac{1}{4}\\ 4t(t-6)\left(\frac{1}{t}+\frac{1}{t-6}\right)=4t(t-6)\frac{1}{4}\\ 4(t-6)+4t=t(t-6)\\ 4t-24+4t=t^2-6t\\ t^2-14t+24=0\\ (t-12)(t-2)=0\\ t=2 \quad \text{or} \quad t=12.\end{gathered}$$

  Hence, it takes Press #1 12 hours and Press #2 6 hours to do the job alone.

Subsection 6

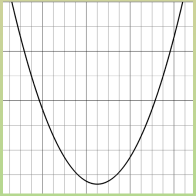
Graph Quadratic Functions Using Properties

We Shall Learn and Practice

- Recognize the graph of a quadratic function.
- Find the axis of symmetry and vertex of a parabola.
- Find the intercepts of a parabola.
- Graph quadratic functions using properties.
- Solve maximum and minimum applications.

Recognize the Graph of a Quadratic Function

- A quadratic function is a function of the form
$$f(x) = ax^2 + bx + c,$$
where a, b, and c are real numbers and $a \neq 0$.
- Every quadratic function has a graph that looks like



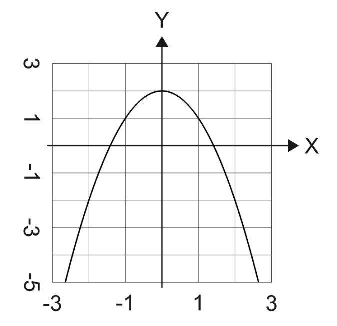
- We call this figure a **parabola**.

Example

- Graph $f(x) = -x^2 + 2$.

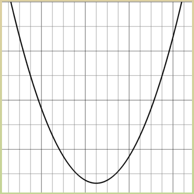
  We create a small table of values and plot the points:

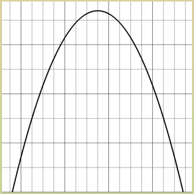
x	$f(x)$
-2	-2
-1	1
0	2
1	1
2	-2



Parabola Orientation

- The parabola corresponding to $f(x) = ax^2 + bx + c$ opens





  - Up, if $a > 0$;
  - Down, if $a < 0$.

Example

- Determine whether the graph of each function is a parabola that opens upward or downward: (a) $f(x) = 2x^2 + 5x - 2$ (b) $f(x) = -3x^2 - 4x + 7$.

(a) In this case $a = 2 > 0$.

So the parabola opens up.

(b) In this case $a = -3 < 0$.

So the parabola opens down.

Axis of Symmetry and Vertex of a Parabola

- The graph of the function

$$f(x) = ax^2 + bx + c$$

is a parabola where:
  - The axis of symmetry is the vertical line $x = -\frac{b}{2a}$;
  - The vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$;
  - The y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation, i.e., we compute $f(-\frac{b}{2a})$.

Example

- For the graph of $f(x) = 2x^2 - 8x + 1$ find: (a) the axis of symmetry (b) the vertex.

(a) The axis of symmetry is the vertical line

$$x = -\frac{b}{2a} = -\frac{-8}{2 \cdot 2} = \frac{8}{4} = 2.$$

(b) The x-coordinate of the vertex is $x = 2$.

To find the y-coordinate, we calculate

$$f(2) = 2 \cdot 2^2 - 8 \cdot 2 + 1 = 8 - 16 + 1 = -7.$$

Thus the vertex is the point $(2, -7)$.

Intercepts of a Parabola

- To find the intercepts of a parabola whose function is

$$f(x) = ax^2 + bx + c$$

  follow the steps:
  - For the y-intercept set $x = 0$ and solve for y;
  - For the x-intercepts set $y = 0$ and solve for x.

Example

- Find the intercepts of the parabola whose function is $f(x) = x^2 + 2x - 8$.

  For the y-intercept, set $x = 0$ and compute:

  $$f(0) = 0^2 + 2 \cdot 0 - 8 = -8.$$

  Thus the y-intercept is $(0, -8)$.

  For the x-intercepts, set $y = 0$ and compute:

  $$\begin{gathered} x^2 + 2x - 8 = 0 \\ (x-2)(x+4) = 0 \\ x = 2 \quad \text{or} \quad x = -4. \end{gathered}$$

  Thus, the x-intercepts are $(-4, 0)$ and $(2, 0)$.

Example

- Find the intercepts of the parabola whose function is $f(x) = 3x^2 + 4x + 4$.
  For the y-intercept, set $x = 0$ and compute:
  $$f(0) = 3 \cdot 0^2 + 4 \cdot 0 + 4 = 4.$$
  Thus the y-intercept is $(0, 4)$.
  For the x-intercepts, set $y = 0$:
  $$3x^2 + 4x + 4 = 0.$$
  We have
  $$D = b^2 - 4ac = 4^2 - 4 \cdot 3 \cdot 4 = 16 - 48 = -32 < 0.$$
  Hence, $3x^2 + 4x + 4 = 0$ does not have any real roots.
  It follows that $f(x)$ has no x-intercepts.

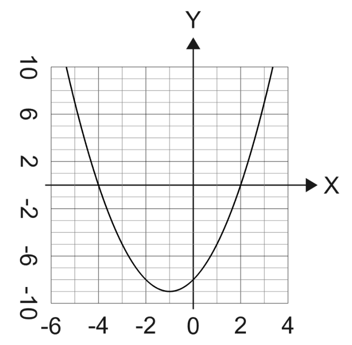
Graph Quadratic Functions Using Properties

- To graph a quadratic function follow the steps:
  - Determine whether the parabola opens up or down;
  - Find the vertex and the axis of symmetry;
  - Find the y- and the x-intercepts;
  - Find additional points if needed;
  - Plot the graph.

Example

- Graph $f(x) = x^2 + 2x - 8$ by using its properties.

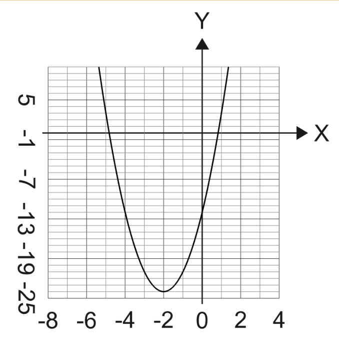
- Opens up ($a = 1 > 0$);
- Axis of symmetry: $x = -\frac{2}{2 \cdot 1} = -1$;
- Vertex $(-1, -9)$;
- y-intercept $(0, -8)$;
- x-intercepts
  $x^2 + 2x - 8 = 0 \Rightarrow (x+4)(x-2) = 0 \Rightarrow x = -4, x = 2$
  Points: $(-4, 0)$ and $(2, 0)$.



Example

- Graph $f(x) = 3x^2 + 12x - 12$ by using its properties.

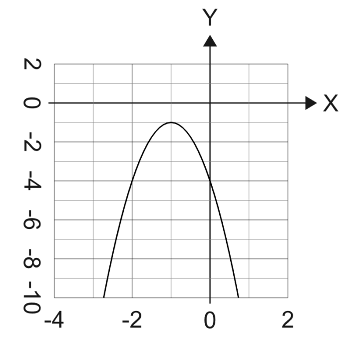
- Opens up $(a = 3 > 0)$;
- Axis of symmetry $x = -\frac{12}{2 \cdot 3} = -2$;
- Vertex $(-2, -24)$;
- y-intercept $(0, -12)$;
- x-intercepts:
  $3x^2 + 12x - 12 = 0 \Rightarrow x^2 + 4x - 4 = 0 \Rightarrow x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot (-4)}}{2 \cdot 1} = \frac{-4 \pm \sqrt{32}}{2} = \frac{-4 \pm 4\sqrt{2}}{2} = -2 \pm 2\sqrt{2}$.
  Points $(-2 - 2\sqrt{2}, 0)$ and $(-1 + 2\sqrt{2}, 0)$.



Example

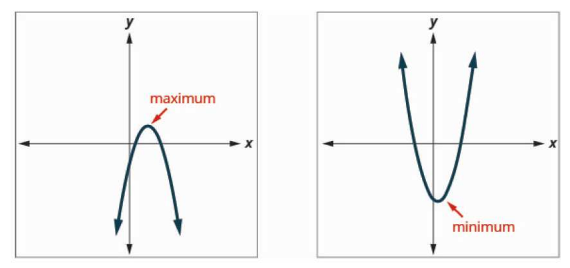
- Graph $f(x) = -3x^2 - 6x - 4$ by using its properties.

- Opens down ($a = -3 < 0$);
- Axis of symmetry $x = -\frac{-6}{2\cdot(-3)} = -1$;
- Vertex $(-1, -1)$;
- y-intercept $(0, -4)$;
- x-intercepts $-3x^2 - 6x - 4 = 0 \Rightarrow 3x^2 + 6x + 4 = 0 \Rightarrow D = b^2 - 4ac = 36 - 4 \cdot 3 \cdot 4 = 36 - 48 = -12 < 0$
- Need some more points: $(-2, -4)$



Solve Maximum and Minimum Applications

- The y-coordinate of the vertex of the graph of a quadratic function is:



  - The **minimum value** of the quadratic equation if the parabola opens upward;
  - The **maximum value** of the quadratic equation if the parabola opens downward.

Example

- Find the maximum or minimum value of the quadratic function $f(x) = x^2 - 8x + 12$.

  The parabola opens up.

  So the minimum value occurs at the vertex.

  $$x = -\frac{b}{2a} = -\frac{-8}{2 \cdot 1} = 4.$$

  Thus, the minimum value is

  $$y_{\min} = f(4) = 4^2 - 8 \cdot 4 + 12 = 16 - 32 + 12 = -4.$$

Physics: Maximum Height

- The formula $h(t) = -16t^2 + v_0 t + h_0$ gives the height in feet, h, of an object shot upwards into the air from an initial height of h_0 feet, with initial velocity, v_0, after t seconds.
- This formula is a quadratic function, so its graph is a parabola.
- By solving for the coordinates of the vertex (t, h), we can find how long it will take the object to reach its maximum height.
- Then we can calculate the maximum height.

Example

- A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function

$$h(t) = -16t^2 + 208t.$$

When will the rocket reach its maximum height? What will be the maximum height?

The maximum height will be reached at

$$t = -\frac{b}{2a} = -\frac{208}{2 \cdot (-16)} = \frac{208}{32} = 6.5 \text{ secs}$$

The maximum height will be

$$h_{\max} = h(6.5) = -16 \cdot 6.5^2 + 208 \cdot 6.5 = -676 + 1352 = 676 \text{ feet}$$

Subsection 7

Graph Quadratic Functions Using Transformations

We Shall Learn and Practice

- Graph quadratic functions of the form $f(x) = x^2 + k$.
- Graph quadratic functions of the form $f(x) = (x - h)^2$.
- Graph quadratic functions of the form $f(x) = ax^2$.
- Graph quadratic functions using transformations.
- Find a quadratic function from its graph.

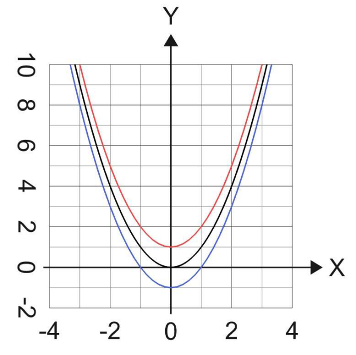
Quadratic Functions of the Form $f(x) = x^2 + k$

- The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.
  - If $k > 0$, shift the parabola vertically up k units;
  - If $k < 0$, shift the parabola vertically down $|k|$ units.

Example

- (a) Graph $f(x) = x^2$, $g(x) = x^2 + 1$, and $h(x) = x^2 - 1$ on the same rectangular coordinate system. (b) Describe what effect adding a constant to the function has on the basic parabola.

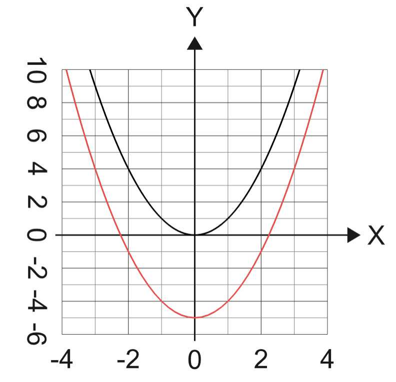
x	$f(x)$	$g(x)$	$h(x)$
-2	4	5	3
-1	1	2	0
0	0	1	-1
1	1	2	0
2	4	5	3



Example

- Graph $f(x) = x^2 - 5$ using a vertical shift.

x	x^2	$f(x)$
-2	4	-1
-1	1	-4
0	0	-5
1	1	-4
2	4	-1



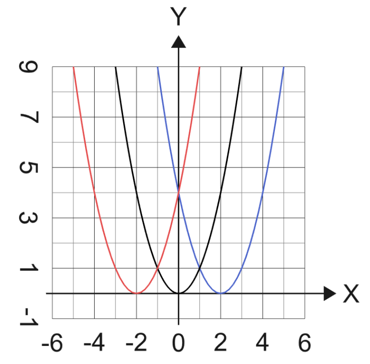
Quadratic Functions of the Form $f(x) = (x - h)^2$

- The graph of $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.
  - If $h > 0$, shift the parabola horizontally right h units;
  - If $h < 0$, shift the parabola horizontally left $|h|$ units.

Example

- (a) Graph $f(x) = x^2$, $g(x) = (x+2)^2$, and $h(x) = (x-2)^2$ on the same rectangular coordinate system. (b) Describe what effect adding a constant to the function has on the basic parabola.

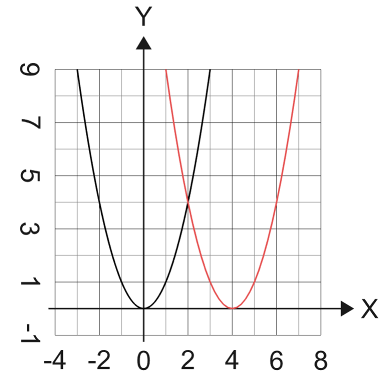
x	$f(x)$	g	h
-2	4	$(-4,4)$	$(0,4)$
-1	1	$(-3,1)$	$(1,1)$
0	0	$(-2,0)$	$(2,0)$
1	1	$(-1,1)$	$(3,1)$
2	4	$(0,4)$	$(4,4)$



Example

- Graph $f(x) = (x - 4)^2$ using a horizontal shift.

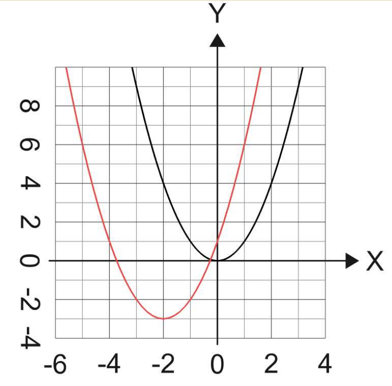
x	x^2	f
-2	4	$(2,4)$
-1	1	$(3,1)$
0	0	$(4,0)$
1	1	$(5,1)$
2	4	$(6,4)$



Example

- Graph $f(x) = (x+2)^2 - 3$ using transformations.

x	x^2	f
-2	4	$(-4, 1)$
-1	1	$(-3, -2)$
0	0	$(-2, -3)$
1	1	$(-1, -2)$
2	4	$(0, 1)$



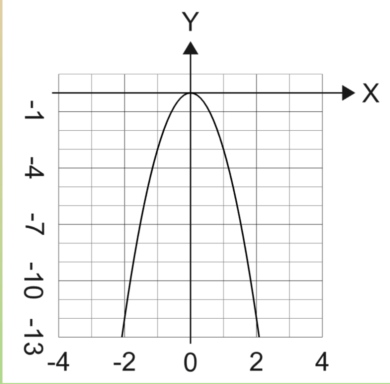
Quadratic Functions of the Form $f(x) = ax^2$

- The coefficient a in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.
  - If $0 < |a| < 1$, the graph of $f(x) = ax^2$ will be "wider" than the graph of $f(x) = x^2$;
  - If $|a| > 1$, the graph of $f(x) = ax^2$ will be "narrower" than the graph of $f(x) = x^2$.

Example

- Graph $f(x) = -3x^2$.

x	$f(x)$
-2	-12
-1	-3
0	0
1	-3
2	-12



Quadratic Functions Using Transformations

- To graph using transformations, follow the steps:
  - Rewrite the function in $f(x) = a(x-h)^2 + k$ form by completing the square;
  - Graph the function using transformations.
- To graph using properties, follow the steps:
  - Rewrite the function in $f(x) = a(x-h)^2 + k$ form;
  - Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$;
  - Find the axis of symmetry, $x = h$;
  - Find the vertex, (h, k);
  - Find the y-intercept (set $x = 0$);
  - Find the x-intercepts (set $y = 0$);
  - Graph the parabola.

Example

- Rewrite $f(x) = -4x^2 - 8x + 1$ in the $f(x) = a(x-h)^2 + k$ form by completing the square.

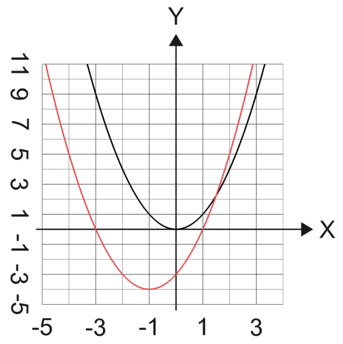
$$\begin{gathered} f(x) = -4x^2 - 8x + 1 \\ f(x) = -4(x^2 + 2x) + 1 \\ f(x) = -4(x^2 + 2x + 1 - 1) + 1 \\ f(x) = -4(x^2 + 2x + 1) + 4 + 1 \\ f(x) = -4(x+1)^2 + 5. \end{gathered}$$

Example

- Graph $f(x) = x^2 + 2x - 3$ by using transformations.
  We first transform into the form $a(x-h)^2 + k$.

$$f(x) = x^2 + 2x - 3$$
$$f(x) = x^2 + 2x + 1 - 1 - 3$$
$$f(x) = (x+1)^2 - 4$$

x	x^2	f
-2	4	$(-3, 0)$
-1	1	$(-2, -3)$
0	0	$(-1, -4)$
1	1	$(0, -3)$
2	4	$(1, 0)$



Example

- (a) Rewrite $f(x) = 3x^2 - 6x + 5$ in $f(x) = a(x-h)^2 + k$ form and (b) graph the function using properties.

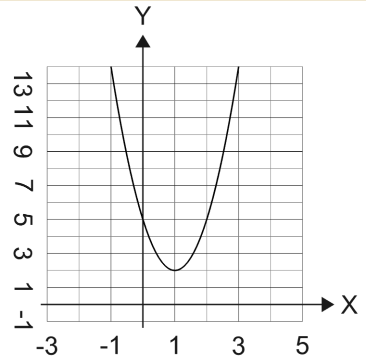
(a)

$$\begin{gathered} f(x) = 3x^2 - 6x + 5 \\ f(x) = 3(x^2 - 2x) + 5 \\ f(x) = 3(x^2 - 2x + 1 - 1) + 5 \\ f(x) = 3(x^2 - 2x + 1) - 3 + 5 \\ f(x) = 3(x-1)^2 + 2. \end{gathered}$$

On the next slide we graph the function using properties.

Example

- We graph $f(x) = 3(x-1)^2 + 2$ using properties.
  - Opens up ($a = 3 > 0$);
  - Axis of symmetry $x = 1$;
  - Vertex $(1, 2)$;
  - y-intercept $(0, 5)$;
  - x-intercepts $3(x-1)^2 + 2 = 0 \Rightarrow (x-1)^2 = -\frac{2}{3} \Rightarrow$ no solutions
  - Need some more points: $(-1, 14)$

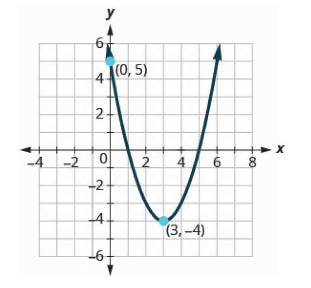


A Quadratic Function from its Graph

- Write the quadratic function in $f(x) = a(x-h)^2 + k$ form whose graph is shown.

  Since the vertex is at $(h, k) = (3, -4)$, we get

  $$f(x) = a(x-3)^2 - 4.$$



Since $(0, 5)$ is a point on the parabola, we must have

$$\begin{aligned} a(0-3)^2 - 4 &= 5 \\ 9a - 4 &= 5 \\ 9a &= 9 \\ a &= 1. \end{aligned}$$

Thus, the equation is $f(x) = (x-3)^2 - 4$.

Subsection 8

Solve Quadratic Inequalities

We Shall Learn and Practice

- Solve quadratic inequalities graphically.
- Solve quadratic inequalities algebraically.

Solve Quadratic Inequalities Graphically

- A **quadratic inequality** is an inequality that contains a quadratic expression.
- The standard form of a quadratic inequality is written:

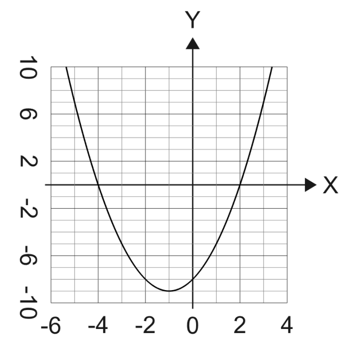
$$ax^2 + bx + c < 0 \qquad ax^2 + bx + c \leq 0$$
$$ax^2 + bx + c > 0 \qquad ax^2 + bx + c \geq 0$$

- To solve a quadratic inequality graphically, follow the steps:
  - Write the quadratic inequality in standard form;
  - Graph the function $f(x) = ax^2 + bx + c$;
  - Determine the solution from the graph.

Example

- (a) Solve $x^2 + 2x - 8 < 0$ graphically and (b) write the solution in interval notation.
  - Opens up $(a = 1 > 0)$;
  - Axis of symmetry: $x = -\frac{2}{2 \cdot 1} = -1$;
  - Vertex $(-1, -9)$;
  - y-intercept $(0, -8)$;
  - x-intercepts $x^2 + 2x - 8 = 0 \Rightarrow (x+4)(x-2) = 0 \Rightarrow x = -4, x = 2$

    Points: $(-4, 0)$ and $(2, 0)$.



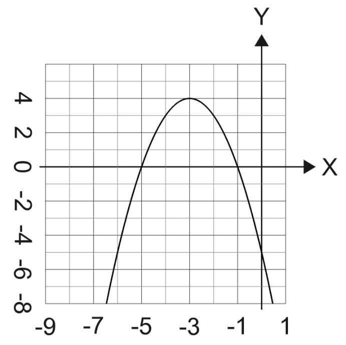
Thus for $x^2 + 2x - 8$ to be negative, we must have $-4 < x < 2$.

In interval notation, the solution set is $(-4, 2)$.

Example

- (a) Solve $-x^2 - 6x - 5 \geq 0$ graphically and (b) write the solution in interval notation.
  - Opens down ($a = -1 < 0$);
  - Axis of symmetry: $x = -\frac{-6}{2\cdot(-1)} = -3$;
  - Vertex $(-3, 4)$;
  - y-intercept $(0, -5)$;
  - x-intercepts $-x^2 - 6x - 5 = 0 \Rightarrow x^2 + 6x + 5 = 0 \Rightarrow (x+1)(x+5) = 0 \Rightarrow x = -5, x = -1$

    Points: $(-5, 0)$ and $(-1, 0)$.



Thus for $-x^2 - 6x - 5$ to be ≥ 0, we must have $-5 \leq x \leq -1$.

In interval notation, the solution set is $[-5, -1]$.

Solve Quadratic Inequalities Algebraically

- To solve a quadratic inequality algebraically, follow the steps:
  - Write the quadratic inequality in standard form.
  - Determine the critical points - the solutions to the related quadratic equation.
  - Use the critical points to divide the number line into intervals.
  - Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.
  - Determine the intervals where the inequality is correct and write the solution in interval notation.

Example

- Solve $x^2 + 2x - 8 \geq 0$ algebraically. Write the solution in interval notation.

  We solve the equation $x^2 + 2x - 8 = 0$ algebraically:

$$\begin{aligned} x^2 + 2x - 8 &= 0 \\ (x+4)(x-2) &= 0 \\ x = -4 \quad \text{or} \quad x &= 2. \end{aligned}$$

  Use these two points to divide the line and then test the sign of the expression $x^2 + 2x - 8$ is each interval formed:

	$(-\infty, -4)$	$(-4, 2)$	$(2, +\infty)$
$x^2 + 2x - 8$	$+$	$-$	$+$

  Since we want $x^2 + 2x - 8$ to be ≥ 0, we get $x \leq -4$ or $x \geq 2$. In interval notation, the solution set is $(-\infty, -4] \cup [2, +\infty)$.

Example

- Solve $-x^2+2x+1 \geq 0$ algebraically. Write the solution in interval notation.

  We solve the equation $-x^2+2x+1=0$ algebraically:

$$\begin{gathered} -x^2+2x+1=0 \\ x^2-2x-1=0 \\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-2)\pm\sqrt{(-2)^2-4\cdot1\cdot(-1)}}{2\cdot1} \\ x=\frac{2\pm\sqrt{8}}{2}=\frac{2\pm2\sqrt{2}}{2}=\frac{2(1\pm\sqrt{2})}{2}=1\pm\sqrt{2}. \end{gathered}$$

  Use these two points to divide the line and then test the sign of the expression $-x^2+2x+1$ is each interval formed:

	$(-\infty,1-\sqrt{2})$	$(1-\sqrt{2},1+\sqrt{2})$	$(1+\sqrt{2},+\infty)$
$-x^2+2x+1$	$-$	$+$	$-$

  Since we want x^2+2x-8 to be ≥ 0, we get $1-\sqrt{2}\leq x\leq 1+\sqrt{2}$. In interval notation $[1-\sqrt{2},1+\sqrt{2}]$.

Example

- Solve and write any solution in interval notation: (a) $-x^2+2x-4\leq 0$ (b) $-x^2+2x-4\geq 0$.
  We solve the equation $-x^2+2x-4=0$ algebraically:
  $$\begin{gathered}-x^2+2x-4=0\\ x^2-2x+4=0\\ D=b^2-4ac=(-2)^2-4\cdot 1\cdot 4=-12<0;\\ \text{So, the equation has no real roots.}\end{gathered}$$
  The line remains undivided. So we test the sign of the expression $-x^2+2x-4$ in the only existing interval:

	$(-\infty,+\infty)$
$-x^2+2x-4$	$-$

(a) Since we want $-x^2+2x-4$ to be ≤ 0, x must be in $(-\infty,\infty)$.
(b) Since we want $-x^2+2x-4$ to be ≥ 0, the inequality has no solutions.