Introduction to Knot Theory

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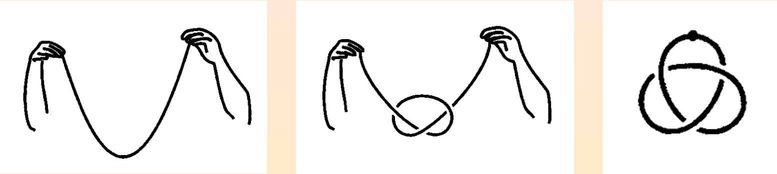
LSSU Math 500

Subsection 1

Introduction

Tying a Knot

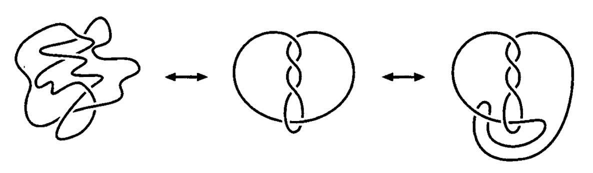
- Take a piece of string.



- Tie a knot in it.
- Now glue the two ends of the string together to form a knotted loop.
- The result is a string that has no loose ends and that is truly knotted.
- Unless we use scissors, there is no way that we can untangle this string.

Knots and Deformations

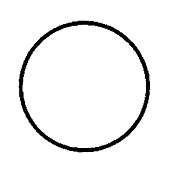
- A **knot** is just such a knotted loop of string, except that we think of the string as having no thickness, i.e., its cross-section is a single point.
- The knot is then a closed curve in space that does not intersect itself anywhere.
- We will not distinguish between the original closed knotted curve and the deformations of that curve through space that do not allow the curve to pass through itself.



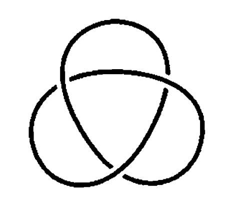
All of these deformed curves will be considered to be the same knot.

The Unknot and the Trefoil Knot

- The simplest knot of all is just the unknotted circle, which we call the **unknot** or the **trivial knot**.



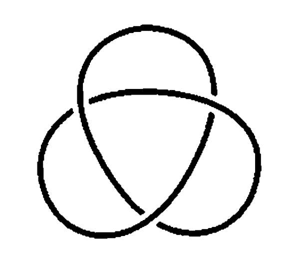
- The next simplest knot is called a **trefoil knot**.
  - How do we know these are actually different knots?
  - How do we know that we couldn't untangle the trefoil knot into the unknot without using scissors and glue, if we tried hard enough?

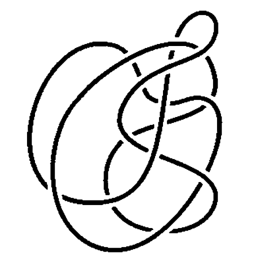


- We may believe by experimentation that untangling the trefoil knot cannot be done, but we would like to be able to prove it mathematically.

Another Picture of the Trefoil Knot

- Another picture of a knot that is actually a trefoil knot, even though it looks completely different from the standard picture of a trefoil.

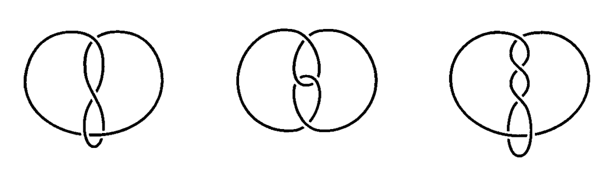




- Can you describe a deformation of the knot on the right into the trefoil knot on the left?

Projection of a Knot

- There are many different pictures of the same knot.
- In the figure, we see three different pictures of a new knot, called the **figure-eight knot**.

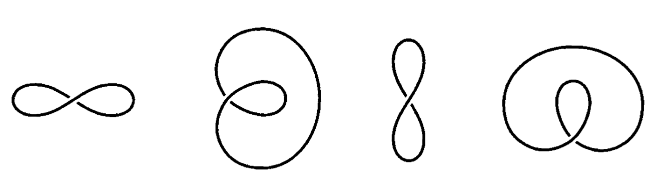


- We call such a picture of a knot a **projection** of the knot.
- The places where the knot crosses itself in the picture are called the **crossings** of the projection.
- We say that the figure-eight knot is a four-crossing knot because:
  - There is a projection of it with four crossings;
  - There are no projections of it with fewer than four crossings.

Non-Trivial Knots and Crossings

- If a knot is to be nontrivial, then it must have more than one crossing in a projection.

  If it only has one crossing, then the four ends of the single crossing must be hooked up in pairs in one of the four ways shown below.

  Any other projection with one crossing can be deformed to look like one of these without undoing the crossing.

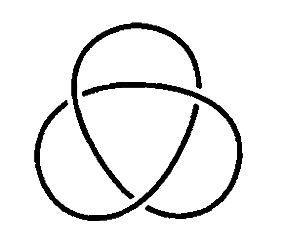
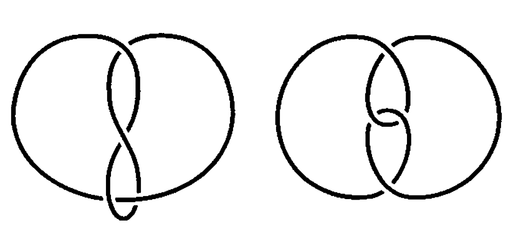
  But in each of these we can untwist the single crossing.

  So each of these represents the trivial knot.

Alternating Knots

- An **alternating knot** is a knot with a projection that has crossings that alternate between over and under as one travels around the knot in a fixed direction.

  Examples: The trefoil knot on the left is alternating.

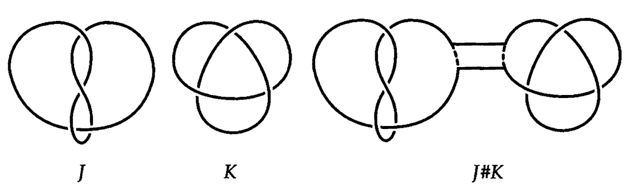
  So is the figure-eight knot on the right.

Subsection 2

Composition of Knots

Composition of Two Knots

- Given two projections of knots, we can define a new knot by:
  - Removing a small arc from each knot projection;
  - Connecting the four endpoints by two new arcs.



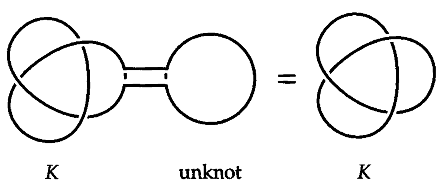
- We call the resulting knot the **composition** of the two knots.
- If we denote the two knots by the symbols J and K, then their composition is denoted by $J\#K$.

Composition of Two Knots (Cont'd)

- We assume that the two projections do not overlap.
- We choose the two arcs that we remove to be on the outside of each projection and to avoid any crossings.
- We choose the two new arcs so they do not cross either the original knot projections or each other.

Composite Knots, Factor Knots and Prime Knots

- We call a knot a **composite knot** if it can be expressed as the composition of two knots, neither of which is the trivial knot.
- The knots that make up the composite knot are called **factor knots**.
- If we take the composition of a knot K with the unknot, the result is again K.



- If a knot is not the composition of any two nontrivial knots, we call it a **prime knot**.

  Example: Both the trefoil knot and the figure-eight knot are prime knots, although this is not obvious.

The Status of the Unknot

- The unknot is not a composite knot.
- This is not obvious, since there could potentially be a way to tangle the unknot up so that we get a projection of it that makes it a composite knot.
- I.e., there could be a picture of the unknot, such that:
  - It has a nontrivial knot on the left;
  - It has a nontrivial knot on the right;
  - It has two strands of the knot joining them;
  - The part of the projection corresponding to the knot on the right somehow untangles that part of the projection corresponding to the knot on the left, resulting in the unknot.

Composition and The Status of the Unknot

- If the unknot were a composite knot, then every knot would be a composite knot.

  Every knot is the composition of itself with the unknot.

  So every knot would be the composition of itself with the nontrivial factor knots that made up the unknot.

Composition of Knots vs Integer Multiplication

- We can think of the indecomposability of the unknot as analogous to the fact that the integer 1 is not the product of two positive integers, each greater than 1.
- Just as an integer factors into a unique set of prime numbers, a composite knot factors into a unique set of prime knots.
- One way that composition of knots differs from multiplication of integers is that there is more than one way to take the composition of two knots.
- We have a choice of where we remove the arc from the outside of each projection and this choice affects the outcome.
- I.e., it is often possible to construct two different composite knots from the same pair of knots J and K.

Orientation

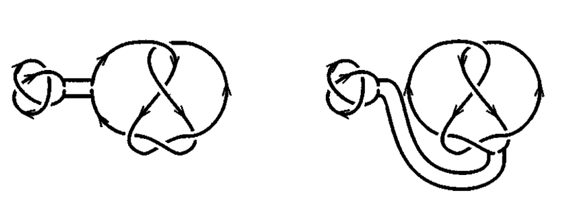
- An **orientation** on a knot is defined by choosing a direction to travel around the knot.
- This direction is denoted by placing coherently directed arrows along the projection of the knot in the direction of our choice.
- We then say that the knot is **oriented**.

Orientation and Composition

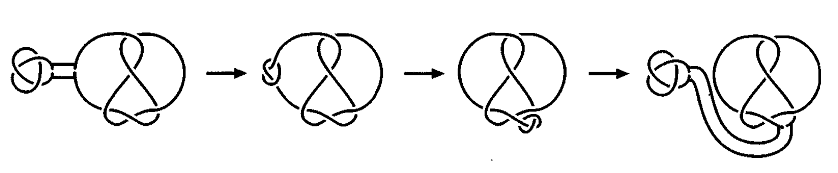
- When we then form the composition of two oriented knots J and K, there are two possibilities:
  - The orientation on J matches the orientation on K in $J\#K$, resulting in an orientation for $J\#K$;
  - The orientation on J and K do not match up in $J\#K$.
- All of the compositions of the two knots where the orientations do match up will yield the same composite knot.
- All of the compositions of the two knots where the orientations do not match up will also yield a single composite knot.
- The two composites, however, one obtained from matching and the other from non-matching orientations, may be distinct.

Example

- The two compositions below give us the same knot.

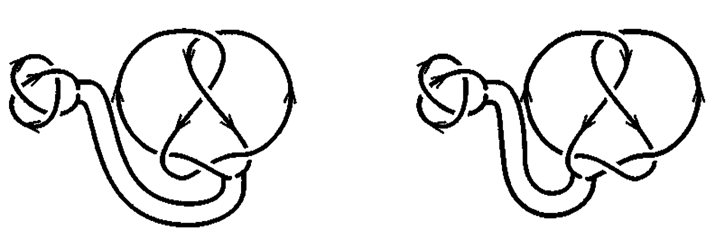


To see this, we can shrink J down in the first picture and then slide it around K until we obtain the second picture.



Example

- Although this will not be the case in general, the two compositions in the following figure where orientations match and orientations differ, respectively, give the same knot.



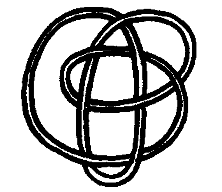
- This occurs because one of the factor knots is invertible.

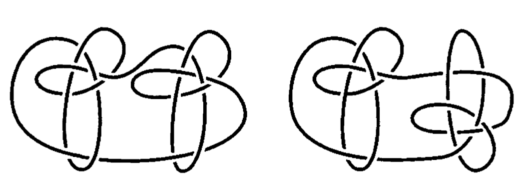
Invertible Knots

- A knot is **invertible** if it can be deformed back to itself so that an orientation on it is sent to the opposite orientation.
- In the case that one of the two knots is invertible, say J, we can always deform the composite knot so that the orientation on K is reversed.
- I.e., we can make the orientations of J and K to always match.
- Therefore, there is only one composite knot that we can construct from the two knots.

Example

- The knot shown on the left, known as Knot 8_{17} is not invertible.





Composing it with itself in the two different ways produces two distinct composite knots that are not equivalent.

- To determine the possible compositions of knots, we need to know which knots are invertible.

Subsection 3

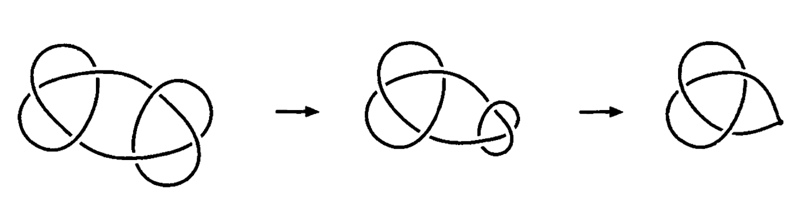
Reidemeister Moves

Ambient Isotopies

- Suppose that we have two projections of the same knot.
- If we made a knot out of string that modeled the first of the two projections, then we should be able to rearrange the string to resemble the second projection.
- This rearranging of the string, i.e., the movement of the string through three-dimensional space without letting it pass through itself, is called an **ambient isotopy**.
  - The word "isotopy" refers to the deformation of the string.
  - The word "ambient" refers to the fact that the string is being deformed through the three-dimensional space that it sits in.

An Important Remark

- In an ambient isotopy we are not allowed to shrink a part of the knot down to a point, as in the figure, in order to be rid of the knot.

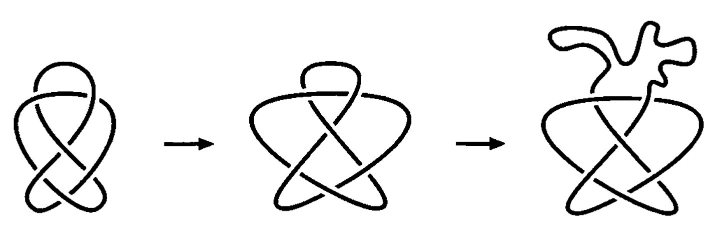


- It's easiest to think of a knot made of string.

  Just as we cannot get rid of a knot in a string by pulling it tighter and tighter, so an ambient isotopy does not allow us to get rid of a knot in this manner.

Planar Isotopies

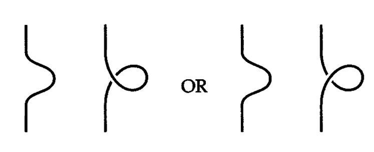
- A deformation of a knot projection is called a **planar isotopy** if it deforms the projection plane as if it were made of rubber with the projection drawn upon it.

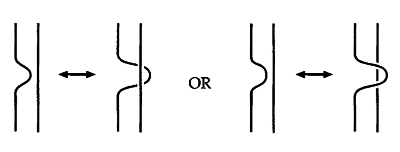


- The word "planar" is used because we are only deforming the knot within the projection plane.

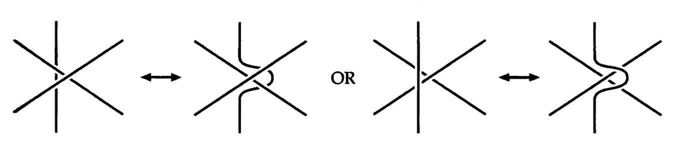
Reidemeister Moves

- A **Reidemeister move** is one of three ways to change a projection of the knot that will change the relation between the crossings.
  - The **first Reidemeister move** allows us to put in or take out a twist.





  - The **second Reidemeister move** allows us to either add two crossings or remove two crossings.
  - The **third Reidemeister move** allows us to slide a strand of the knot from one side of a crossing to the other side of the crossing.



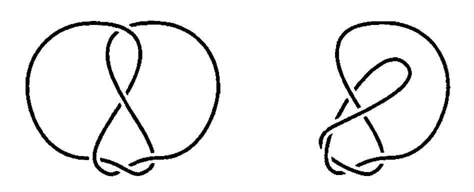
Reidemeister's Theorem

- Although each of these moves changes the projection of the knot, it does not change the knot represented by the projection.
- Each such move is an ambient isotopy.
- The German mathematician Kurt Reidemeister proved:

  If we have two distinct projections of the same knot, we can get from the one projection to the other by a series of Reidemeister moves and planar isotopies.

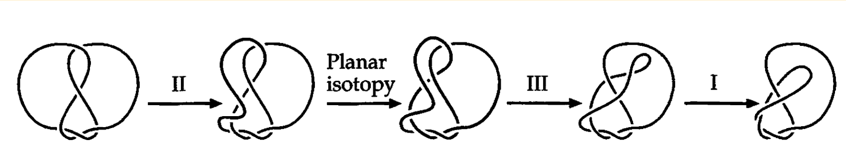
Example

- The two projections in the figure correspond to the same knot.



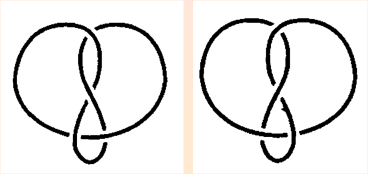
Therefore, according to Reidemeister, there is a series of Reidemeister moves that takes us from the first projection to the second.

- We show one series of moves that demonstrates the equivalence.

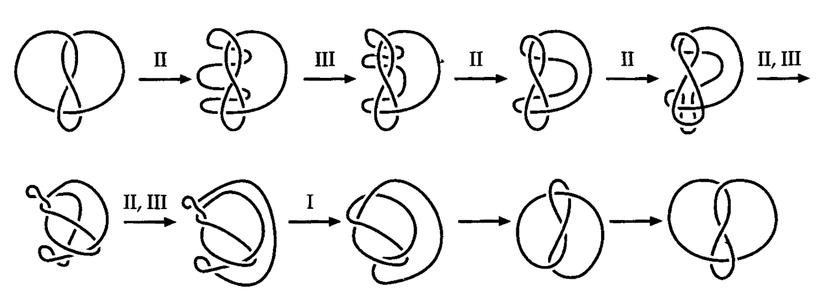


Mirror Equivalence of the Figure-Eight Knot

- The figure-eight knot is known to be equivalent to its mirror image.
- This is the knot obtained by changing every crossing in the knot to the opposite crossing.



- We show a sequence of Reidemeister moves for this equivalence.



Amphichiral Knots

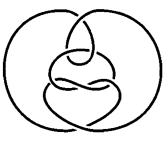
- A knot that is equivalent to its mirror image is called **amphichiral** by mathematicians and **achiral** by chemists.
- We consider a knot and its mirror image to be distinct knots unless the knot is amphichiral.

Limitation of Reidemeister Moves

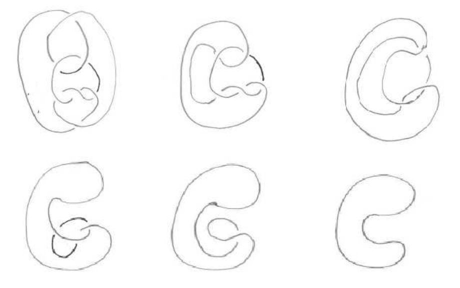
- Reidemeister moves and planar isotopy suffice to get us from any one projection of a knot to any other projection of that knot.
- So it might seem that the problem of determining whether two projections represent the same knot would be easy.
- We have to check whether or not there is a sequence of Reidemeister moves to get us from the one projection to the other.
- Unfortunately, there is no limit on the number of Reidemeister moves that it might take us to get from one projection to the other.
- E.g., if the two original projections have 10 crossings each, it is conceivable that, in the process of performing the Reidemeister moves, we will have to increase the number of crossings to 1000, before the moves simplify the projection back down to 10 crossings.

A Nasty Unknot

- Consider the knot shown:



- Show that this knot actually represents the unknot.
- Find a sequence of Reidemeister moves to untangle the knot.



Subsection 4

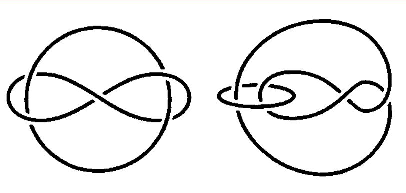
Links

Links

- A **link** is a set of knotted loops all tangled up together.
- Two links are considered to be the **same** if we can deform the one link to the other link without ever having anyone of the loops intersect itself or any of the other loops in the process.

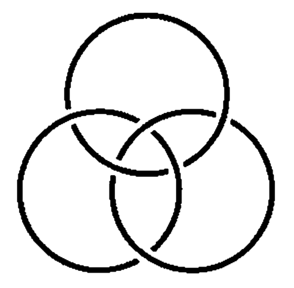
  Example: The picture shows two projections of one of the simplest links, known as the **Whitehead link**.

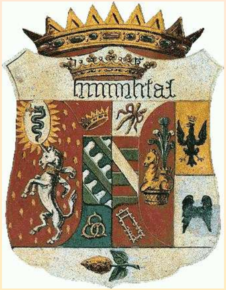
  Since it is made up of two loops knotted with each other, we say that it is a **link of two components**.



Example: Borromean Rings

- The following link has three components, called the **Borromean rings**.





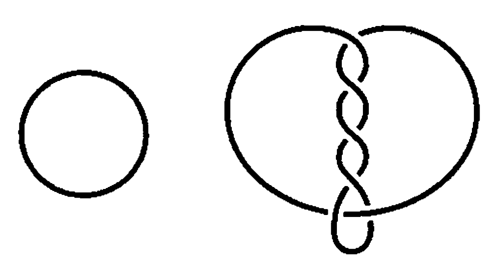
- This link is named after the Borromeo, an Italian family from the Duchy of Milan, who used this pattern of interlocking rings on their family crest.

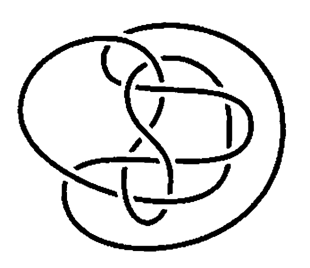
Reidemeister Moves for Links

- A knot will be considered a link of one component.
- Almost everything we have said about knots holds true for links.
- For instance, if two projections represent the same link, there must be a sequence of Reidemeister moves to get from the one projection to the other.

Splittable Links

- A link is called **splittable** if the components of the link can be deformed so that they lie on different sides of a plane in three-space.
- Sometimes it is obvious when a link is splittable, as on the left.



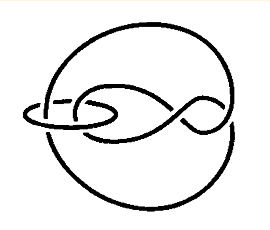


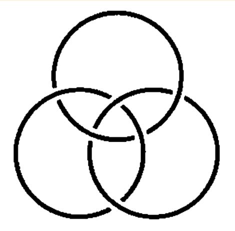
- It is often the case that a link is splittable, but we cannot easily tell that by looking at the projection, as on the right.

Telling Certain Links Apart

- There is one quick way for telling certain links apart:
  - Count the number of components in the link;
  - If the numbers are different, the two links have to be different.

Example: The trefoil knot, the Whitehead link, and the Borromean rings all have to be distinct links.

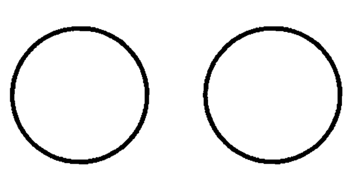
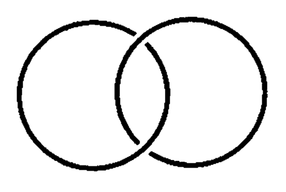




Telling Apart Links with Same Number of Components

- If we have two projections of links, each with the same number of components, we would like to be able to tell if they represent the same link.

  Example: Consider the two simplest links of two components.

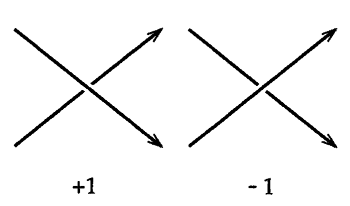
  The first is the **unlink** (or **trivial link**) **of two components**.

  The second is the **Hopf link**.

  One difference between these two links is that:
  - The unlink is splittable (components can be separated by a plane).
  - In the Hopf link, the two components do link each other once.

Orientations and Crossings

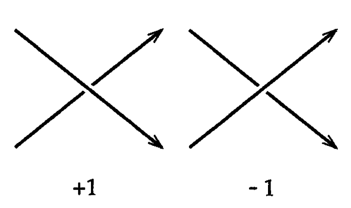
- Let M and N be two components in a link.
- Choose an orientation on each of them.
- Then at each crossing between the two components, one of the following two cases will arise.



- We count a:
  - $+1$ for each crossing of the first type;
  - -1 for each crossing of the second type.

Orientations and Crossings (Cont'd)

- Sometimes it is hard to determine from the picture



whether a crossing is of the first type or the second type.
  - If a crossing is of the first type, then rotating the understrand clockwise lines it up with the overstrand so that their arrows match.
  - If a crossing is of the second type, then rotating the understrand counterclockwise lines the understrand up with the overstrand so that their arrows match.

The Linking Number

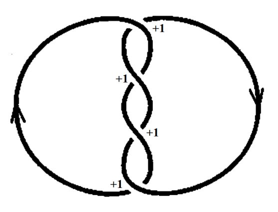
- We define the **linking number** of the link with components M and N as the sum of the $+1$s and -1s over all the crossings between M and N divided by 2.
- We do not count the crossings between a component and itself.

  Example:

  For the unlink, the linking number of the two components is 0.

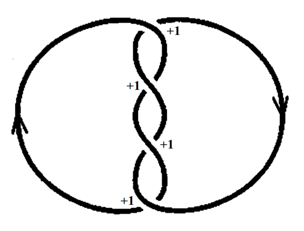
  For the Hopf link, the linking number will be 1 or -1, depending on the orientations on the two components.

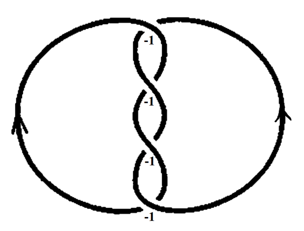
  The two components in the oriented link pictured on the right have linking number 2.



Linking Number and Orientation

- If we reverse the orientation on one of the two components, but not the other, the linking number of these two components is multiplied by -1.





- The absolute value of the linking number, however, is independent of the orientations on the two components.

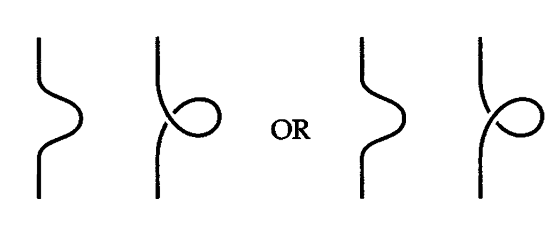
Linking Number and Projections

- Notice that we use a particular projection of the link in order to compute the linking number.
- In fact, we can show that the computed linking number will always be the same, no matter what projection of the link we use to compute it.
- We show this by proving that the Reidemeister moves do not change the linking number.
- This is because we will know that:
  - We can get from anyone projection of a link to any other via a sequence of Reidemeister moves;
  - No Reidemeister move changes the linking number.

  So it will follow that two different projections of the same link yield the same linking number.

Linking Number and Type I Moves

- We look at the effect of the first Reidemeister move on the linking number.

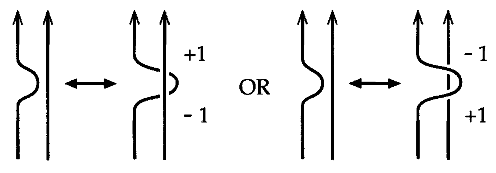


It can create or eliminate a self-crossing in one of the two components. This will not affect the crossings that involve both components. So the move leaves the linking number unchanged.

Linking Number and Type II Moves

- We look at what a Type II Reidemeister move does.

  In the figure we have chosen orientations on the strands.



We assume that the strands correspond to two different components, because otherwise the move has no effect on the linking number.

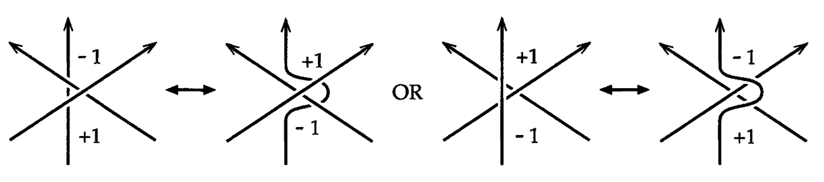
One of the new crossings contributes a $+1$ to the sum, and the other crossing contributes a -1. So the net contribution is 0.

Even if we change the orientation on one of the strands, we will still have one $+1$ and one -1 contribution.

So Type II moves leave the linking number unchanged.

Linking Number and Type III Moves

- Finally we look at Type III moves.



Suppose orientations are chosen for each of the three strands and $+1$s and -1s are assigned to each of the crossings.

It is clear that sliding the strand over in the Type III move does not change the number of $+1$s or -1s.

So the linking number is preserved.

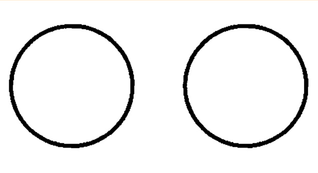
Invariants

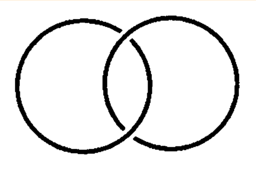
- We say that the linking number is an **invariant** of the oriented link.
- This means that, once the orientations are chosen on the two components of the link, the linking number is unchanged by ambient isotopy.
- It remains invariant when the projection of the link is altered.
- Another invariant of links that we have already mentioned is simply the number of components in the link.
- It is also unchanged by ambient isotopies of the link.

Distinguishing Links Using the Linking Number

- We can use the linking number to distinguish links.
- We want to distinguish links that do not already have orientations.
- So we use the absolute value of the linking number.
- Any two links with two components that have distinct absolute values of their linking numbers have to be different links.

  Example:

  The trivial link of two components has linking number 0.

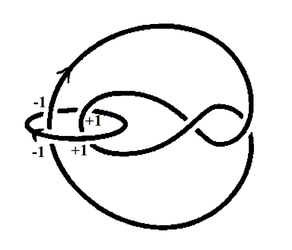
  The absolute value of the linking number of the Hopf link is 1.

  So the Hopf link cannot be the trivial link.

Limitation of the Linking Number

- Unfortunately, there are different links that have equal linking numbers.

  Example:

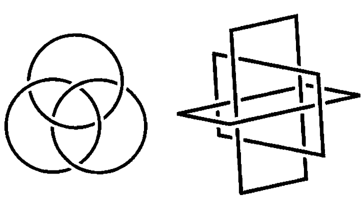


  The linking number for the Whitehead link is 0, just like the trivial link of two components.

- So, based on the linking numbers, we cannot even show that the Whitehead link is different from the trivial link of two components.

Brunnian Links

- Consider again the Borromean rings.



- If we removed anyone of the three components of this link, the remaining two components would become two trivial unlinked circles.
- The fact that these three rings are locked together relies on the presence of all three components.
- A link is called **Brunnian** if:
  - The link itself is nontrivial;
  - The removal of anyone of the components leaves us with a set of trivial unlinked circles.

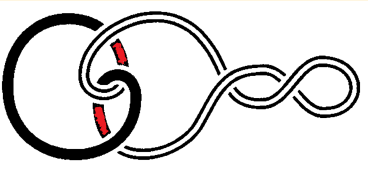
Subsection 5

Tricolorability

Tricolorability

- A **strand** in a projection of a link is a piece of the link that goes from one undercrossing to another with only overcrossings in between.
- A projection of a knot or link is **tricolorable** if each of the strands in the projection can be colored one of three different colors, so that at each crossing, either three different colors come together or all the same color comes together.
- In order that a projection be tricolorable, we further require that at least two of the colors are used.

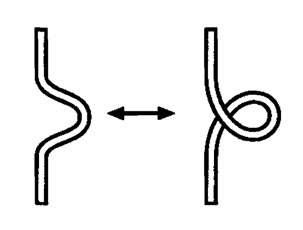
  Example: Two projections of the trefoil knot that are tricolorable:

Tricolorability and Reidemeister Moves (Type I)

Claim: If a projection of a knot is tricolorable, then the Reidemeister moves will preserve the tricolorability.

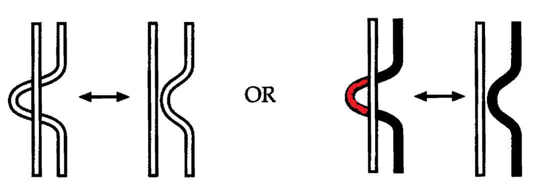
Suppose we do a Type I move and introduce a crossing.
We can just leave all the strands involved the same color.
The new crossing will satisfy the requirements for tricolorability.



Similarly, removing a crossing by a Type I move preserves tricolorability.

Tricolorability and Reidemeister Moves (Type II)

- Similarly, using a Type II move to reduce the number of crossings by two will also preserve tricolorability.



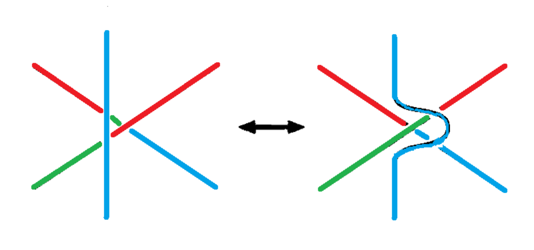
  - If all of the strands. that appear in the diagram for the Reidemeister move are the same color, we can color the strands that result from the Reidemeister move that same color;
  - If three distinct colors come together at each of the two crossings, we can color the two resulting strands as on the right figure.
- In both these cases, since the original projection was colored with at least two distinct colors, the resulting projection will also be colored with at least two colors.

Tricolorability and Reidemeister Moves (Type III)

- Similarly, we can show that Type III Reidemeister move preserves tricolorability.

  There are several cases to check.

  One of them is shown below.



Tricolorability and Projections

- Reidemeister moves leave tricolorability unaffected.
- So whether or not a projection is tricolorable depends only on the knot given by the projection.
- Either every projection of a knot is tricolorable or no projection of that knot is tricolorable.

  Example: The unknot is not tricolorable.

  Consider the usual projection of the unknot.

  It does not have two distinct strands.

  So we cannot use at leat two colors on it.
- So any knot that is tricolorable must be distinct from the unknot.

The Trefoil Knot and the Unknot

- Every projection of the trefoil knot is tricolorable.

- We saw that the unknot is not tricolorable.
- Thus, the trefoil knot and the unknot must be distinct.
- So there is at least one other knot besides the unknot.

Tricolorability and Composition

- The composition of any knot with a tricolorable knot yields a new tricolorable knot.

  Consider any tricolored projection of a tricolorable knot.

  Suppose, in the composition process, the small arc removed from it is on a blue strand.

  Then color all strands of the other knot blue and compose.

  This gives a tricolored projection of the composition knot.

  Example: The trefoil knot is tricolorable.

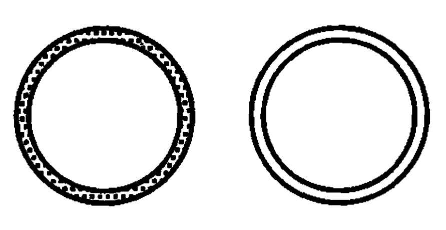
  So the composition of the trefoil knot with any other knot is tricolorable.

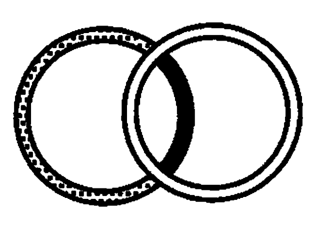
  Since the unknot is not tricolorable, the unknot cannot be the composition of the trefoil knot with any other knot.

Tricolorability for Links

- Tricolorability for links of two components is slightly different.

  Example: The trivial link of two components is tricolorable.

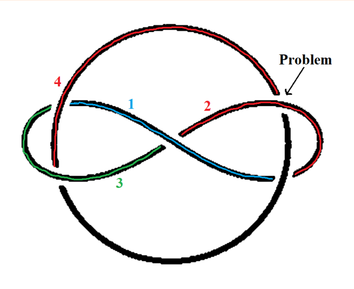




- This is the reverse of what happened for tricolorability for knots.
- If we have a link of two components that is not tricolorable, we know it cannot be the unlink.

The Whitehead Link Revisited

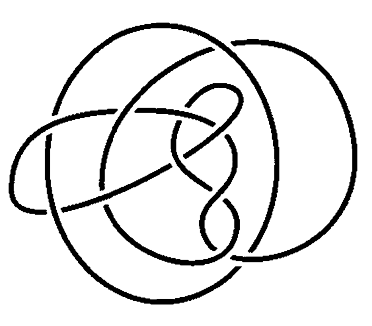
- The Whitehead link is not tricolorable.

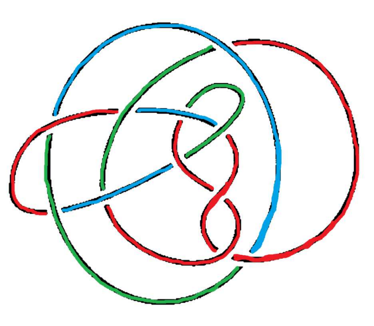


- So it is not the trivial link of two components.
- The linking number was not enough to show this before.

Example: A Tricolorable Link

- We can show that the link in the figure is tricolorable.



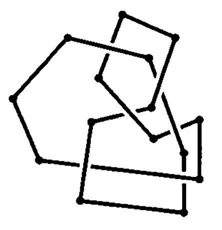


Subsection 6

Knots and Sticks

Knots and Sticks

- Suppose given a collection of straight sticks.
- The sticks can be any length that we want.
- We want to glue them together end to end in order to make a nontrivial knot.



- How many sticks will it take to make a nontrivial knot?
- Three sticks are not enough

  They would just form a triangle that lies in a plane.

  If we looked down at the plane, we would see a projection of the knot with no crossings.

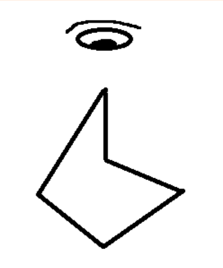
  So it would have to be the unknot.

Using Four Sticks

- If we view the four sticks from any direction, we will see a projection of the corresponding knot.
- If two of the sticks are attached to each other at their ends, they cannot cross each other in the projection.
  - Two straight lines can cross at most once.
  - Here, the crossing point is the point where the sticks are attached.
- So in the projection, each stick can only cross the one stick that is not attached to either one of its ends.
- Therefore, there can be at most two crossings in the projection.
- The only knot with a projection of two or fewer crossings is the unknot.

Using Five Sticks

- We form a knot formed using five sticks.
- We look on the projection resulting by looking straight down one of the five sticks.
- In the projection of the knot that we see, we will only be able to see four of the sticks, since the fifth stick is vertical.

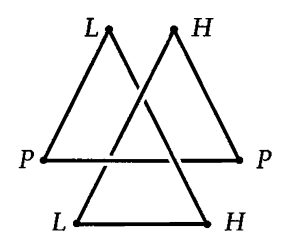


- For the same reason as in the previous paragraph, the four sticks that we see can have at most two crossings.
- So the knot we see must the unknot.
- This implies that the full knot is the unknot.

Six Sticks to Form an Unknot

- We showed it must take at least six sticks to make a knot.
- In fact, it is possible to make a trefoil knot with six sticks.

- The picture looks believable, but we must verify that we can really make a trefoil knot in space out of straight sticks like this.
- The sticks should not be bent or warped to fit together in this way.



- Let the vertices labeled P lie in the xy plane.
- The vertices labeled L lie low, underneath the plane.
- The vertices labeled H lie high, above the plane.
- Then it is clear that the knot can be constructed from sticks.

The Stick Number

- Define the **stick number** $s(K)$ of a knot K to be the least number of straight sticks necessary to make K.
- It was shown by Adams et.al. in 1997 that, if J and K are knots,

$$s(J \# K) \leq s(J) + s(K) - 3.$$