Introduction to Knot Theory

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 500

1. Tabulating Knots
   - The Dowker Notation for Knots
   - Conway's Notation
   - Knots and Planar Graphs

Subsection 1

The Dowker Notation for Knots

Alternating Knots

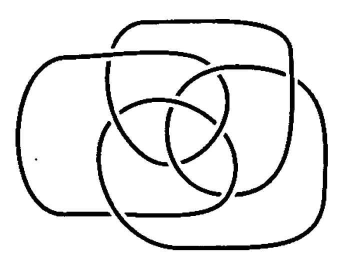
- A knot diagram is **alternating** if the crossings alternate under, over, under, over, as one travels along the knot.
- A knot is **alternating** if it has an alternating diagram.

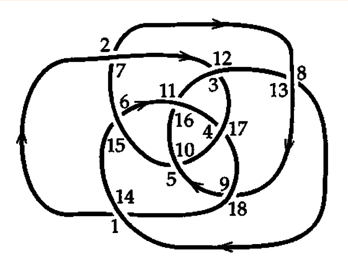
Dowker Notation: Alternating Knots

- The **Dowker notation** is a simple way to describe a projection of a knot.
- We handle first alternating knots.
- Suppose we have a projection of an alternating knot.
- Choose an orientation on the knot.
  - Pick any crossing and label it 1.
  - Leaving that crossing along the understrand in the direction of the orientation, label the next crossing that you come to with a 2.
  - Continue through that crossing on the same strand of the knot, and label the next crossing with a 3.
  - Continue to label the crossings with the integers in sequence until you have gone all the way around the knot once.
- When you are done, each crossing will have two labels on it, since the knot passes through each crossing twice.

Dowker Notation: Alternating Knots (Illustration)

- Consider the alternating knot on the left:

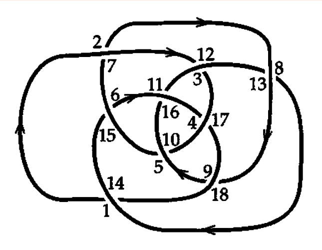




- Following the process outlined in the preceding slide we get the numbering of the crossing as on the right.
- Notice that each crossing has one even number and one odd number labeling it.

Compactifying the Notation

- We can think of this labeling as giving us a pairing between the odd numbers and the even numbers from 1 to 18



- In this case, we get

1	3	5	7	9	11	13	15	17
14	12	10	2	18	16	8	6	4

- As a shorthand, we could just write

$$14 \quad 12 \quad 10 \quad 2 \quad 18 \quad 16 \quad 8 \quad 6 \quad 4,$$

  the meaning being that 1 is paired with 14, 3 with 12, 5 with 10, etc.
- We obtain a sequence of even integers, where the number of even integers is exactly the number of crossings in the knot.

Reversing the Process

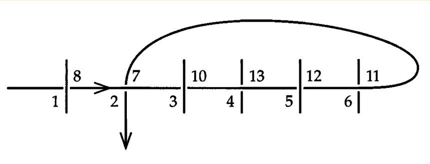
- Given a sequence of even integers that represents a projection of an alternating knot, we draw the projection.

  Example: Consider the sequence 8 10 12 2 14 6 4.

  This is shorthand for

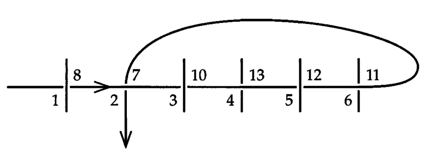
$$\begin{matrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 8 & 10 & 12 & 2 & 14 & 6 & 4 \end{matrix}.$$

  Start by drawing just the first crossing, labeling it with a 1 and an 8.



Reversing the Process (Cont'd)

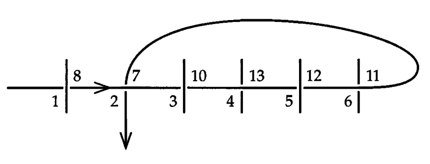
- We extend the understrand of the knot and then draw in the next crossing, which corresponds to 2.



Since 2 is paired with 7, we label this crossing with a 2 and a 7.

Because the knot is alternating, we know that the strand that we are on goes over this crossing.

Reversing the Process (Cont'd)



- We continue the overstrand through this crossing to the next crossing where it becomes the understrand, labeling the new crossing with a 3 and the integer that is paired with 3, namely 10.
- We continue this process until the next integer that should be placed on a crossing already labels an existing crossing.
- We then know that the knot must now circle around to pass through that crossing.

Reversing the Process (Cont'd)

- We have two choices as to how to circle around

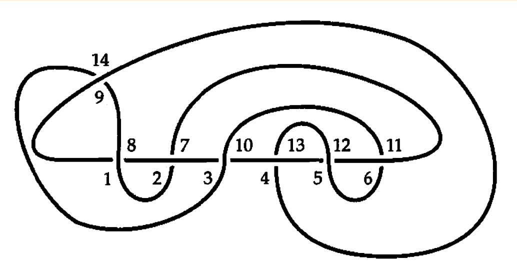
  Either circling to the right or to the left in order to pass back through the previously drawn crossing.

  For the time being, we ignore this ambiguity and just choose either direction for circling around.

Reversing the Process (Conclusion)

- We continue in this manner.
  - If neither of the labels on the next crossing has occurred before, then we make a new crossing.
  - If one of the labels has occurred before, we circle the knot through that crossing.

All the way along, we will be sure that the crossings alternate as we progress along the knot.



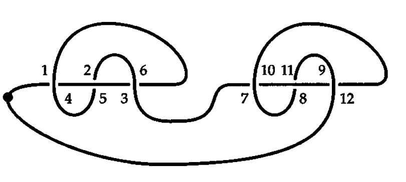
Choice of Circling Direction

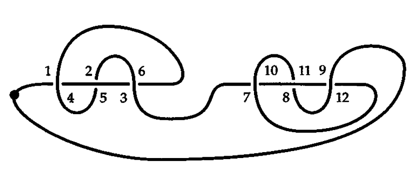
- The ambiguity in our choice of how the knot circles around is significant.
- Our choice can change the resulting knot.

  Example: The sequence

  $$4 \quad 6 \quad 2 \quad 10 \quad 12 \quad 8$$

  represents two distinct knots:



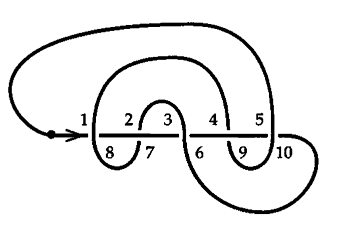


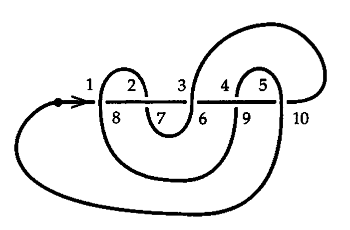
Dowker Notation: Non-Uniqueness

- The two knots represented by the sequence 4 6 2 10 12 8 are composite knots.
- This is reflected in the fact that the sequence 4 6 2 10 12 8 is actually a shuffling of the three numbers 2, 4, 6 and then a shuffling of the three numbers 8, 10, and 12.
- When the permutation of the even numbers can be broken into two separate subpermutations, the resulting knots are composite (assuming each of the factor knots is nontrivial).
- In this case, the knot is not completely determined by the Dowker notation.

Dowker Notation: Uniqueness

- If we restrict ourselves to sequences of even numbers that cannot be split into subpermutations, either a particular knot or its mirror image results.

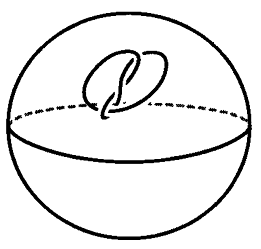




- When the knot is amphicheiral, only one knot can be the result.
- Although the possible projections look different, they will all correspond to the same pair of knots.

Projecting Onto a Sphere

- The best way to see that the possible projections will correspond to the same pair of knots is to think of projecting the knot onto a sphere rather than onto a plane.

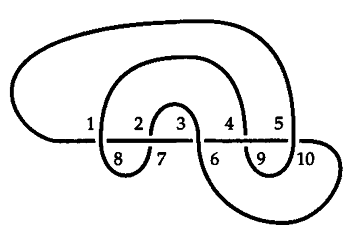


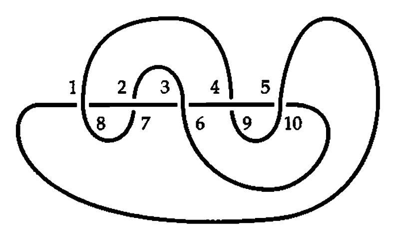
- The advantage is that there is no special outer region with infinite area as there is in a projection onto the plane.

Example

- The figure contains two projections described by

$$8 \quad 6 \quad 10 \quad 2 \quad 4.$$





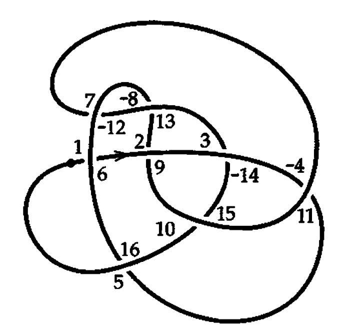
- These are distinct as projections on the plane.
- They are equivalent projections on the sphere.

Extending the Notation to Non-Alternating Knots

- To extend the system to knots that are not alternating, we add in $+$ and $-$ signs to our sequence of even numbers.
- When traversing the knot using the labeling system that we have described, we assign an even integer and an odd integer to each crossing.
  - If the even integer is assigned to the crossing while we are on the overstrand at that crossing, we leave the even integer positive.
  - If the even integer is assigned to the crossing while we are on the understrand of that crossing, we make the corresponding even number negative.

Example

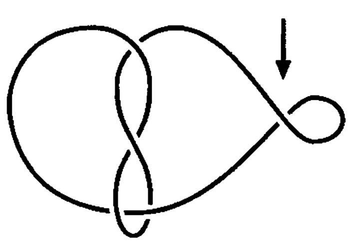
- Consider the knot of the figure.

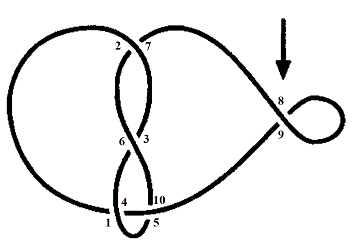


- The numbers 14, 12, 4, and 8 become negative.

Dowker Notation and Trivial Crossings

- We detect from the sequence of numbers that a projection has a trivial crossing in it like the one on the left.





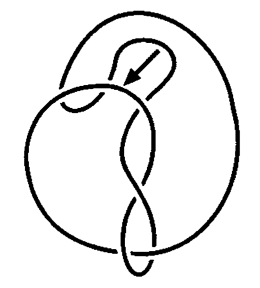
- Suppose the projection has n crossings.
- Let its Dowker Notation be

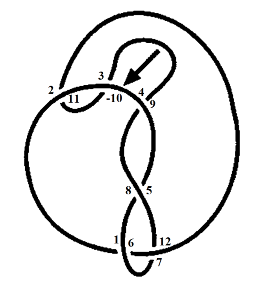
$$a_1 \quad a_3 \quad \cdots \quad a_{2n-1}.$$

- The projection has a trivial crossing if there exists a k, $1 \leq k \leq n$, such that $a_{2k-1} = 2k - 2$ or $2k$ modulo $2n$.

Dowker Notation and Type II Reidemeister Moves

- We detect from the sequence of numbers a Type II Reidemeister move that will reduce the number of crossings by two.





- There must exist k, $1 \leq k \leq n$, such that at least one of the following four pairs (two shown and horizontal reflections) arise:

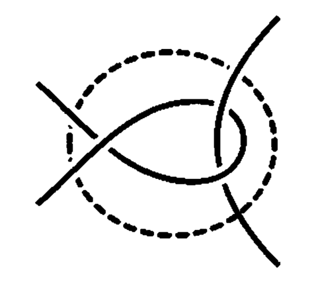
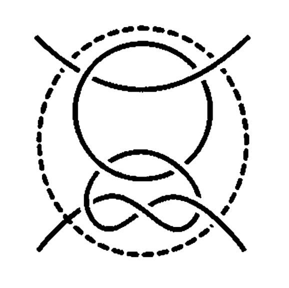
$$\begin{array}{cccccc} \cdots & 2k-1 & \cdots & a_{2k-1}+1 & \cdots \\ \cdots & \pm a_{2k-1} & \cdots & \mp 2k & \cdots \end{array} \qquad \begin{array}{cccccc} \cdots & 2k-1 & \cdots & a_{2k-1}-1 & \cdots \\ \cdots & \pm a_{2k-1} & \cdots & \mp 2k & \cdots \end{array}$$

Subsection 2

Conway's Notation

Tangles

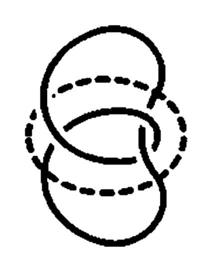
- A **tangle** in a knot or link projection is a region in the projection plane surrounded by a circle such that the knot or link crosses the circle exactly four times.

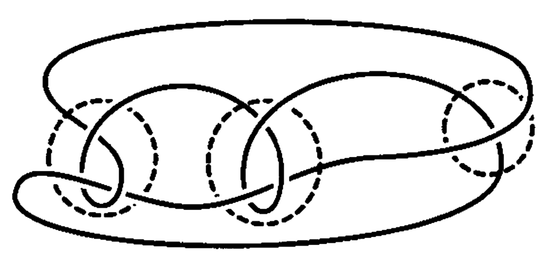
 

- We will always think of the four points where the knot or link crosses the circle as occurring in the four compass directions NW, NE, SW, and SE.

Tangles and Knots

- We can use tangles as the building blocks of knot and link projections.

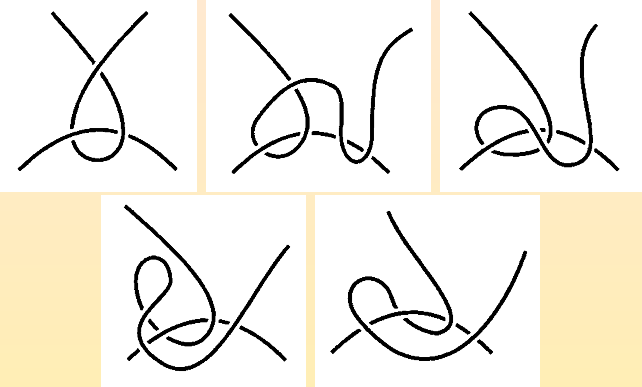




- Therefore, understanding tangles will be very helpful in understanding knots.

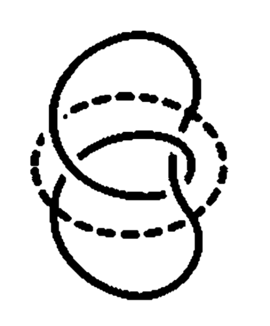
Equivalence of Tangles

- We will say two tangles are **equivalent** if we can get from one to the other by a sequence of Reidemeister moves while:
  - The four endpoints of the strings in the tangle remain fixed;
  - The strings of the tangle never journey outside the circle defining the tangle.



Remark on Equivalence of Knots

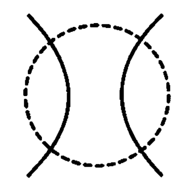
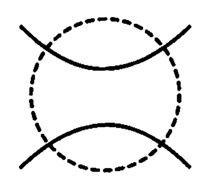
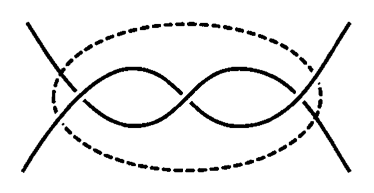
- Suppose we form a knot from a single tangle by gluing together the ends in pairs as in the figure.



- Two such knots are equivalent whenever the corresponding tangles are equivalent.

Simple Tangles and Notation

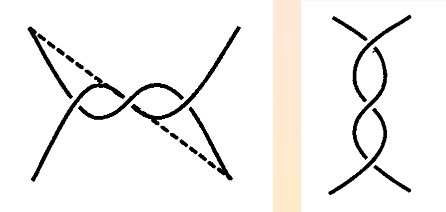
- One of the simplest tangles is two vertical strings, as on the left, denoted by ∞.

- We denote the tangle consisting of two horizontal strings as the 0 tangle.
- We could wind two horizontal strings around each other to get the third tangle. This is denoted by the number of left-handed twists we put in (in this case 3).
- If we twist the other way, we denote the resulting tangle by -3.
- In a positive-integer twist, the overstrand always has a positive slope.

Forming More Complicated Knots

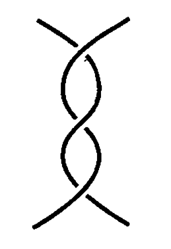
- We form a more complicated tangle, starting from the 3 tangle.

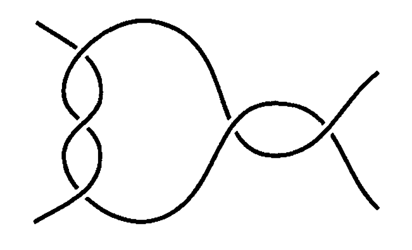


- First, we reflect the tangle through the NW-SE diagonal line.
- Think of this reflection as if we reflected in a mirror that was perpendicular to the projection plane and intersected the projection plane along the NW-SE diagonal line.
- Note that:
  - The two ends of the tangle along the diagonal are fixed;
  - The two ends of the tangle not on the diagonal are switched.

The Tangle 3 2

- We continue with another step.

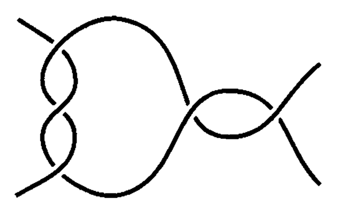


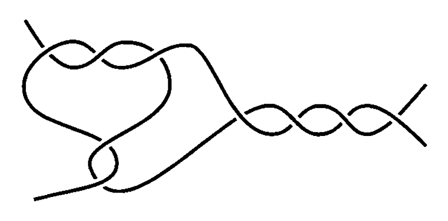


- We wind the two right-hand ends of the tangle around each other to get the right figure.
- This tangle is denote by 3 2, since it results from the original tangle, having:
  - Three twists of the horizontal strings followed by a reflection;
  - Then two twists of the horizontal strings.

The Tangle 3 2 -4

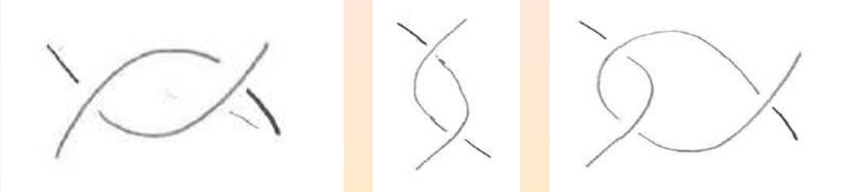
- We take the tangle 3 2, as on the left.

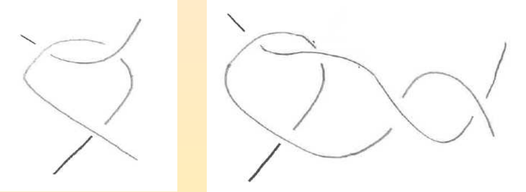




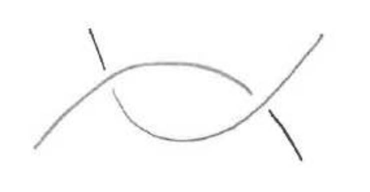
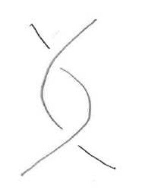
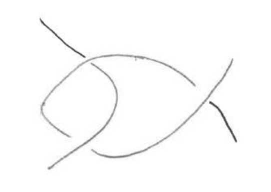
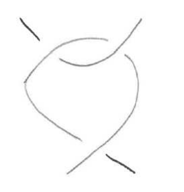
- We again reflect about the NW-SE diagonal.
- Then we add -4 twists to the right-hand strings, as on the right.
- We denote this tangle 3 2 -4.

The Tangle 2 1 −2





The Tangle 1 1 1 1

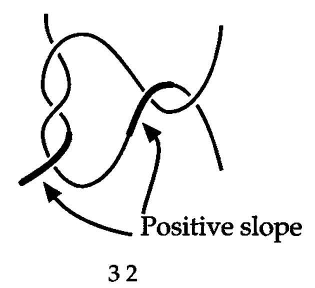

Rational Tangles: The Even Case

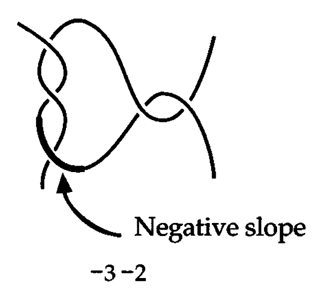
- We call any tangle that we could construct in the manner outlined in the preceding slides a **rational tangle**.
- Let the rational tangle be represented by an even number of integers.
- We can think of constructing it by:
  - Starting with two vertical strings (the ∞ tangle);
  - Twisting the two bottom endpoints around each other some number of times, while holding the top two endpoints fixed;
  - Twisting the two right-hand endpoints around each other while keeping the left-hand endpoints fixed;

  We continue by alternately twisting the bottom two endpoints and the right two endpoints to create the tangle.

The ± Signs in the Notation

- A positive-integer twist always gives the overstrand a positive slope, regardless of whether the twist is occurring in two vertical strands or two horizontal strands





Rational Tangles: The Odd Case

- Let the rational tangle be represented by an odd number of integers.
- We can construct it by:
  - Starting with two horizontal strings (the 0 tangle);
  - Twisting the two right-hand endpoints appropriately;
  - Twisting the two bottom endpoints appropriately;

    ⋮

Continued Fractions Corresponding to Tangles

- Consider two tangles given by the sequences $-2\ 3\ 2$ and $3\ -2\ 3$.
  The **continued fraction** corresponding to $-2\ 3\ 2$ is formed as follows:

$$\frac{1}{-2} \to 3 + \frac{1}{-2} \to \frac{1}{3 + \frac{1}{-2}} \to 2 + \frac{1}{3 + \frac{1}{-2}}.$$

  We simplify:

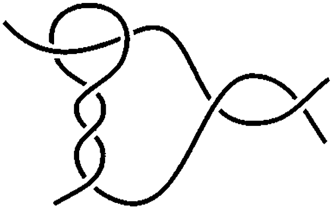
$$2 + \frac{1}{3 + \frac{1}{-2}} = 2 + \frac{1}{\frac{5}{2}} = 2 + \frac{2}{5} = \frac{12}{5}.$$

  The continued fraction corresponding to $3\ -2\ 3$ is

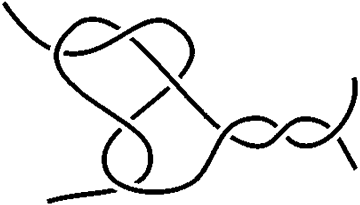
$$3 + \frac{1}{-2 + \frac{1}{3}} = \frac{12}{5}.$$

Equivalence of Tangles via Continued Fractions

- If the continued fractions corresponding to two rational tangles are equal, then the two tangles are equivalent.
- Since the continued fractions of $-2\ 3\ 2$ and $3\ -2\ 3$ are equal, the two rational tangles are equivalent.



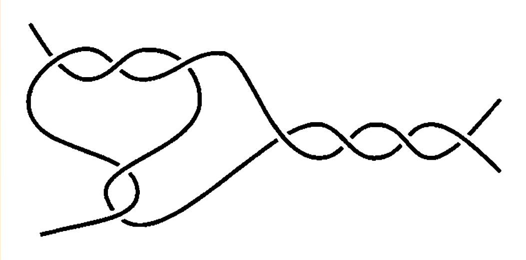
-2 3 2



3 -2 3

Inequivalence Using Continued Fractions

- Conversely, if two rational tangles are equivalent, then the continued fractions corresponding to the two tangles are equal.
- Compute the continued fraction of 3 2 -4.



$$-4+\frac{1}{2+\frac{1}{3}}=-4+\frac{1}{\frac{7}{3}}=-4+\frac{3}{7}=-\frac{25}{7}.$$

- So 3 2 -4 is distinct from -2 3 2 and 3 -2 3.

The Equivalence Criterion Using Continued Fractions

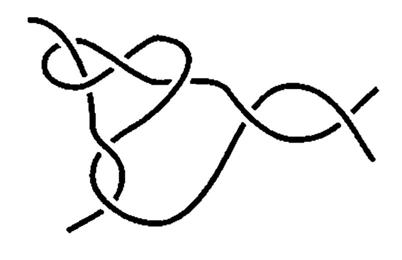
- Suppose we have two rational tangles given by the sequences of integers $ijk\ldots\ell m$ and $npq\ldots rs$.
- We can compute the corresponding continued fractions

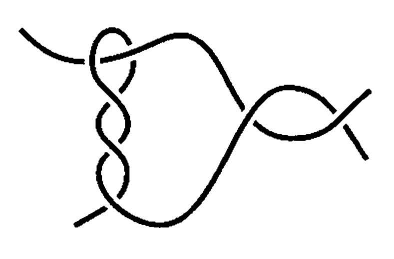
$$m+\cfrac{1}{\ell+\cfrac{1}{\ddots+\cfrac{1}{k+\cfrac{1}{j+\cfrac{1}{i}}}}} \quad \text{and} \quad s+\cfrac{1}{r+\cfrac{1}{\ddots+\cfrac{1}{q+\cfrac{1}{p+\cfrac{1}{n}}}}}.$$

- These fractions are both rational numbers.
- The two tangles are equivalent if and only if these two rational numbers are the same.
- The proof of this result is difficult.

Example

- Determine whether the following tangles are equivalent.





- The left is the tangle $-2\ -2\ -2\ -2$.
- The right is the tangle $2\ -3\ 2$.

Example

- We consider the tangles $-2\ -2\ -2\ -2$ and $2\ -3\ 2$.
- We compute the corresponding continued fractions:

$$
\begin{aligned}
-2+\frac{1}{-2+\frac{1}{-2+\frac{1}{-2}}} &= -2+\frac{1}{-2-\frac{2}{5}} = -2-\frac{5}{12} = -\frac{29}{12}; \\
2+\frac{1}{-3+\frac{1}{2}} &= 2-\frac{2}{5} = \frac{8}{5}.
\end{aligned}
$$

- Since the continued fractions are not equal, the two tangles are not equivalent.

Example

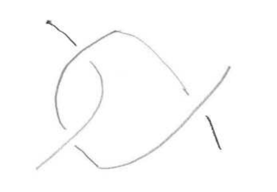
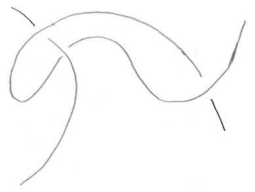
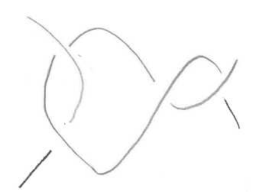
- We show that the rational tangle $2\ 1\ a_1\ a_2\ \ldots\ a_n$ is equivalent to the rational tangle $-2\ 2\ a_1\ a_2\ \ldots\ a_n$ by using continued fractions.
- We have

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \frac{1}{1+\frac{1}{2}}}} = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \frac{1}{2+\frac{1}{-2}}}}.$$

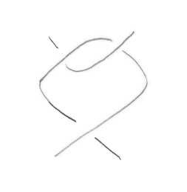
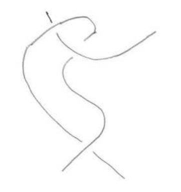
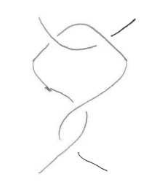
- So the two rational tangles are equivalent.

Example (Cont'd)

- We show that the rational tangle $2\ 1\ a_1\ a_2\ \ldots\ a_n$ is equivalent to the rational tangle $-2\ 2\ a_1\ a_2\ \ldots\ a_n$ by drawing a picture.
- If n is even, we have

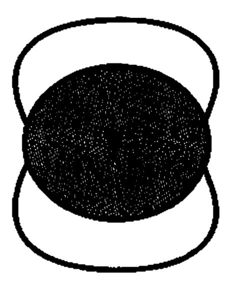
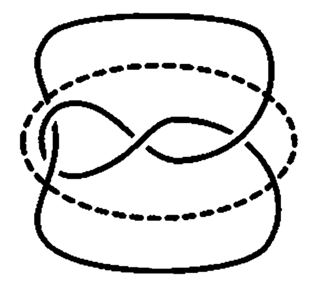




- If n is odd, we have

Conway's Notation

- If we close off the ends of a rational tangle we call the resulting link a **rational link**.

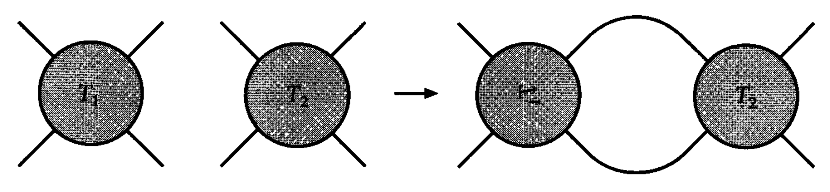
Example: The figure-eight knot is a rational knot.

It results from the rational tangle 2 2.

- We can use the notation for rational tangles to denote the corresponding rational knot.
- The resulting notation is called **Conway's notation**.

Multiplication of Rational Tangles

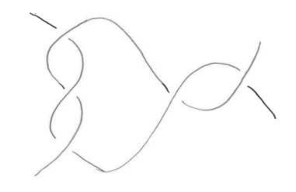
- We will define a way to "multiply" two tangles to obtain a new tangle.



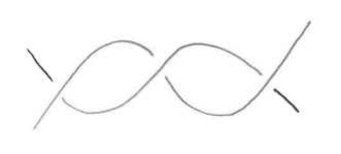
  - We reflect the first tangle across its NW-SE diagonal line;
  - We glue it to the second tangle.
- Note that this definition of multiplication fits in nicely with our definition of a rational tangle.

Example

- Multiplication of rational tangles fits in nicely with our definition of a rational tangle.
- Consider the rational tangle 3 2.

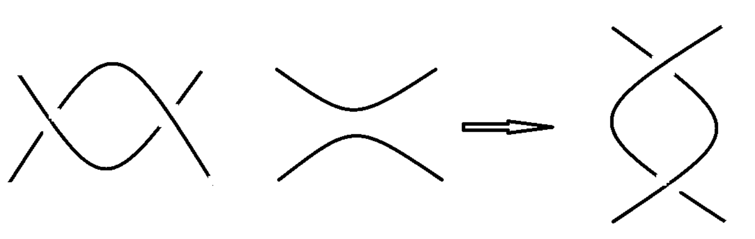


- It results from multiplying together the two tangles 3 and 2.



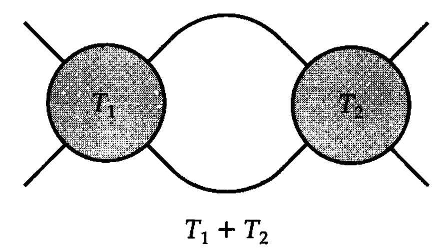
Properties of Multiplication

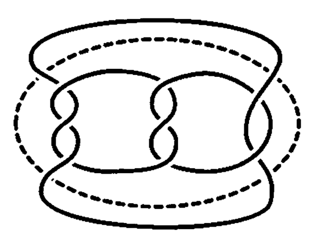
- Multiplying together two rational tangles will always generate a rational tangle.
- If we ever want to reflect a tangle across its NW-SE diagonal line, we can simply multiply it by the tangle 0.



Addition of Tangles

- We can also "add" together two tangles.





Example: The right figure depicts the knot 8_5.

It corresponds to the tangle

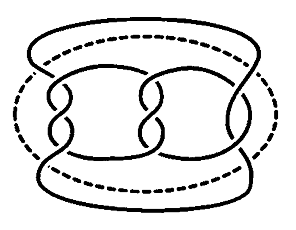
$$3\ 0 + 3\ 0 + 2\ 0.$$

So it can be written as $3\ 0 + 3\ 0 + 2\ 0$.

Pretzel Knots

- If we multiply each tangle in a sequence of tangles by 0, and then add them all together, we denote the resultant tangle by the sequence of numbers that stand for the original tangles, only now separated by commas.

  Example: We would denote the tangle for 8_5 by 3, 3, 2.



- A knot obtained from a tangle represented by a finite number of integers separated by commas is often called a **pretzel knot**.

Algebraic Tangles and Algebraic Links

- We defined the operations:
  - Addition of tangles;
  - Multiplication of tangles.
- We call any tangle obtained by the operations of addition and multiplication on rational tangles an **algebraic tangle**.
- An **algebraic link** (sometimes called an **arborescent link**) is simply a link formed when we connect the NW string to the NE string and the SW string to the SE string on an algebraic tangle.
- We denote the link the same way we denote the corresponding tangle.

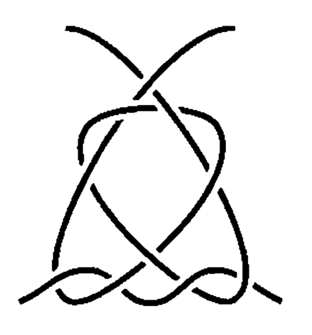
Algebra of Tangles

- Algebraic tangles are behaving a lot like the real numbers.
- We can add two of them or multiply two of them.
- But there are differences between the structure of the real numbers and the structure of algebraic tangles.
  - Multiplication on tangles is not commutative.
  - Multiplication on tangles is not associative.
  - Addition of tangles does not have inverses.
    That is, given a tangle T, in general, there is no inverse tangle that, when added to T, gives back the trivial tangle 0.

Not All Tangles Are Algebraic

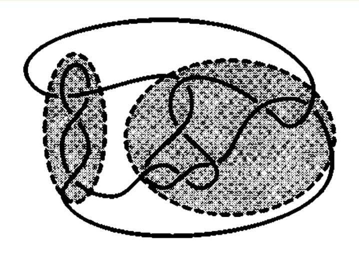
- There are tangles that are not algebraic.

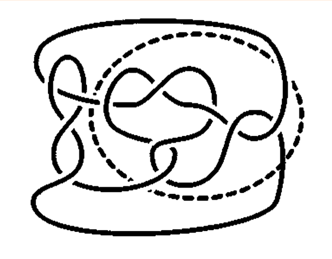
  Example: The tangle in the figure is not algebraic.



Mutation

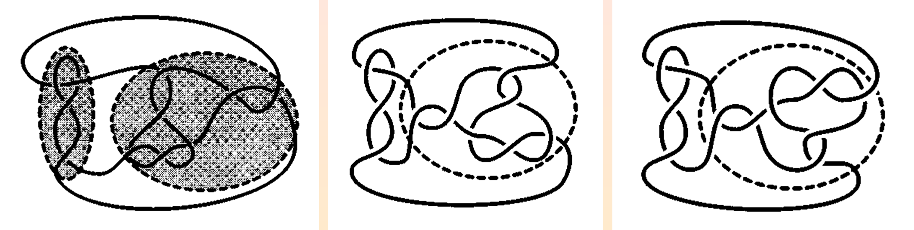
- Another way to obtain new knots is called **mutation**.
- Let K be a knot that we think of as being formed from two tangles.





- We form a new knot by:
  - Cutting the knot open along four points on each of the four strings coming out of T_2;
  - Flipping T_2 over;
  - Gluing the four strings back together.

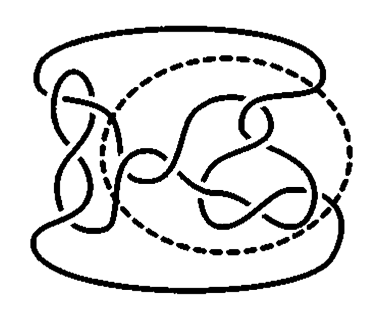
Mutation (Cont'd)

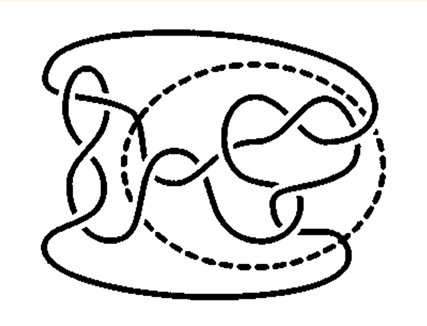


- We could also:
  - Cut the four strings coming out of T_2
  - Flip T_2 left to right;
  - Glue the strings back together.
- If we did both operations in turn, it's as if we rotated the tangle 180 degrees and then reglued it.
- Any of these three operations is called a **mutation**.
- The three resultant knots together with the original knot are called **mutants** of one another.

The Kinoshita-Terasaka Mutants

- The figure shows two famous mutants called the **Kinoshita-Terasaka mutants**.



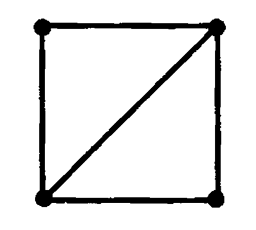
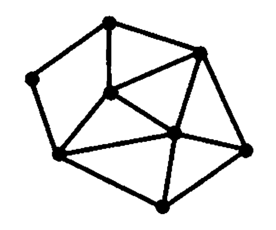
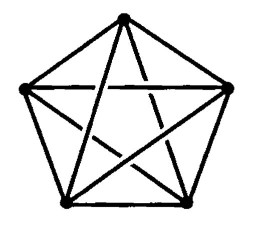


Subsection 3

Knots and Planar Graphs

Graphs and Planar Graphs

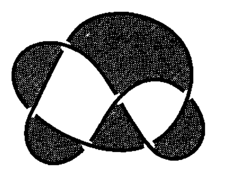
- A **graph** consists of:
  - A set of points called **vertices**;
  - A set of **edges** that connect them.

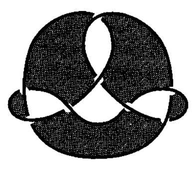




- A graph is **planar** if it lies in the plane, as in the first two figures above.

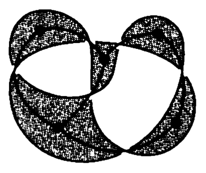
The Graph Corresponding to a Link Projection

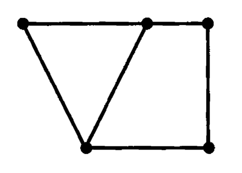
- From a projection of a knot or link, we create a corresponding planar graph in the following way.
  - First shade every other region of the link projection so that the infinite outermost region is not shaded.





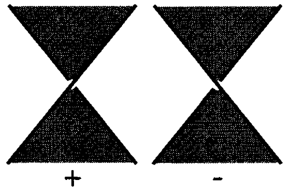
  - Put a vertex at the center of each shaded region.
  - Connect with an edge any two vertices that are in regions that share a crossing .

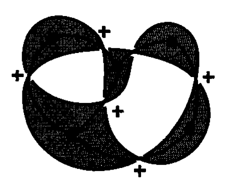


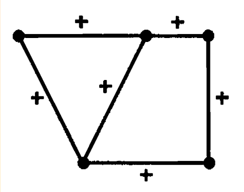


The Signed Graph Corresponding to a Link Projection

- The graph obtained in the preceding slide does not depend in any way on whether a crossing is an overcrossing or an undercrossing.
- We define crossings to be **positive** or **negative** depending on which way they cross.



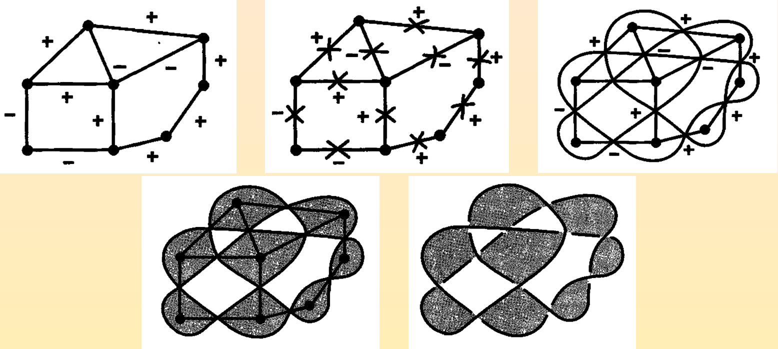




- Now we label each edge in the planar graph with a:
  - $+$ if the edge passes through a $+$ crossing;
  - $-$ if the edge passes through a $-$ crossing.
- We call the result a **signed planar graph**.

From Signed Planar Graphs to Link Projections

- Starting with a signed planar graph:
  - Put an x across each edge.
  - Connect the edges inside each region of the graph.
  - Shade those areas that contain a vertex.
  - Then, at each of the x's, put in a crossing corresponding to whether the edge is a $+$ or a $-$ edge.



Knot and Link Projections and Signed Planar Graphs

- We can turn knot and link projections into signed planar graphs.
- We can turn Reidemeister moves into operations on signed planar graphs.
- The question of whether knot projections are equivalent under Reidemeister moves becomes one of whether signed planar graphs are equivalent under operations that the Reidemeister moves become.