Introduction to Knot Theory

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

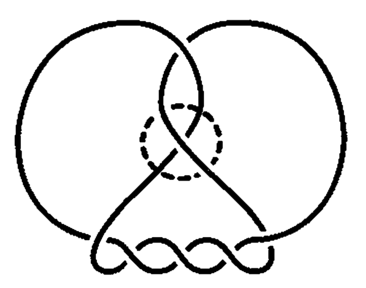
LSSU Math 500

Subsection 1

Unknotting Number

Example: Unknotting Number

- If we changed the crossing circled in the figure



  the knot depicted would become the unknot.
- The one change of crossing completely unknots the knot.
- We say that this knot has unknotting number 1.

Unknotting Number

- We say that a knot K has **unknotting number** n if the following hold:
  - There exists a projection of the knot such that changing n crossings in the projection turns the knot into the unknot;
  - There is no projection such that fewer changes would have turned it into the unknot.
- We denote the unknotting number of a knot K by $u(K)$.

Unknotting a Projection of a Knot

Claim: Every projection of a knot can be changed into a projection of the unknot by changing some subset of the crossings in the projection.

Given a projection of a knot:

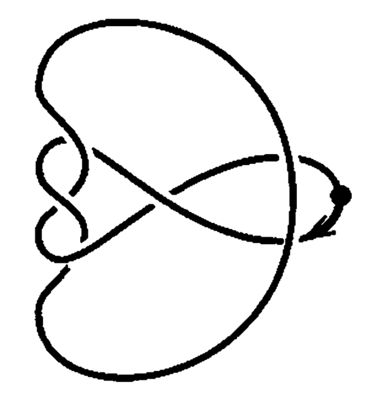
- Pick a starting point on the knot that for convenience is not at a crossing;
- Pick a direction to traverse the knot.

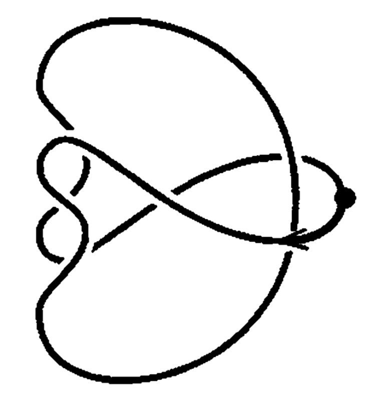
Now, beginning at that point, we head along the knot in our chosen direction.

- The first time that we arrive at a particular crossing, we change the crossing if necessary so that the strand that we are on is the overstrand.
- If we come to a crossing that we have already been through once, we do not change that crossing, but rather continue through it on what must necessarily be the understrand.

Once we have returned to our initial starting point, we have a projection of a knot that was obtained from our original knot by changing crossings and that will in fact be the trivial knot.

Illustrating the Unknotting of a Knot





Unknotting: Proof of Triviality

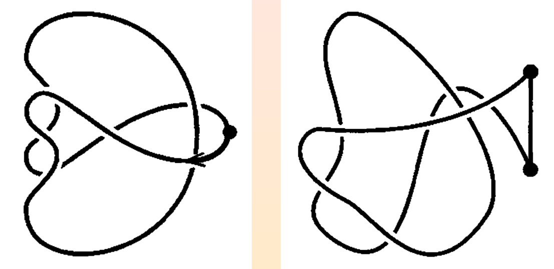
- To see that this is the trivial knot, we view it in three-space.
  Think of the z-axis as coming upward out of the projection plane.
  - Starting at the initial point again, we place that point in three-space with z-coordinate $z = 1$.
  - As we traverse the knot, we decrease the z-coordinates of each of the points on the knot until we get almost back to where we started.
  - That last point will have z-coordinate $z = 0$.
  - But, since we gave the initial point and the last point z-coordinates $z = 0$ and $z = 1$, and these are supposed to be the same point, we had better put in a vertical bar from one to the other to complete the knot

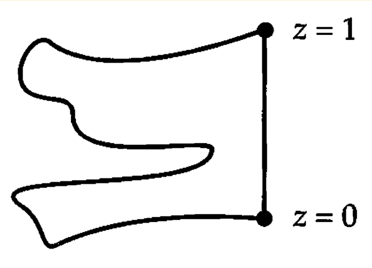
  Note then that when we look straight down the z-axis at our knot, we see the projection that we had changed the crossings to create.

  But when we look at our projection from the side, we see a projection with no crossings.

  Hence this knot is a trivial knot.

Unknotting: Illustration of the Proof of Triviality

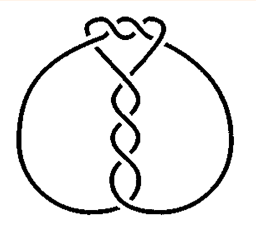




Hardness of Determining the Unknotting Number

- It is very hard to find the unknotting number of a knot.

  Example: Consider the projection of the knot on the right.

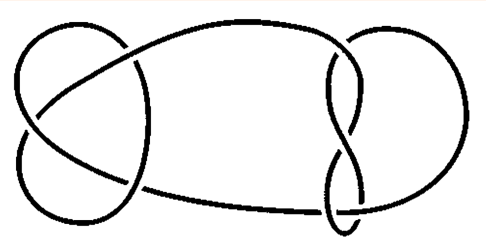
  

  If we change crossings, it looks like the unknotting number is 2 (which it is).

  But how do we know that there is no other projection of this knot that can be unknotted by only one crossing change?
- Kanenobu and Murakami proved that the unknotting number of the knot is 2 by applying the Cyclic Surgery Theorem, due to Culler and Shalen, and Gordon and Luecke, to prove that no such projection exists.

Unknotting Composite Knots

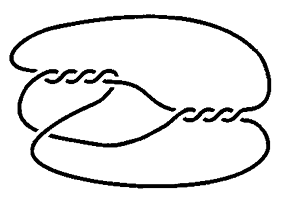
- Can a composite knot have unknotting number 1?



- We might expect the answer to be no.
  - Changing one crossing might allow us to untangle one of the two factor knots that make up the composite knot;
  - It seems unlikely that the one crossing change would allow us to untangle both factor knots.
- Scharlemann proved that a knot with unknotting number 1 is prime.

Unknotting Number and Number of Crossings

- Bleiler and Nakanishi independently discovered an example of a knot whose unknotting number is not realized in a projection of the knot with a minimal number of crossings!
- The figure shows a knot with Conway notation 5 1 4.
- It is known that this knot cannot be drawn with fewer crossings, so its minimal crossing number is 10.



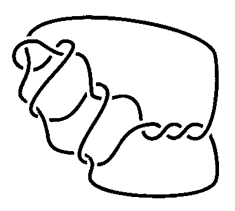
- It is also known that this is the only projection (up to planar isotopy and mirror reflection) of this knot with 10 crossings.
- Tt takes at least three crossing changes in the projection in the figure to unknot this knot.

Unknotting Number and Number of Crossings (Cont'd)

- Another projection of the same knot has Conway notation

$$2 \quad -2 \quad 2 \quad -2 \quad 2 \quad 4.$$

- We can check that the two continued fractions give the same rational number.
- This projection of the knot in can be unknotted by changing only two crossings.
- One can prove that the unknotting number of the knot in this figure is in fact 2.

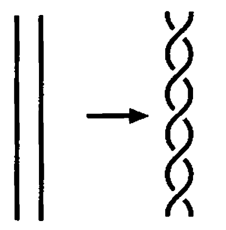


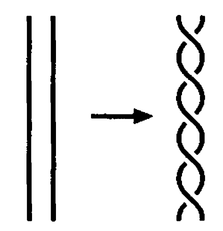
- The unknotting number is realized by a projection that is not minimal!

k-Moves

- Consider a given a projection of a knot.
- Define a k-**move** to be a local change in the projection that replaces two untwisted strings with two strings that twist around each other with k crossings in a right-handed manner.
- Define a $-k$-**move** in the same way, except that the twist is a left-handed twist.

Example: A 5-move and a -5-move.



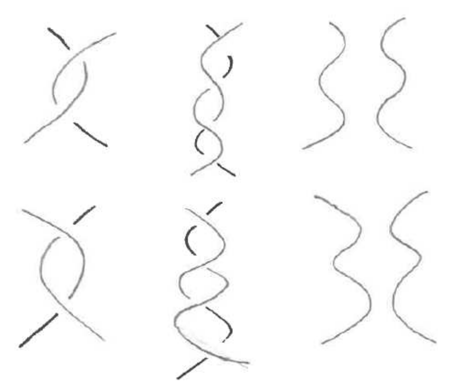


k-Equivalence

- We say that two knots or links are k-**equivalent** if we can get from a projection of one to a projection of the other through a series of k-moves and $-k$-moves.
- We allow ourselves to change the projections via ambient isotopies between the various moves that we perform.

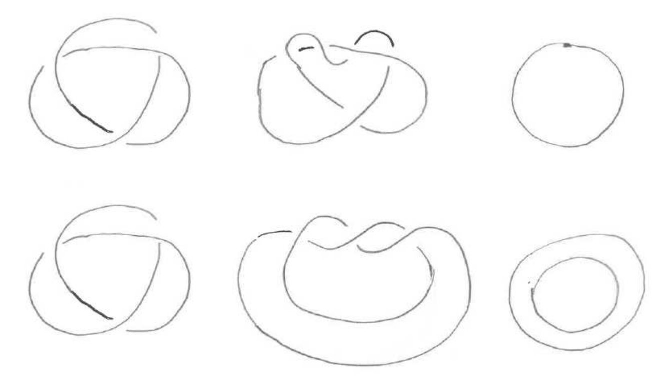
Links and 2-Equivalence

- We can show that every link is 2-equivalent to the trivial link with the same number of components.
- The key moves are the ones shown below.



Example

- The trefoil knot is 2-equivalent to the trivial knot and 3-equivalent to the trivial link.

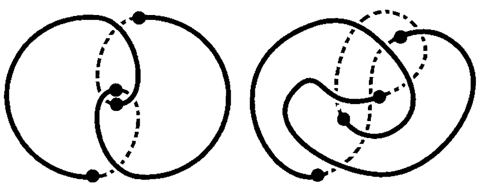


Subsection 2

Bridge Number

Example: Bridge Number

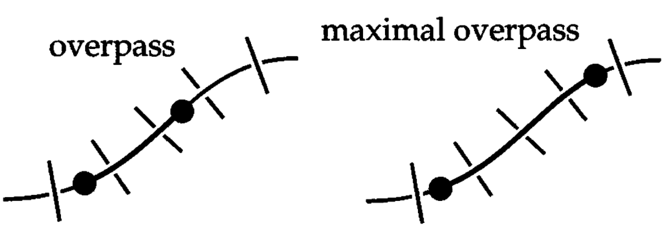
- In the figure, we show a pair of unusual projections of the trefoil and figure-eight knots, respectively.



- Think of the knots as cutting through the projection plane with:
  - The solid portions of the knots as lying above the plane;
  - The dashed portions as lying below the plane.
- Each knot intersects the plane in four vertices.
- In both pictures, there are two unknotted arcs from each knot lying above the plane.
- This is the least number of such unknotted arcs in any projection of these knots.
- We say these knots both have bridge number 2.

Overpasses and Maximal Overpasses

- Consider a projection of a knot to a plane.
- Define an **overpass** to be a subarc of the knot that goes over at least one crossing but never goes under a crossing.



- A **maximal overpass** is an overpass that could not be made any longer.
- Both endpoints of a maximal overpass occur just before we go under a crossing.

The Bridge Number

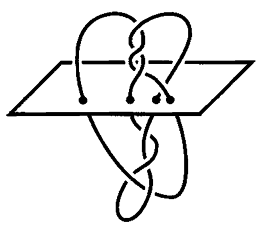
- The **bridge number of the projection** is the number of maximal overpasses in the projection (those maximal overpasses forming the bridges over the rest of the knot).
- Note that each crossing in the projection must have some maximal overpass that crosses over it.
- The **bridge number of** K, denoted $b(K)$, is the least bridge number of all of the projections of the knot K.
- If a knot has bridge number 1, it must be the unknot.

Two-Bridge Knots

- Knots that have bridge number 2 are known as **two-bridge knots**.
- Suppose we cut a two-bridge knot open along the projection plane.
- We would be left with:
  - Two unknotted untangled arcs from the knot above the plane, corresponding to the two maximal overpasses;
  - Two unknotted untangled arcs from the knot below the plane.
- These are unknotted and untangled, since they can have no crossings with each other.
- All of the crossings came from a maximal overpass and one of these arcs.

Two-Bridge Knots (Cont'd)

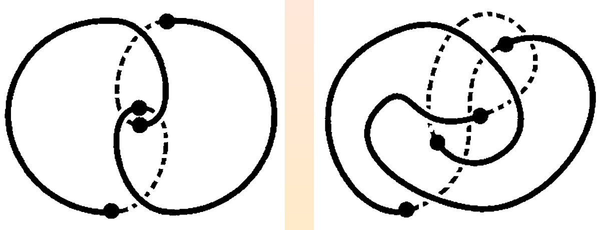
- To construct all possible two-bridge knots, we just glue the endpoints of two unknotted untangled strings above the plane to the endpoints of two unknotted untangled strings below the plane.



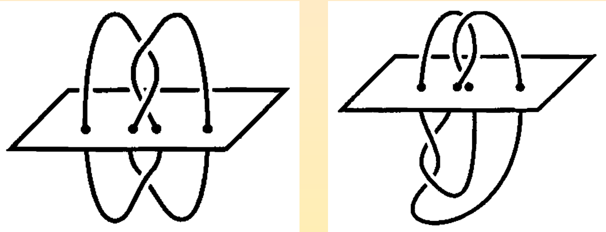
- Although the strings to each side of the plane are individually unknotted, they can twist around each other and themselves.

Trefoil and Figure-Eight Knots Revisited

- The trefoil and figure-eight knots:

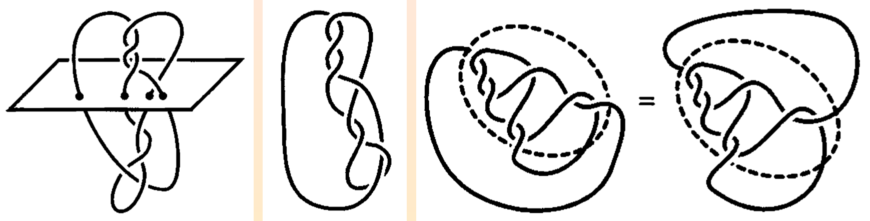


- The side view of these two-bridge representations:



Two-Bridge Knots and Rational Knots

- Consider a picture of a two-bridge knot.



- We can always free one of the strings and redraw our projection as in the second figure.
- Now we can see that this two-bridge knot is in fact simply a rational knot, by turning every other integer tangle horizontal, starting with the bottom one.
- The two-bridge knots are exactly the rational knots.

Subsection 3

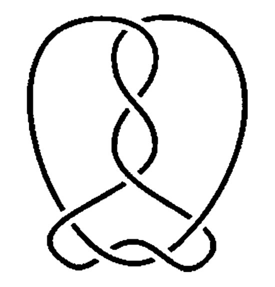
Crossing Number

Crossing Number

- The **crossing number** of a knot K, denoted $c(K)$, is the least number of crossings that occur in any projection of the knot.
- To determine the crossing number of a knot K:
  - Find a projection of the knot K, with some number of crossings n.
  - Then we know the crossing number is n or smaller.
  - Suppose all of the knots with fewer crossings than n are known.
  - If K does not appear in the list of knots of fewer than n crossings, then K must have crossing number n.

Example

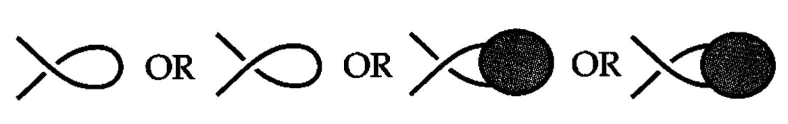
- Consider the knot shown.



- It has crossing number ≤ 7 since it has a projection with 7 crossings.
- To see that it has crossing number exactly 7, we must show that it is distinct from all the knots of fewer than 7 crossings.
- The latter is very difficult in general.

A General Result on Crossing Numbers

- Call a projection of a knot **reduced** if there are no easily removed crossings, as in the figure.



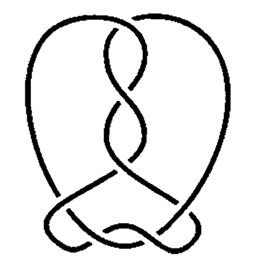
- Kauffman, Murasugi and Thistlethwaite proved that an alternating knot in a reduced alternating projection of n crossings has crossing number n.
- There cannot be a projection of such a knot with fewer crossings.
- They utilized the Jones polynomial for knots, which will be discussed in the last set of slides.

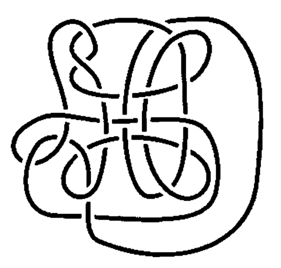
Determining the Crossing Number of Alternating Knots

- We can tell the crossing number of any alternating knot.
  - We can tell by just looking at an alternating projection whether or not it is reduced;
  - We can lower the number of crossings if it is not reduced.

Example: Crossing Number of Alternating Knots

- Consider the knot on the left.
  The figure is a reduced alternating projection of 7 crossings.
  So the crossing number of the knot on the left is 7.





- Consider, also, the knot on the right.
  It is in a reduced alternating projection of 23 crossings.
  Hence its crossing number is 23.
  So there is no projection of this knot with fewer than 23 crossings.