Introduction to Knot Theory

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LSSU Math 500

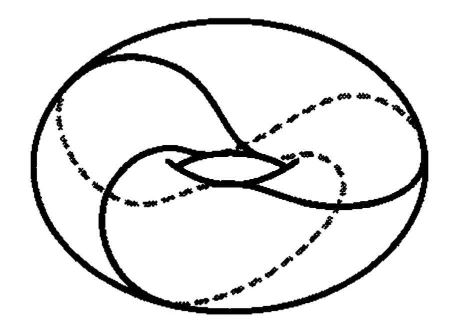
Subsection 1

Torus Knots

Torus Knots

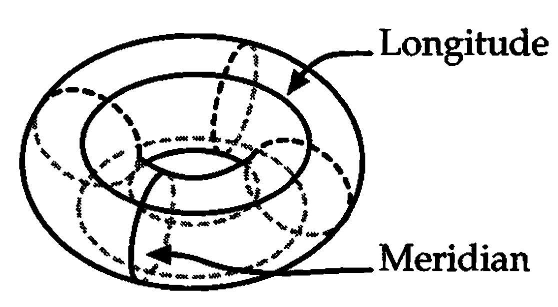
- **Torus knots** are knots that lie on an unknotted torus, without crossing over or under themselves as they lie on the torus.

  Example: The figure is a picture of the trefoil knot on a torus.



Meridian Curves and Longitude Curves

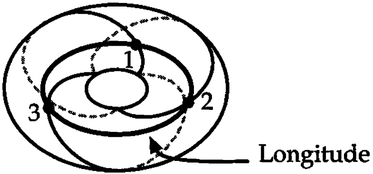
- We call a curve that runs once the short way around the torus a **meridian curve**.

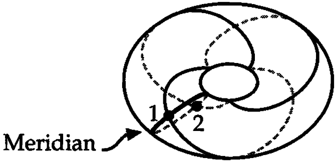


- A curve that runs once around the torus the long way is called a **longitude curve**

Meridians, Longitudes and the Trefoil Curve

- The trefoil knot wraps around the torus:
  - Three times meridionally;
  - Twice longitudinally.
- To see that these wrapping numbers are correct:
  - Add the meridian and longitude curves to the torus;
  - Count the number of times the trefoil crosses each.

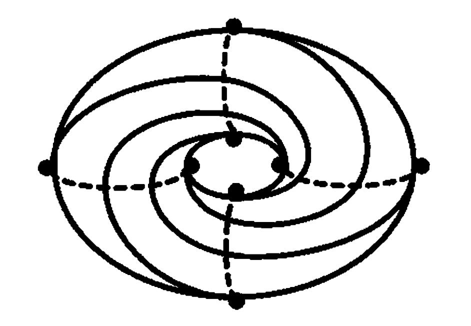




  - The trefoil crosses the longitude three times.
    To do so, it must wrap around the torus meridionally three times.
  - The trefoil crosses the meridian twice.
    So it must wrap around the torus longitudinally two times.
- We call the trefoil knot a $(3, 2)$-**torus knot**.

Another Example

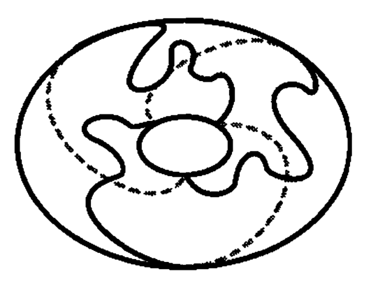
- The figure shows a $(4,3)$-torus knot.

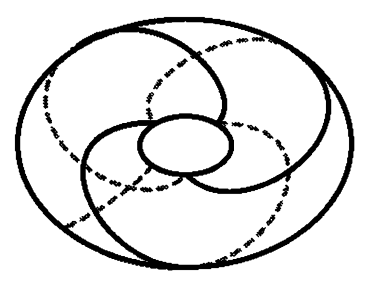


- Every torus knot is a (p,q)-torus knot for some pair of integers.
- The two integers will always be relatively prime (that is, their greatest common divisor is 1).

Deformation of Torus Knots

- The knot on the left lies on a torus.

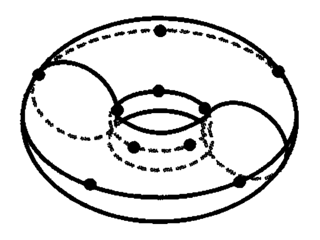


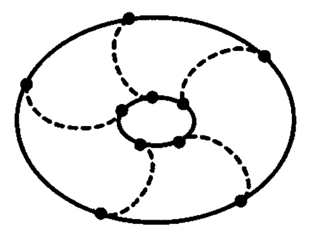


- It does not look like the (p, q)-torus knots we have drawn.
- However, we can deform it until it looks more like the torus knot that it is.

Drawing a $(5,3)$-Torus Knot

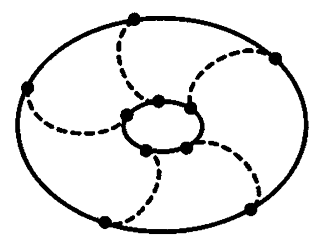
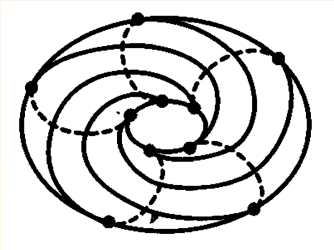
- Suppose we want to draw a $(5,3)$-torus knot.
- It wraps five times meridionally around the torus.
- So it should cross the longitude five times.
- We mark five points on the outside equator of the torus and five points on the inside equator.





- We attach each point that we marked on the outside equator of the torus to the corresponding point on the inside equator, utilizing a strand that runs directly across the bottom of the torus.

Drawing a $(5, 3)$-Torus Knot (Cont'd)

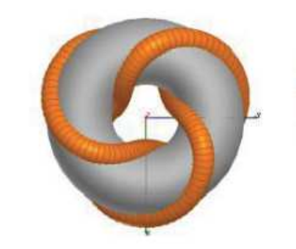
- We also want the knot to wrap three times longitudinally.
- We attach each point on the outside equator to the point on the inside equator that is a $3/5$ turn clockwise from the outside point.
- I.e., we jump ahead three points, utilizing a strand that runs over the top of the torus.
- The result is a knot that travels three times longitudinally around the torus and five times meridionally.

Drawing a (p, q)-Torus Knot

- To draw a (p, q)-torus knot:
  - We place p points around the inside and outside equators of the torus;
  - Attach the inside and outside points directly across the bottom of the torus;
  - Attach each outside point to the inside point that is clockwise q points ahead, using a strand that goes over the top of the torus.

Crossings on the Projection of a (p, q)-Torus Knot

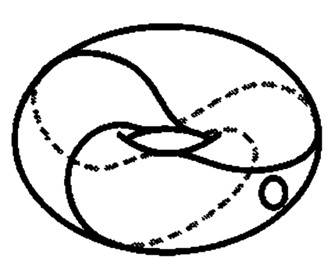
- A (p, q)-torus knot always has a projection with $p(q-1)$ crossings.
- As drawn on a torus the knots wraps:
  - Meridionally p times;
  - Longitudinally q times.
- The p segments drawn over the top do not intersect.
- Each of these, however, connects to an inside point that is clockwise q points ahead of it.
- Thus, in a projection is would create $(q-1)$ crossings.

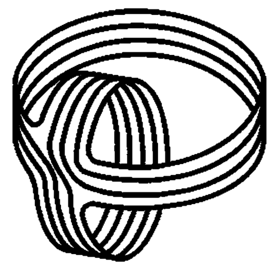


- So the projection will consist of $p(q-1)$ crossings.

Every (p, q)-Torus Knot is a (q, p)-Torus Knot

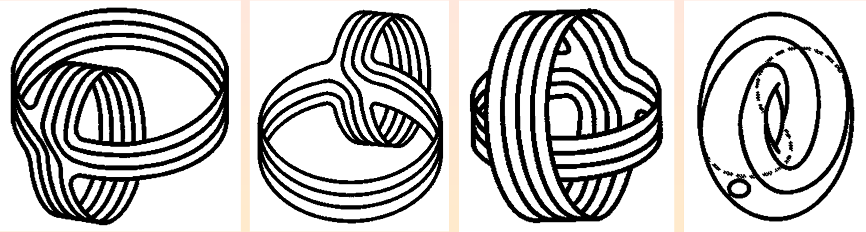
- Consider the trefoil knot, a $(3, 2)$-torus knot.
- Remove a disk that does not touch the knot from the torus.





- We can deform a torus with one boundary component into two bands that are attached to one another.
  - The shorter band corresponds to a meridian of the torus;
  - The longer band corresponds to a longitude of the torus.
- As the deformation occurs, we carry along the knot.

Every (p, q)-Torus Knot is a (q, p)-Torus Knot (Cont'd)



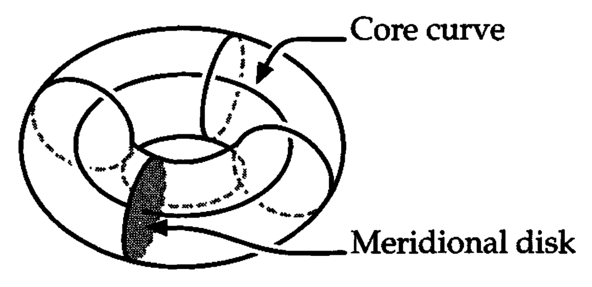
- Take the longer band and turn it inside out.
- Then take the shorter band and turn it inside out also.
- We can now deform the bands back into a torus with one boundary component, but with the roles of the two bands reversed.
  - The band that originally corresponded to a longitude now corresponds to a meridian and vice-versa.
- Since the meridian and the longitude have been exchanged, the knot is now a $(2, 3)$-torus knot on the new torus.

Remarks on the Crossing Number

- The process outlined in the preceding slides works just as well to show that any (p, q)-torus knot is also a (q, p)-torus knot.
- This implies that a (p, q)-torus knot has a projection with $p(q-1)$ crossings and a projection with $q(p-1)$ crossings.
- Therefore, the crossing number for a (p, q)-torus knot is at most the smaller of $p(q-1)$ and $q(p-1)$.
- Murasugi proved that in fact the smaller of $p(q-1)$ and $q(p-1)$ is exactly the crossing number of a (p, q)-torus knot.

The Solid Torus

- A **solid torus** is a doughnut where we include both the interior of the doughnut as well as the surface.

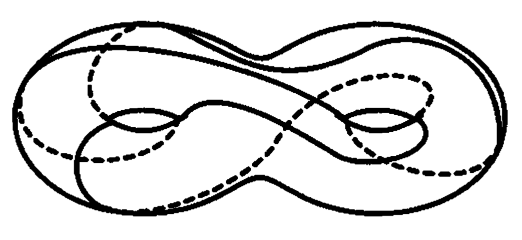


- The **core curve** of a solid torus is the trivial knot that runs once around the center of the doughnut.
- A **meridional disk** of the solid torus is a disk in the solid torus that has a meridian curve as its boundary.

Two-Embeddable Knots

- A standardly embedded genus n surface, is one that is embedded unknotted in space.
- There are knots that cannot be placed on a standardly embedded torus but can be placed on a standardly embedded genus two surface.
- We call these **two-embeddable knots**, since they can be embedded (placed without any crossings) on a standardly embedded genus two surface.

  Example: The figure-eight knot is a two-embeddable knot.



n-Embeddable Knots

- We say that a knot K is an n-**embeddable knot** if:
  - K can be placed on a genus n standardly embedded surface without crossings;
  - K cannot be placed on any standardly embedded surface of lower genus without crossings.

  Claim: Any knot is an n-embeddable knot for some n.

  Take a projection for the knot and have the strands at a crossing run over and under a handle of the surface.

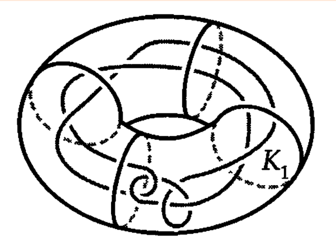
  The surface need only be isotopic to a standardly embedded surface.
- The Claim shows that we can have a hierarchy of knots, depending on the minimal genus of a standardly embedded surface that they lie on.

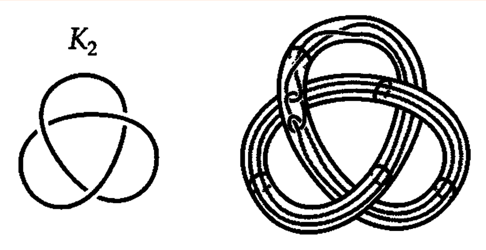
Subsection 2

Satellite Knots

Satellite Knots

- Let K_1 be a knot inside an unknotted solid torus.





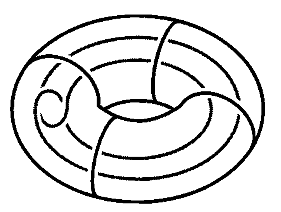
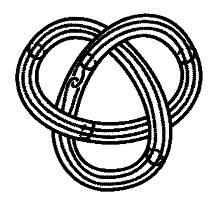
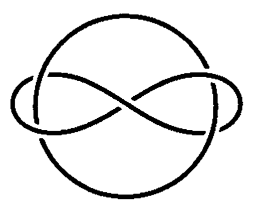
- We knot that solid torus in the shape of a second knot K_2.
- This will take the knot K_1 that lies inside the original solid torus to a new knot inside the knotted solid torus.
- We call this new knot, K_3, a **satellite knot**.
- The knot K_2 is called the **companion knot** of the satellite knot.

Satellite Knots: Assumptions

- If the companion knot was trivial, then $K_3 = K_1$.
- So we always assume that the companion knot is a nontrivial knot.
- We also always assume that the knot K_1 hits every meridional disk of the solid torus, and it cannot be isotoped to miss any of them.
- We think of the satellite knot as a knot that stays within a solid torus that has the companion knot as its core curve (just as a satellite stays within orbit around a planet).

Whitehead Doubles

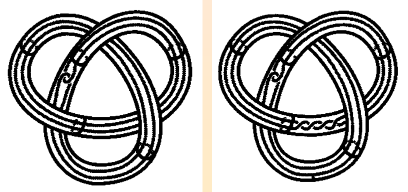
- Take the original knot K_1 to be an unknot, but sitting inside the solid torus twisted up as in the figure on the left.

- Then the resulting satellite knot is called a **Whitehead double** of the companion knot.
- The name refers to the fact that the knot K_1 here resembles the Whitehead link.

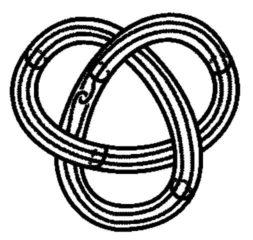
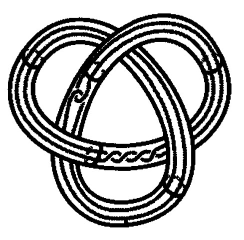
Non-Uniqueness of a Whitehead Double

- A Whitehead double is not unique.
- This can be shown by the following process.
- Cut the solid torus open along a meridional disk;
- Twist one end some number of times;
- Glue the meridional disks back together again.



- We obtain a homeomorphic copy of the solid torus.
- But now two strands of K_1 are twisted around each other.
- Then, when we knot this solid torus as a trefoil knot, we obtain a second Whitehead double of the trefoil.

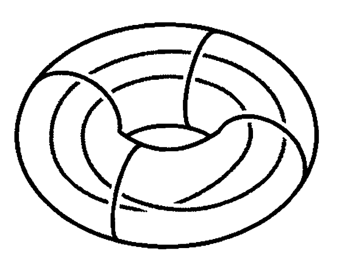
Non-Uniqueness of a Whitehead Double (Cont'd)

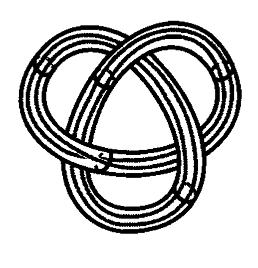
 

- Both of these Whitehead doubles have a common property.
- Suppose we cut open three-space along the knotted torus.
- Then we get two pieces:
  - One is the solid torus with K_1 in it;
  - One is three-space with the interior of a solid torus knotted as a trefoil knot removed from it.

Two-Strand Cables

- Suppose now that the original knot K_1 is again unknotted, but sitting inside the solid torus as in the figure on the left.

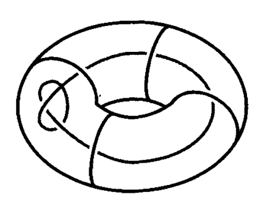
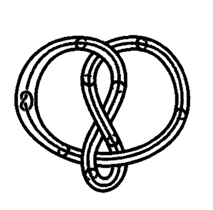
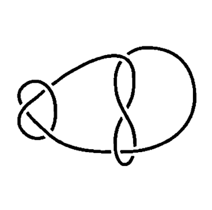




- Then the resulting satellite knot is called a **two-strand cable** of the companion knot.
- It's as if we had a cable that ran twice around the companion knot.
- The two-strand cable will not be unique, as we can add twists to it.

Satellite Knots and Composition

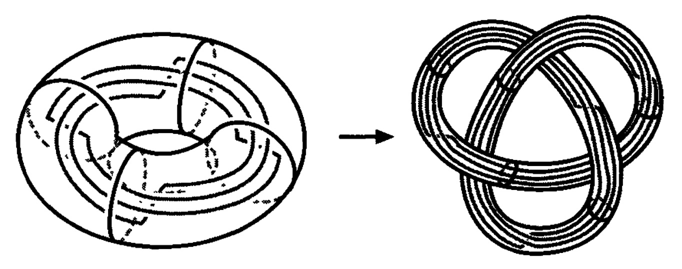
- The operation of forming a satellite knot can be thought of as a generalization of the idea of composition.
- Suppose K_1 only has one strand that reaches longitudinally around the solid torus.

- Consider the satellite knot formed by knotting the solid torus like K_2.
- This is in fact the composite knot $K_1 \# K_2$.
- Notice the *swallow-follow torus* mentioned previously.

Cable Knots

- If the knot K_1 that we start with is a torus knot, then we call the resulting satellite knot with companion K_2 a **cable knot** on K_2.



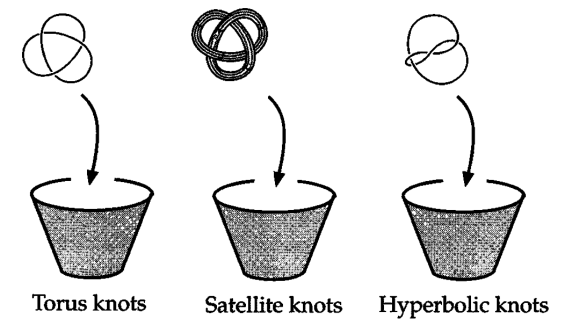
- We can think of it as taking a cable that wraps around the knot K_2 a total of p times meridionally and q times longitudinally.

Subsection 3

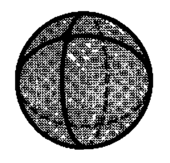
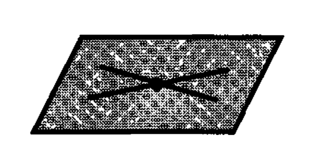
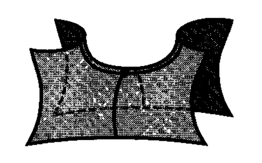
Hyperbolic Knots

Introduction

- Hyperbolic Knots were introduced in 1974.
- Thurston proved in 1978 that the only knots that are not hyperbolic knots are torus knots and satellite knots.
- It follows that every knot falls into exactly one of the three categories.



Positive, Zero and Negative Curvature

- A sphere has positive curvature.
  Pick a point on the surface of the sphere.
  Take cross sections in several directions through that point.
  All cross sections are circles that curve in the same direction.
- A plane has zero curvature.
  Pick a point and take cross sections in several directions through it.
  We always get a line, which has no curvature.
- But a saddle has negative curvature.
  Consider the central point.
  Take cross sections in two different directions through the point.
  We obtain two parabolas, one opening up and one opening down.

Hyperbolic Three-Space

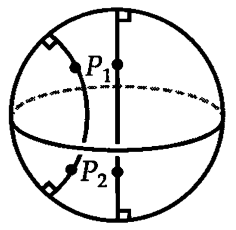
- We aim at endowing the complement of a knot with a **hyperbolic metric** having curvature -1.
- The geometry that results is called **hyperbolic geometry**.
- The simplest example of a three-dimensional space that has a hyperbolic metric is called **hyperbolic three-space**, denoted by H^3.
- A particular model of H^3 that we consider is called the **Poincaré model**.

The Poincaré Model

- The points are the points in three-space inside the unit ball,

$$H^3 = \{(x, y, z) : x^2 + y^2 + z^2 < 1\}.$$

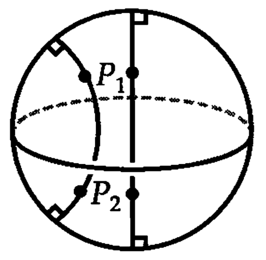
- Let P_1 and P_2 be two such points in H^3.
- Let C be part of a circle in H^3 that:
  - Has both of its endpoints on the unit sphere;
  - Is perpendicular to the unit sphere at its endpoints;
  - Passes through the two points P_1 to P_2.



- If P_1 and P_2 do not both lie on a line segment that is a diameter of the unit sphere, there is always a unique such arc of a circle.
- If P_1 and P_2 do lie on the same diameter, we replace the arc of a circle with that line segment that is a diameter passing through P_1 and P_2.

Geodesics in H^3

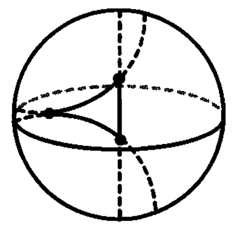
- It will turn out that the shortest path in hyperbolic three-space from P_1 to P_2 is the path, say w, within the arc of a circle or vertical line from P_1 to P_2.
- Any arc of a circle or diameter in H^3 that is perpendicular to the unit sphere is called a **geodesic** in H^3.
- A geodesic is a curve that has the property that for any two points P_1 and P_2 within it, the shortest path from P_1 to P_2 also lies in the curve.



- Notice that straight lines are geodesics in Euclidean space, as the shortest path between any two points in a line also lies in the line.

Angles of Hyperbolic Triangle

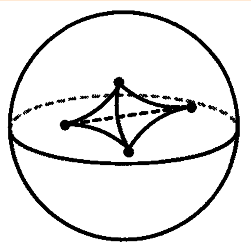
- Suppose we form a triangle in hyperbolic space such that each of its three edges comes from segments of geodesics.
- The sum of the angles of the triangle is less than the sum of the angles of the corresponding Euclidean triangle with the same vertices.



- The sum of the angles of the Euclidean triangle is exactly 180°.
- So the sum of the angles of the hyperbolic triangle is strictly less than 180°.
- This amazing fact holds universally.
- The angles of any triangle in hyperbolic three-space add up to less than 180°.

Geodesic Planes and Tetrahedra in H^3

- We now describe how we can use pieces of hyperbolic three-space in order to obtain so-called hyperbolic manifolds.



- The pieces that we use are tetrahedra.
- The edges of the tetrahedra are geodesics in hyperbolic space.
- The faces are geodesic planes in hyperbolic space.
- It is no big surprise that these geodesic planes are one of two kinds:
  - Pieces of spheres, perpendicular to the unit sphere bounding H^3;
  - Pieces of disks contained in planes that pass through the origin in H^3.

Gluing Tetrahedra

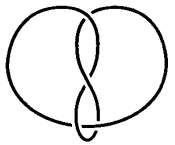
- We constructed surfaces by gluing together pairs of edges in a set of triangles until every edge had been glued to some other edge.
- Analogously, it is possible to glue together pairs of faces in a set of tetrahedra until every face has been glued to some other face.
- When done correctly, the result can sometimes be a knot complement.

Hyperbolic Knots

- Suppose the tetrahedra that we glue together are hyperbolic tetrahedra, in that they sit inside hyperbolic space.
- Suppose we glue them together along their faces so that their faces match without distortion.
- Thus, the hyperbolic measurements of distances within the individual tetrahedra match.
- The result is a hyperbolic knot complement.
- We can use the hyperbolic method for measuring distance within the individual tetrahedra in order to obtain a hyperbolic method for measuring distance in the entire knot complement.
- We then say that the knot is a **hyperbolic knot**.

Hyperbolic Volumes

- Every hyperbolic knot has a **hyperbolic volume**.
- It is the sum of the volumes of the individual hyperbolic tetrahedra that make up the knot complement.
- It gives the volume of the complement of the knot, as measured by our hyperbolic metric.
- Although it appears that the volume of three-space minus the knot would be infinite, it is in fact finite when measured using this hyperbolic method of measuring volume.
- The hyperbolic volume is an invariant for the hyperbolic knots. It depends only on the knot and not on any particular projection.



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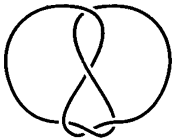

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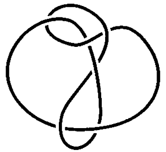

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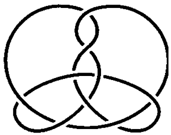

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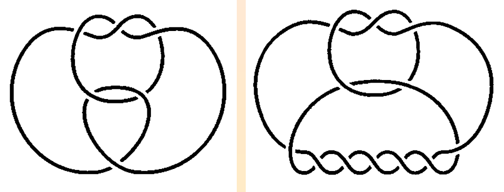
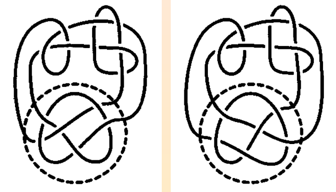
Different Knots with Same Volume

- In general, there are distinct knots that have the same volume.

  Example: The two knots on the left have the same volume.





- More generally, if we flip a tangle in a hyperbolic knot to produce a mutant knot, the mutant will also be a hyperbolic knot and it will have the same volume.

  Example: The two knots on the right cannot be distinguished by volume.

Computing the Volume of a Knot

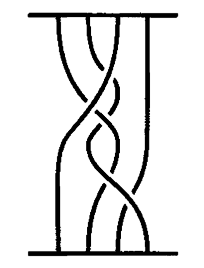
- To compute the volume of a knot:
  - We first cut the complement of the knot into a finite set of tetrahedra.
  - We then place the tetrahedra in hyperbolic space.
  - In order that these hyperbolic tetrahedra glue together correctly to give the hyperbolic metric, a set of equations must be satisfied.
  - For n tetrahedra, we obtain n polynomial equations in n variables. (The variables are in fact complex variables.)
  - We use numerical methods to solve this system numerically.
  - The solution determines the hyperbolic metric on each tetrahedron.
  - We can then compute the hyperbolic volume of each of the tetrahedra.
  - We add the volumes to get the volume of the knot complement.
- This work can be done using Jeffrey Weeks' computer program `SnapPea`.

Subsection 4

Braids

Braids

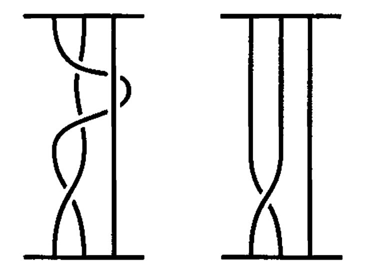
- A **braid** is a set of *n* strings, all of which are attached to a horizontal bar at the top and at the bottom.



- Each string always heads downward as we move along anyone of the strings from the top bar to the bottom bar.
- Another way to say the same thing is that each string intersects any horizontal plane between the two bars exactly once.

Equivalence of Braids

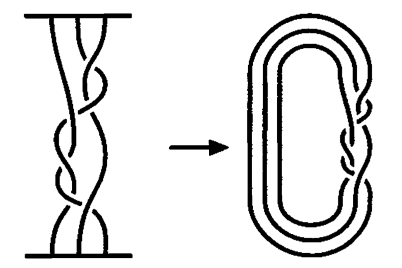
- We consider two braids to be **equivalent** if we can rearrange the strings in the two braids to look the same:
  - Without passing any strings through one another or themselves;
  - Keeping the bars fixed and keeping the strings attached to the bars.



- We are not allowed to pull the strings over the top of the upper bar or beneath the bottom bar.

The Closure of a Braid

- We can pull the bottom bar around and glue it to the top bar.
- The resulting strings form a knot or link.
- This is called the **closure** of the braid.



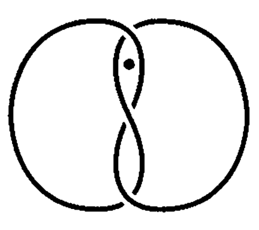
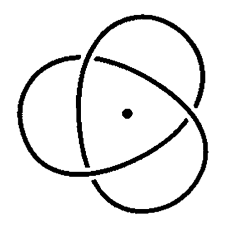
- Therefore every braid corresponds to a particular knot or link.
- We can think of there being an axis coming right out of the page, around which the closure of the braid is wrapped.

The Closed Braid Representation of a Knot

- We say that we have a **closed braid representation** of the knot if there is a choice of orientation on the knot so that, as we traverse the knot in that direction, we always travel clockwise around the axis without any backtracking.

  Example: A projection of the trefoil with an axis is shown on the left.

  This is not a closed braid representation around its axis.

Consider, also, the projection and axis shown on the right.

This is a closed braid representation around its axis.

Links and Closed Braids

Claim: Every knot or link is a closed braid.

Let L be our knot or link in a particular projection.

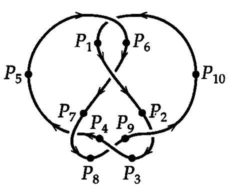
Orient each of the components of L.

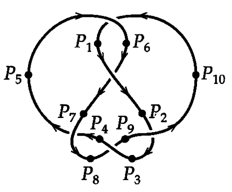
For any strand of the knot between an overcrossing and an undercrossing, choose a point on the strand.

We start with a point P_1 that occurs after an undercrossing.

As we traverse the knot or link in the direction of the orientation, we label the chosen points P_1 through P_n.

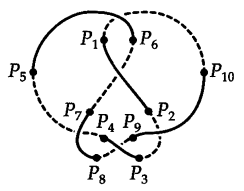
We can think of these labeled points as being the intersection of the projection plane with the knot or link, and the strands above and below the projection plane as the bridges.



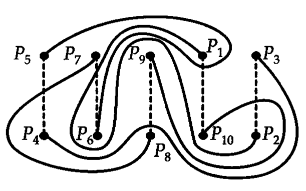


Links and Closed Braids (Cont'd)

- The strand of the knot connecting P_1 to P_2 lies above the projection plane. The strand connecting P_2 to P_3 lies beneath the projection plane. The strand connecting P_{2i-1} to P_{2i} always lies above the projection plane, and the strand connecting P_{2i} to P_{2i+1} always lies below.



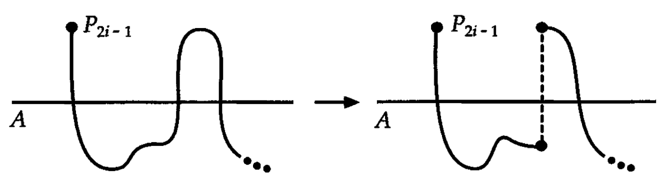
  We isotope (rearrange without cutting and pasting) the projection so that the n strands beneath the projection plane are lined up. We arrange the strands so that the even-numbered points are all next to one another.



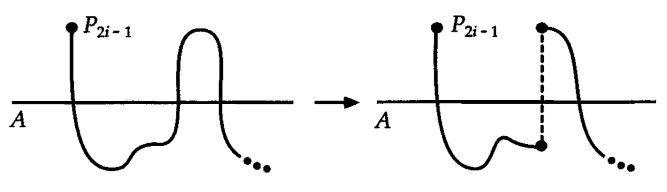
  We can performe this rearrangement under the projection plane, since we are just sliding nonintersecting arcs around a bit.

Links and Closed Braids (Cont'd)

- Let A be a straight line in the projection plane that is a perpendicular bisector of all of the lower bridges.
  Each of the upper bridges will then cross A an odd number of times, since an upper bridge starts at a point P_{2i-1} that is north of segment A and ends at the point P_{2i} that is south of segment A.
  We would like the upper bridges to cross A only once.
  So we will do a little bridge work to make it so.
  Suppose one of the upper bridges does cross A more than once.
  Take the second point on it where it crosses A after leaving P_{2i-1}.
  Push the upper bridge below the projection plane at that point.



Links and Closed Braids (Cont'd)



- The one upper bridge now splits into two new upper bridges and a new lower bridge.
  - The first new upper bridge crosses A once.
  - The second new upper bridge crosses A two fewer times than the original upper bridge did.

  We can repeat this process with the new upper bridges.

  We repeat eventually with the other remaining upper bridges until every upper bridge crosses the line A exactly once.

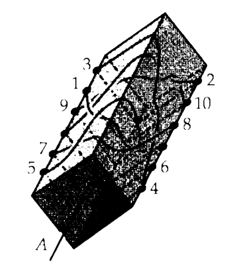
Links and Closed Braids (Cont'd)

- We now draw our link as a closed braid with axis A.

  We begin each of the upper bridges at its starting point P_{2i-1} in the projection plane.

  As we draw the bridge, we have it increase in height above the projection plane until it is directly above the line A.

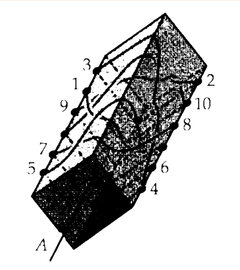
  At this time we are at the point on the bridge that used to be where it crossed A.



Links and Closed Braids (Conclusion)

- After that point, we have the bridge decrease in height until it reaches the point P_{2i} back in the projection plane.

  Similarly, starting from the even-numbered points, we have the lower bridges decrease in height until they are directly beneath A.



  Now we are at the point on them where they used to cross under A.

  We then have them increase in height until they reach the odd-numbered points back in the projection plane.

The Braid Index

- Every link can be represented as a closed braid
- We are interested in representing the link as a simple braid with as few strings as possible.
- We define the **braid index** of a link to be the least number of strings in a braid corresponding to a closed braid representation of the link.

  Example:
  - The braid index of the unknot is 1.
  - The braid index of the trefoil knot is 2.

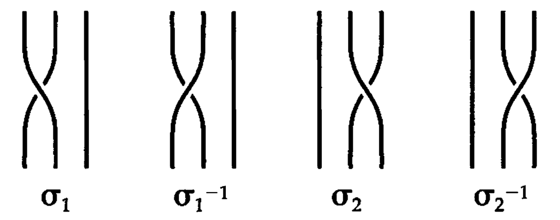
On the Braid Index

- The braid index is an invariant for knots and links, but in general it is difficult to compute.
  - Putting a knot or link in braid form and then counting the strings gives an upper bound on the braid index.
  - In the next chapter, we see one way to get a lower bound on the braid index using knot polynomials.
- Yamada related the braid index to the number of Seifert circles.

  In particular, he proved that the braid index of a knot is equal to the least number of Seifert circles in any projection of the knot.
- Ohyama proved that if L is a nonsplittable link, $c(L)$ is its crossing number, and $b(L)$ is its braid index, then $c(L) \geq 2(b(L) - 1)$.
- For the figure-eight knot, which has braid index 3, this is actually an equality.

Description of Braids

- A projection of a braid can be described by listing which of the strings cross over and under each other as we move down the braid.
- We can arrange it so that no two crossings in the braid occur at exactly the same height.
- We look at three-string braids.



  - If the first string crosses over the second, we call it a σ_1 crossing.
  - If the first string crosses under the second, we call it a σ_1^{-1} crossing.
  - If the second string crosses over the third, we call that an σ_2 crossing.
  - If it crosses under the third string, it is a σ_2^{-1} crossing.

Description of Braids (Cont'd)

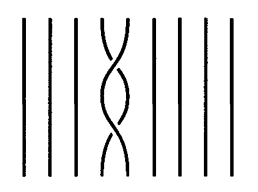
- Consider the braid shown on the right.
- It is described completely by listing the crossings in order from top to bottom as

$$\sigma_2\sigma_1\sigma_1\sigma_2^{-1}\sigma_1\sigma_1.$$

- We call this a **word** for the braid.
- More generally, if we have a braid with n strings, we denote by:
  - σ_i the i-th string crossing over the $(i+1)$-st string;
  - σ_i^{-1} the i-th string crossing under the $(i+1)$-st string.

Simplifying Braids Using Words

- Suppose a braid has $\sigma_i^{-1}\sigma_i$ as part of the word that describes it.
- Then geometrically, this pair of crossings looks like the figure.



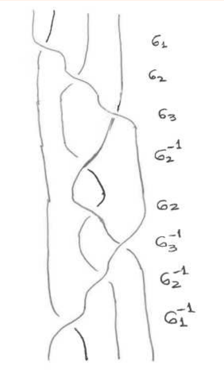
- A simple Type II Reidemeister move eliminates both crossings, but leaves us with an equivalent braid.
- The effect on the word is to eliminate $\sigma_i^{-1}\sigma_i$.
- The same phenomenon occurs also for $\sigma_i\sigma_i^{-1}$.

Example

- Consider the word

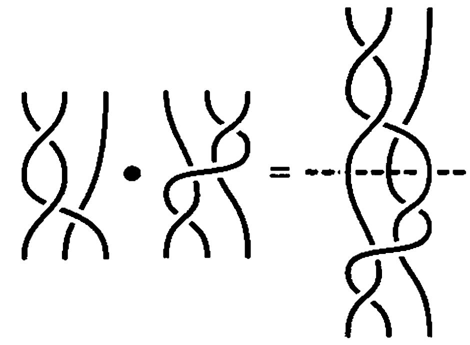
$$\sigma_1\sigma_2\sigma_3\sigma_2^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}.$$

- It collapses down to nothing.
- So the braid that it represents is equivalent to the trivial braid of four vertical strings that do not cross.



Product of Braids

- Suppose we have two n-string braids.
- We can stack them on top of each other to create a new braid.



- This is the **product** of the original two braids.

Properties of the Braid Product

- The following properties apply to the product of braids:
  (a) The trivial n-string braid I_n, consisting of n strings that do not cross, acts as an identity when it multiplies any other n-string braid.
  I.e., multiplying a braid B by I_n results in the braid B.
  (b) Let a braid B be described by the word

$$\sigma_{i_1}^{e_1}\sigma_{i_2}^{e_2}\cdots\sigma_{i_k}^{e_k},$$

  where:
  - $i_1, \ldots, i_k \in \{1, 2, \ldots, n-1\}$;
  - $e_1, \ldots, e_k \in \{-1, 1\}$.

  Let B^{-1} be the braid described by

$$\sigma_{i_k}^{-e_k}\cdots\sigma_{i_2}^{-e_2}\sigma_{i_1}^{-e_1}.$$

  Then B^{-1} has the property that $BB^{-1} = I_n$ and $B^{-1}B = I_n$.
  (c) Multiplication of n-string braids is associative, namely

$$B_1(B_2B_3) = (B_1B_2)B_3,$$

  for any three n-string braids B_1, B_2 and B_3.

The Group of n-String Braids

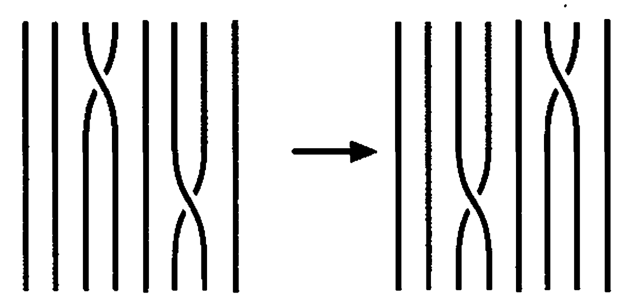
- A **group** is a set of elements, in this case the n-string braids, and a way to multiply elements such that:
  1. There exists an identity element that, when multiplying any element, does not change it;
  2. Every element has an inverse;
  3. The multiplication is associative, that is, $a(bc) = (ab)c$, for any three elements a, b, c of the group.
- Let B_n denote the group of n-string braids.
- A particular element of B_n is a braid together with all the other braids that are equivalent to it.
- We say that an element is an equivalence class of braids, although we will sometimes refer to it as a single braid.

Rules for Equivalence of Words

- An element of B_n has many different projections and many different words that represent it.
- We would like to know when two different words written out in the letters $\sigma_1^{\pm 1}, \ldots, \sigma_n^{\pm 1}$ represent the same braid.
- The following rules may be applied without changing the equivalence class of two words.
  - We may add or delete $\sigma_i \sigma_i^{-1}$ or $\sigma_i^{-1} \sigma_i$ from a word.
    This corresponds to a kind of Type II Reidemeister move.
  - The two braids $\sigma_i \sigma_{i+1} \sigma_i$ and $\sigma_{i+1} \sigma_i \sigma_{i+1}$ are equivalent.
    This is a kind of Type III Reidemeister move.
  - If a braid contains $\sigma_1 \sigma_j$, where $|i - j| > 1$, then we can switch the order of σ_1 and σ_j, replacing $\sigma_i \sigma_j$ in our word with $\sigma_j \sigma_i$.

Third Rule: Illustration

- If a braid contains $\sigma_1\sigma_j$, where $|i - j| > 1$, then we can switch the order of σ_1 and σ_j, replacing $\sigma_i\sigma_j$ in our word with $\sigma_j\sigma_i$.



Equivalence of Words

- Two words w_1 and w_2 represent the same braid if and only if we can get from the one word to the other by a sequence of these three operations.

  Example: Consider $w_1 = \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_2\sigma_4$ and $w_2 = \sigma_2\sigma_1\sigma_2^2$.

  These represent the same five-string braid since we can get from the first word to the second word by the following set of applications of the three rules.

$$
\begin{aligned}
w_1 = \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_2\sigma_4 &\xrightarrow{\text{Rule 3}} \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_4\sigma_2 \\
&\xrightarrow{\text{Rule 3}} \sigma_1\sigma_2\sigma_4^{-1}\sigma_4\sigma_1\sigma_2 \\
&\xrightarrow{\text{Rule 1}} \sigma_1\sigma_2\sigma_1\sigma_2 \\
&\xrightarrow{\text{Rule 2}} \sigma_2\sigma_1\sigma_2\sigma_2 = w_2.
\end{aligned}
$$

Markov Equivalence

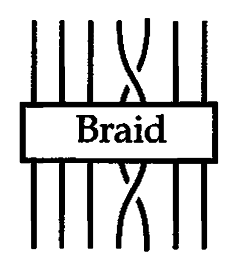
- To make braids useful for knot and link theory we would like to be able to determine when the closures of two braids represent the same oriented link.
- We call two braids **Markov equivalent** if their closures yield the same oriented link.
- We would like to have a set of moves on braids that give all equivalences on the corresponding closed braids.

Markov's Theorem

- Markov's Theorem says that two braids are Markov equivalent if and only if they are related through a sequence of:
  - The three operations that we have already seen (which are operations that give us back the same open braid);
  - Two additional operations.
- Note the following:
  - None of the operations so far change the number of strings in the corresponding braid;
  - There are closed-braid representations of the same link that do not have the same number of strings.
- These facts make obvious the need for at least one more operation (on top of the three already encountered) to determine Markov equivalence.

First Operation: Conjugation

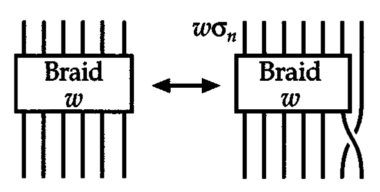
- The first new operation is called **conjugation**.
- On the word for a braid, conjugation is the operation of multiplying the word at the beginning by σ_j and at the end by σ_j^{-1} or at the beginning by σ_j^{-1} and at the end by σ_j.



- It is easy to see that conjugation does not change the oriented link corresponding to the closed braid.
- It corresponds to a Type II Reidemeister move on the link projection.

Second Operation: Stabilization

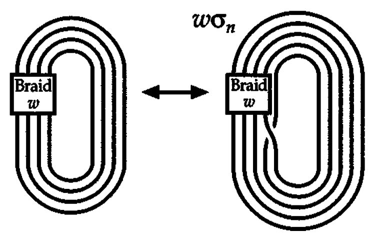
- The next operation is called **stabilization**.
- We add or delete a loop in the closed braid.
- In terms of the word describing a braid, this operation takes a word w corresponding to an n-string braid and replaces it with the word $w\sigma_n$ or $w\sigma_n^{-1}$, each of which corresponds to an $n+1$-string braid.
- We also allow the inverse operation, where a word of the form $w\sigma_n$ or $w\sigma_n^{-1}$ is replaced with just the word w, assuming w does not contain the letters σ_n or σ_n^{-1} within it.



- The resulting word w then corresponds to a braid with one less string.

Stabilization (Cont'd)

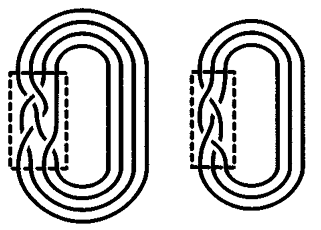
- Note the oriented link corresponding to the closed braid remains unchanged by either of these operations.
- The operations correspond to a Type I Reidemeister move on the link projection.

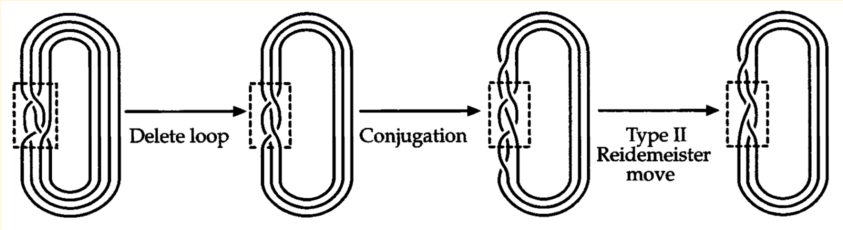


- The two operations of conjugation and stabilization, together with the three operations mentioned previously, suffice to get us from anyone closed-braid representation of an oriented link to any other closed-braid representation of the same oriented link.

Example

- Here are two closed-braid representatives of the figure-eight knot.
- Hence, there must be a sequence of these Markov moves (together with the three equivalence moves for a given open braid) that take us from the first braid to the second braid.





Subsection 5

Almost Alternating Knots

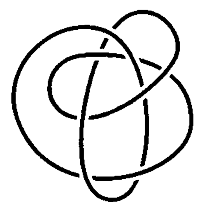
Almost Alternating Knots

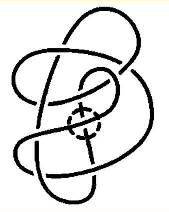
- We call a projection of a link an **almost alternating projection** if one crossing change in the projection would make it an alternating projection.
- We call a link an **almost alternating link** if:
  - It has an almost alternating projection
  - It does not have an alternating projection.

Example: The knot in the figure is an almost alternating knot.

It is known to be nonalternating.

It has the almost alternating projection on the right.

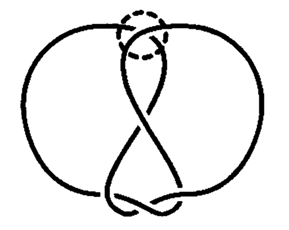


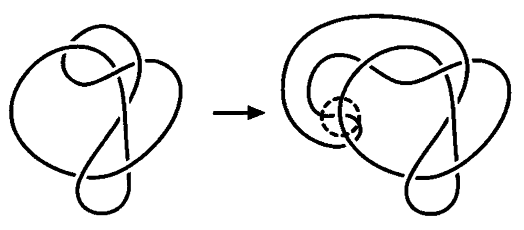


Knots and Links with Almost Alternating Projections

- An amazing array of knots and links have almost alternating projections.

  Example: The unknot has an almost alternating projection.

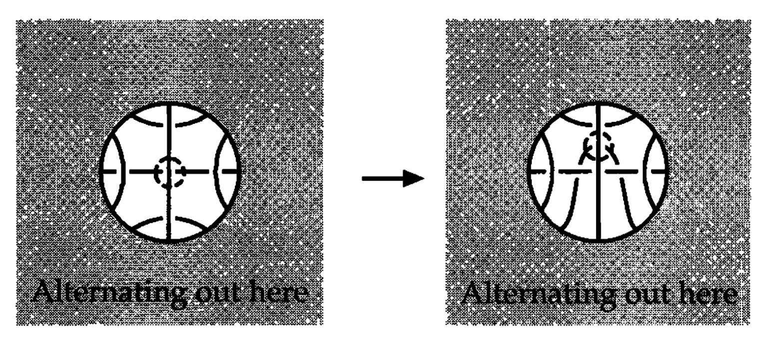




- Even more amazing is the fact that every alternating knot or link has an almost alternating projection.
- We can simply perform a Type II Reidemeister move to an alternating projection to obtain an almost alternating projection.

"Tongue Moves" for Adding Crossings

- Given an almost alternating projection, we can always complicate it by a "tongue move".



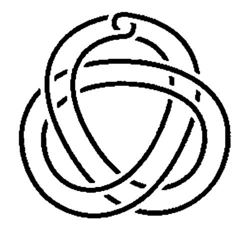
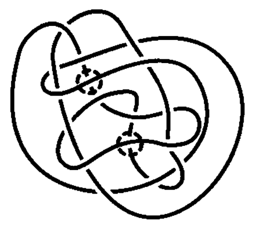
- We just push a part of the link up over the nonalternating crossing.
- The result is a new almost alternating projection with two more crossings.

m-Almost Alternating Knots

- Define an m-**almost alternating knot** to be a knot that:
  - Has a projection where m crossing changes would make the projection alternating;
  - Has no projection that could be made alternating in fewer than m crossing changes.
- We consider alternating knots to be 0-almost alternating.
- Almost alternating knots are one-almost alternating.

Example

- An example of a two-almost alternating knot is the following Whitehead double of the trefoil.

- It is not alternating or almost alternating because it is a satellite knot.
- On the right, a projection that is two-almost alternating is shown.

m-Almost Alternating Classification

Claim: Every knot is m-almost alternating for some m.

Let K be a given knot.

Consider the set

$$N = \{n : K \text{ has an } n\text{-almost alternating projection}\}.$$

Then N is a subset of the set of natural numbers.

Therefore, N has a minimum element m.

By definition:

- K has an m-almost alternating projection;
- K does not have a projection that could be made alternating with fewer than m crossing changes.

So K is m-almost alternating.

The Value of m Versus the Unknotting Number

- We have divided all knots into separate categories, depending on their value of m.
- This number m measures how far a knot is from being alternating.
- It is similar to the unknotting number in that the unknotting number is the least number of crossing changes necessary in any projection to make the knot into the unknot.
- The unknotting number measures how far a knot is from being the unknot.
- In some sense, these two measurements, the almost alternating number and the unknotting number, are the two extremes.
  - The alternating knot is "the most complicated" knot we can create by changing crossings in a projection;
  - The trivial knot is the simplest knot we can create by changing crossings in a projection.