Introduction to Knot Theory

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LSSU Math 500

Subsection 1

The Bracket Polynomial and the Jones Polynomial

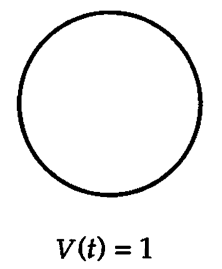
Knot Polynomials as Invariants

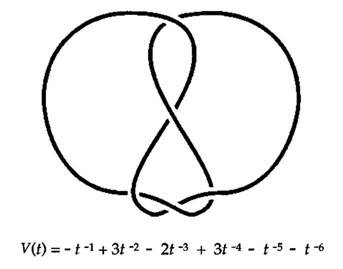
- To each knot, we associate a Laurent polynomial, i.e., one that can have both positive and negative powers of the variable t.
- We compute the polynomial from a projection of the knot.
- Any two projections of the same knot yield the same polynomial.
- So the polynomial is an invariant of the knot.
- If we have two pictures of two knots and the computed polynomials are different, then the two knots have to be distinct.

Example

- E.g., for one of the polynomials $V(t)$ that we compute:
  - The unknot has polynomial 1;
  - The knot on the right has polynomial

$$V(t) = -t^{-1} + 3t^{-2} - 2t^{-3} + 3t^{-4} - t^{-5} - t^{-6}.$$





- Therefore, the unknot and the knot on the right are distinct.

The Bracket Polynomial (Rule 1)

- Suppose that we are trying to create a polynomial invariant for knots and links.
- Let us use the notation $\langle K \rangle$ to denote the bracket polynomial of a knot K.
- We have a few requirements for the polynomial.
- First, we would like the polynomial of the trivial knot to be 1.
- Then our first rule becomes:

$$\text{Rule 1:} \qquad \langle \bigcirc \rangle = 1.$$

The Bracket Polynomial (Rule 2)

- Second, we want a method for obtaining the bracket polynomial of a link in terms of the bracket polynomials of simpler links.
- Given a crossing in our link projection, we split it open vertically and horizontally, in order to obtain two new link projections, each of which has one fewer crossing.
- We make the bracket polynomial of our link projection a linear combination of the bracket polynomials of our two new link projections, with not yet determined coefficients:

$$\text{Rule 2:} \qquad \begin{aligned} \langle \times \rangle &= A\langle)(\rangle + B\langle \asymp \rangle \\ \langle \times \rangle &= A\langle \asymp \rangle + B\langle)(\rangle. \end{aligned}$$

- The second equation here is just the first equation, but looked at from a perpendicular view.
- So we do not actually consider these two equations as distinct rules.

The Bracket Polynomial (Rule 3)

- Finally, we would like a rule for adding in a trivial component to a link (the result of which will always be a split link):

$$\text{Rule 3:} \qquad \langle L \cup \bigcirc \rangle = C\langle L \rangle.$$

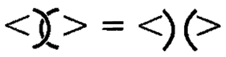
- Each time we add in an extra trivial component that is not tangled up with the original link L, we just multiply the entire polynomial by C.
- As with A and B, we consider C a variable in the polynomial, for the time being.

Invariance

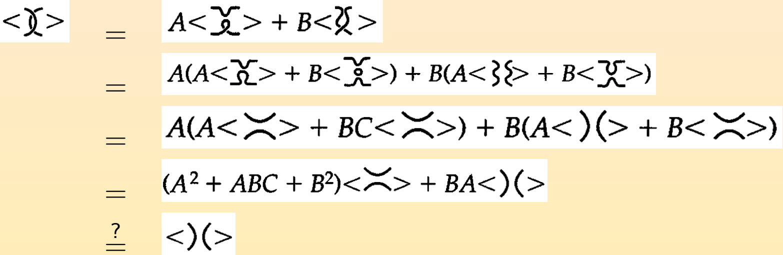
- The most important criterion for our polynomial is that it be an invariant for links.
- Equivalently, the calculation of the polynomial cannot depend on the particular projection that we start with.
- It must be unchanged by the Reidemeister moves.

Investigating Invariance (Type II Move)

- We begin by looking at what happens to our polynomial when we apply a type II Reidemeister move.
- We want



- We have:



Equating the Coefficients

- In order that the polynomial be unchanged by this move, we are forced to make $B = A^{-1}$, so that the coefficient in front of the vertical tangle is one.
- With $B = A^{-1}$, it is apparent that we also need $A^2 + C + A^{-2} = 0$ so that the coefficient in front of the horizontal tangle is zero.
- So we should make $C = -A^2 - A^{-2}$.
- The three rules for computing the bracket polynomial become:

$$\begin{array}{lrcl}
\text{Rule 1:} & \langle \bigcirc \rangle & = & 1; \\
\text{Rule 2:} & \langle \times \rangle & = & A\langle)(\rangle + A^{-1}\langle \asymp \rangle; \\
 & \langle \times \rangle & = & A\langle \asymp \rangle + A^{-1}\langle)(\rangle; \\
\text{Rule 3:} & \langle L \cup \bigcirc \rangle & = & (-A^2 - A^{-2})\langle L \rangle.
\end{array}$$

Investigating Invariance (Type III Move)

- Now, let's see what effect the third Reidemeister move has on the polynomial:

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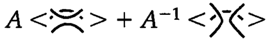
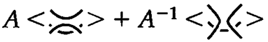


$\overset{\text{Type II}}{=}$



$=$

- Thus, Type III Reidemeister moves have no effect on the polynomial.
- Once we have fixed it so that the Type II moves leave the polynomial unchanged, the Type III move comes for free.

Example I

- We just use Rules 1 and 3 to calculate the polynomial for the usual projection of the trivial link of two components.

  We have

$$\begin{aligned} \langle \bigcirc \cup \bigcirc \rangle &\overset{\text{Rule 3}}{=} -(A^{-2}+A^{2})\langle \bigcirc \rangle \\ &\overset{\text{Rule 1}}{=} -(A^{2}+A^{-2})1. \end{aligned}$$

Example II

- We compute the bracket polynomial of a projection of the simplest nontrivial link on two components, the Hopf link.
- This time, we use all three rules.

$\langle$  $\rangle$

$= A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

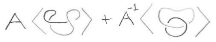
$= A\,(A\langle$  $\rangle + A^{-1}\langle$  $\rangle)$ $+ A^{-1}\,(A\langle$  $\rangle + A^{-1}\langle$  $\rangle)$

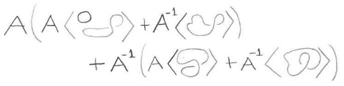
$$= A\,(A(-(A^2+A^{-2})) + A^{-1}\,(1)) + A^{-1}\,(A(1) + A^{-1}\,(-(A^2+A^{-2})))$$

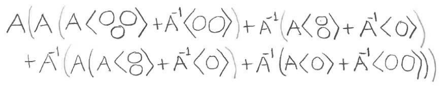
$$= -A^4 - A^{-4}$$

Example III

- We compute the bracket polynomial of a projection of the trefoil.
- Again, we use all three rules.

$=$ 

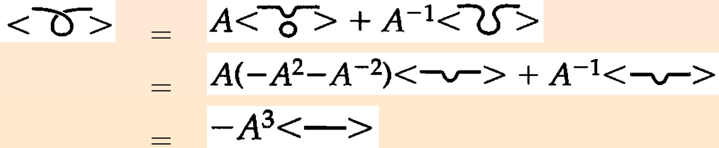
$=$ 

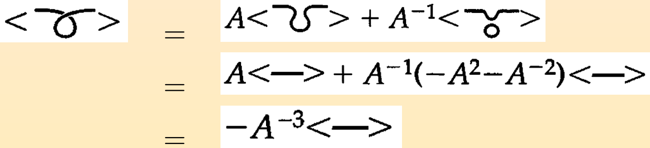
$=$ 

$$\begin{aligned}
&= A^3(-A^{-2}-A^2)^2 + A(-A^{-2}-A^2) + A(-A^{-2}-A^2) + A^{-1} \\
&\quad + A(-A^{-2}-A^2) + A^{-1} + A^{-1} + A^{-3}(-A^{-2}-A^2) \\
&= A^{-1} + 2A^3 + A^7 - A^{-1} - A^3 - A^{-1} - A^3 + A^{-1} \\
&\quad - A^{-1} - A^3 + A^{-1} + A^{-1} - A^{-5} - A^{-1} \\
&= -A^{-5} - A^3 + A^7.
\end{aligned}$$

Investigating Invariance (Type I Move)

- We look at what happens to the polynomial when we apply a Type I Reidemeister move.

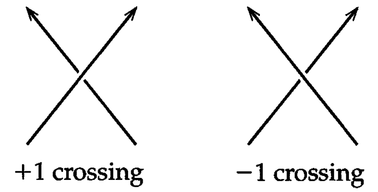


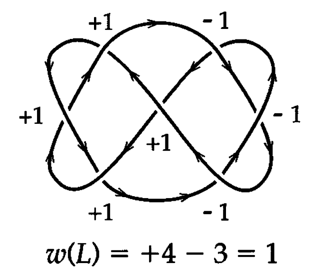


- The polynomial has been changed by a Type I move.
- Our polynomial depends on what projection of the knot we have.
- We try next to fix this problem.

The Writhe of an Oriented Link Projection

- We give an orientation to our knot or link projection L.
- At every crossing of the projection, we have either a $+1$ or -1, as in the left figure.



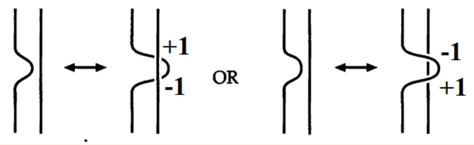


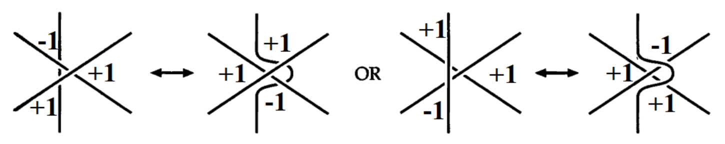
- The sum of all these $+1$s and -1s is called the **writhe** of the oriented link projection L and denoted by $w(L)$.
- This is also sometimes called the **twist** of the projection.

  Example: We can calculate the writhe of the oriented link projection shown in the figure on the right.

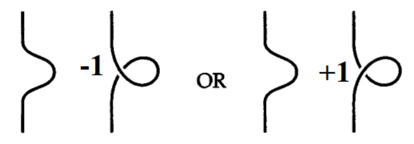
Writhe and Reidemeister Moves

- TThe writhe of a link projection is invariant under Reidemeister moves II and III.





- Notice that Reidemeister move I always changes the writhe by ± 1.

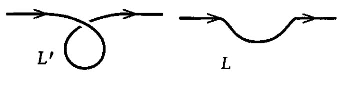


The X Polynomial

- We define a new polynomial called the X polynomial.
- It is a polynomial of oriented links.
- It is defined to be
$$X(L) = (-A^3)^{-w(L)}\langle L\rangle.$$
- Both $w(L)$ and $\langle L\rangle$ are unaffected by moves II and III.
- So $X(L)$ is unaffected by moves II and III.

The X Polynomial (Cont'd)

- We look at a Reidemeister move of Type I.
- Suppose we had a strand as in the figure and took out the twist.



- Then $w(L') = w(L) + 1$.

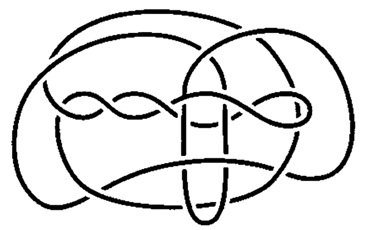
$$\begin{aligned} X(L') &= (-A^3)^{-w(L')}\langle L'\rangle \\ &= (-A^3)^{-(w(L)+1)}\langle L'\rangle \\ &= (-A^3)^{-w(L)-1}((-A)^3\langle L\rangle) \\ &= (-A^3)^{-w(L)}\langle L\rangle \\ &= X(L). \end{aligned}$$

- Thus, $X(L)$ is unaffected by this Type I Reidemeister move.
- Similarly, it is unaffected by the other version of a Type I move.
- Therefore, $X(L)$ is an invariant for knots and links.

Example

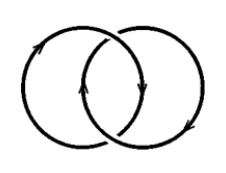
- As an example, consider the link $\bigcirc\ \bigcirc$.
- As we previously computed, $\langle \bigcirc\bigcirc \rangle = -A^2 - A^{-2}$.
- Since the writhe of this link is 0, we have that $X(\bigcirc\bigcirc) = -A^2 - A^{-2}$.
- This result is independent of the projection of the link.

- We could take a really nasty projection of this link, like the one in the figure.
- If we calculated the X polynomial for this projection, we would find that the answer was exactly the same, namely $-A^2 - A^{-2}$.



Example I

- We compute $X(L)$ for the oriented knot shown.



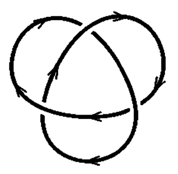
- We know that

$$= -A^{-4} - A^{4}.$$

- Moreover, we have
$$w(L) = +2.$$
- So we obtain
$$X(L) = (-A^3)^{-w(L)}\langle L\rangle = (-A^3)^{-2}(-A^{-4} - A^4) = -A^{-10} - A^{-2}.$$

Example II

- We compute $X(L)$ for the oriented knot shown.



- We know that

 $= -A^{-5} - A^{3} + A^{7}.$

- Moreover, we have
$$w(L) = -3.$$

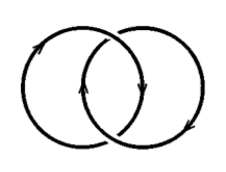
- So we obtain
$$X(L) = (-A^3)^{-w(L)} \langle L \rangle = (-A^3)^3 (-A^{-5} - A^3 + A^7) = A^4 + A^{12} - A^{16}.$$

The Jones Polynomial

- The original **Jones polynomial** is obtained from $X(L)$ by replacing each A in the polynomial with $t^{-1/4}$.
- The resulting polynomial with variable t (and powers that are not necessarily integers) is exactly the polynomial that Jones first came up with in 1984.
- We denote this polynomial by $V(L)$, and sometimes $V(t)$, when the link involved is clear.
- Using the Jones polynomial (or equivalently the X polynomial) we can distinguish every prime knot of 9 or fewer crossings, i.e., they all have distinct Jones polynomials.

Example I

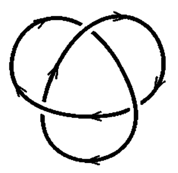
- Write the Jones polynomial of the knot shown.



- We have
$$X(L) = -A^{-10} - A^{-2}.$$
- Therefore,
$$V(L) = -(t^{-1/4})^{-10} - (t^{-1/4}) - 2 = -t^{5/2} - t^{1/2}.$$

Example II

- Write the Jones polynomial of the trefoil knot.



- We know

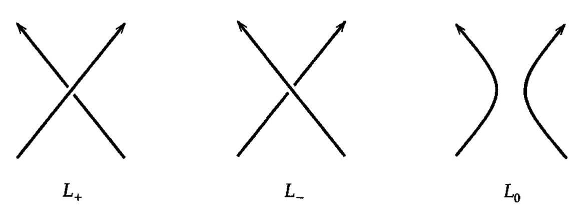
$$X(L) = A^4 + A^{12} - A^{16}.$$

- So we get

$$V(L) = (t^{-1/4})^4 + (t^{-1/4})^{12} - (t^{-1/4})^{16} = t^{-1} + t^{-3} - t^{-4}.$$

A Relation Satisfied by the Jones Polynomial

- Let L_+, L_-, and L_0 be three oriented link projections that are identical except where they appear as in the following figure.



- We can use the relation of the bracket polynomial to show that the Jones polynomials of the three links are related by the equation

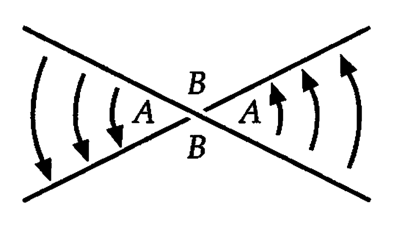
$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0.$$

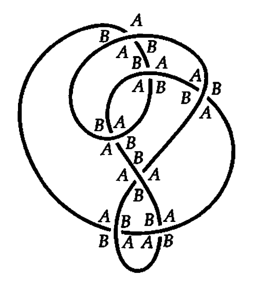
Subsection 2

Polynomials of Alternating Knots

Labeling the Regions in a Knot Projection

- We develop a second way to think about the bracket polynomial.
- Consider the four regions of the projection plane that come together at a crossing.





We label two of them with an A and two of them with a B by the following simple rule:

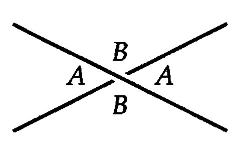
- Rotate the overstrand counterclockwise, passing over two of the regions.
- Label these two regions with an A and the remaining two with a B.

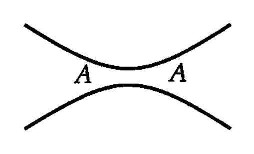
A-Splits and B-Splits

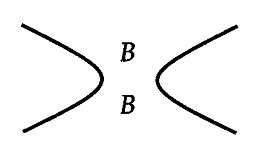
- Recall Rule 2 for calculating the bracket polynomial:

$$\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle.$$

Consider a crossing that is labeled.







We split open the crossing in two different ways.

- The first splitting opens a channel between the two regions labeled A at the crossing. We call this an A-**split**.
- The second splitting opens a channel between the two regions labeled B at the crossing. We call this a B-**split**.

States of a Link and Bracket Polynomial

- We call a choice of how to split all of the n crossings in the projection of L a **state**.
- The bracket polynomial of L then depends on the bracket polynomials for all of the possible states of the projection of L.
- A particular state of L turns L into a corresponding link L'.
- L' has no crossings.
- So L' must be a set of nonoverlapping unknotted loops in the plane.
- Set $|S|$ be the number of loops in L'.
- Then the bracket polynomial of L' is simply

$$(-A^2 - A^{-2})^{|S|-1}.$$

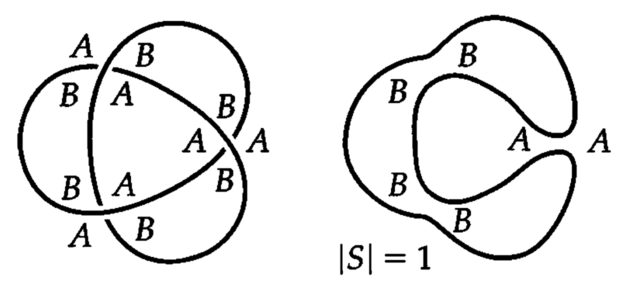
States of a Link and Bracket Polynomial (Cont'd)

- Each time we split at a crossing, the polynomials of the two resultant links were multiplied:
  - By A if the split was an A-split;
  - By A^{-1} if the split was a B-split.
- So the polynomial of L' is multiplied by $A^{a(S)}A^{-b(S)}$, where:
  - $a(S)$ is the number of A-splits in S;
  - $b(S)$ is the number of B-splits in S.
- It follows that the total contribution to the bracket polynomial by the state S is

$$A^{a(S)}A^{-b(S)}(-A^2 - A^{-2})^{|S|-1}.$$

Example

- Consider the particular state of the trefoil knot shown in the figure:



- It contributes A^1A^{-2} to the bracket polynomial of this projection of the trefoil knot.

Computing the Bracket Polynomial of a Link Projection

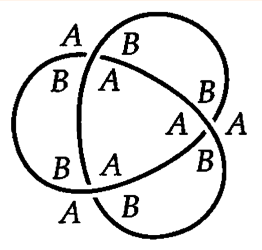
- The bracket polynomial of the projection of the link L is the sum over all of the possible states of the contributions of all individual states,

$$\langle L \rangle = \sum_S A^{a(S)} A^{-b(S)} (-A^2 - A^{-2})^{|S|-1}.$$

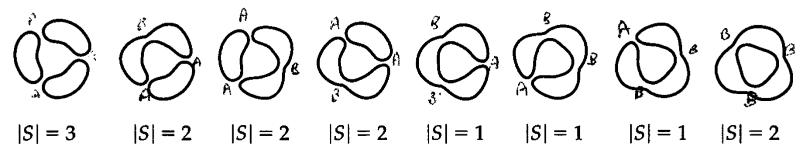
- If we want to compute the bracket polynomial of a given projection of L, we do the following:
  - List all of the links obtained by splitting all of the crossings of L in every possible combination;
  - Compute the contribution to the polynomial of each term.

Example: The Bracket Polynomial of a Link Projection

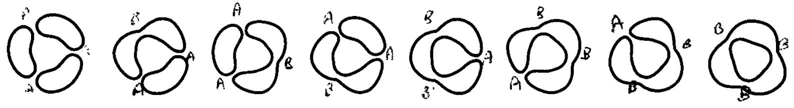
- We recompute the bracket polynomial of the trefoil projection shown on the right.



- Since there are three crossings in the projection, there will be $2^3 = 8$ states.
- For each state S of the eight states, we compute $|S|$ by simply counting how many circles there are in the corresponding link.



Example (Cont'd)



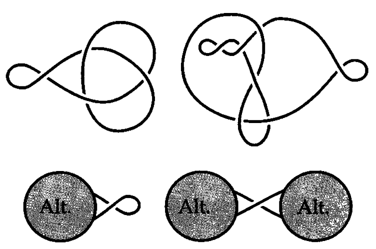
$|S| = 3$ $\quad |S| = 2$ $\quad |S| = 2$ $\quad |S| = 2$ $\quad |S| = 1$ $\quad |S| = 1$ $\quad |S| = 1$ $\quad |S| = 2$

$$\begin{aligned}
\langle K \rangle &= A^3A^0(-A^2 - A^{-2})^{3-1} + A^2A^{-1}(-A^2 - A^{-2})^{2-1} \\
&\quad + A^2A^{-1}(-A^2 - A^{-2})^{2-1} + A^2A^{-1}(-A^2 - A^{-2})^{2-1} \\
&\quad + A^1A^{-2}(-A^2 - A^{-2})^{1-1} + A^1A^{-2}(-A^2A^{-2})^{1-1} \\
&\quad + A^1A^{-2}(-A^2 - A^{-2})^{1-1} + A^0A^{-3}(-A^2 - A^{-2})^{2-1} \\
&= A^3(-A^2 - A^{-2})^2 + 3A(-A^2 - A^{-2}) \\
&\quad + 3A^{-1} + A^{-3}(-A^2 - A^{-2}) \\
&= A^7 - A^3 - A^{-5}.
\end{aligned}$$

Reduced Alternating Projections

- We have defined an **alternating knot** to be any knot that has a projection such that if you traverse the knot in a particular direction, you alternately pass over and then under crossings.
- We call the projection an **alternating projection**.

- We will call an alternating projection **reduced** if there are no unnecessary crossings in the projection, as in the figure.
- An unreduced alternating projection can be simplified it to a reduced one, thereby lowering the number of crossings.



- But if an alternating projection is reduced, there is no obvious way to lower the number of crossings.

The Kauffman, Thistlethwaite, Murasugi Theorem

Theorem

- Two reduced alternating projections of the same knot have the same number of crossings.
- A reduced alternating projection of a knot has the least number of crossings for any projection of that knot.

- The theorem implies that we can determine the crossing number for any alternating knot.
  - We take an alternating projection and simplify it until it is reduced.
  - Now we know that:
    - All reduced alternating projections have the same number of crossings;
    - The least number of crossings occurs in a reduced alternating projection.
  - So the least number of crossings is the number of crossings in this projection.

The Span of a Polynomial

- The **span** of a polynomial is the difference between the highest power that occurs in the polynomial and the lowest power that occurs in the polynomial.

  Example: Consider the polynomial

  $$A^3 - 2A^2 + 1 - A^1 - 7A^{-2}.$$

  The span of this polynomial is $3 - (-2) = 5$.

The Span of the Bracket Polynomial

Claim: Even though the bracket polynomial is not an invariant for knots, the span of the bracket polynomial is an invariant.

I.e., for a knot K, if we calculate the bracket polynomial from any projection, and then take the span, we always get the same answer.

Suppose we have two different projections P_1 and P_2 of the knot K.

There is a series of Reidemeister moves that take us from P_1 to P_2.

We have already seen that the Reidemeister moves of Types II and III do not change the bracket polynomial at all.

So these moves leave the span of the bracket polynomial unchanged.

So it remains to examine Type I moves.

The Span of the Bracket Polynomial (Cont'd)

- Type I Reidemeister moves can change the bracket polynomial.

  We show that they leave the span unaffected.

  We saw that a Type I move multiplies the polynomial by A^3 or A^{-3}.
  - If we multiply by A^3, this increases the highest power in the polynomial by 3 and increases the lowest power in the polynomial by 3. Hence the difference of those two, which gives the span, is unchanged.
  - Similarly, multiplying the entire polynomial by A^{-3} also leaves the span unchanged.

  Thus, all three Reidemeister moves leave the span of the bracket polynomial unchanged.

  We conclude that the span of the bracket polynomial must be the same for all projections of the knot K.

  So the span of the bracket polynomial is an invariant of the knot.

Number of Regions in a Link Projection

Claim: The number of regions R (including the region outside the knot) in a connected knot or link projection is always two more than the number of crossings .

Use the fact that the Euler characteristic of a disk is always 1.

Alternatively, draw a knot, keeping count of the number of regions created whenever a new crossing is created.

Span of Reduced Alternating Projection

Lemma

If K has a reduced alternating projection of n crossings, then $\text{span}(\langle K\rangle) = 4n$.

- We know that the span of the bracket polynomial of K does not depend on the projection of the knot that we use. So we may use the reduced alternating projection given in the statement of the lemma. Since the span is simply the difference between the highest power and the lowest power, we look at each of these two quantities.
  Each state S contributes a term $A^{a(S)}A^{-b(S)}(-A^2 - A^{-2})^{|S|-1}$.
  The highest power of A in this term is $A^{a(S)}A^{-b(S)}(A^2)^{|S|-1}$.
  Among all the states we therefore want to find the one that has the highest value of $a(S) - b(S) + 2(|S| - 1)$.
  That highest value will be the highest power of A that occurs in the bracket polynomial.

Span of Reduced Alternating Projection (Cont'd)

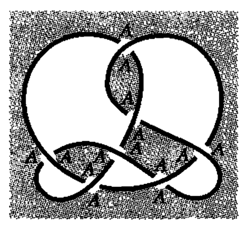
- In order to make $a(S) - b(S) + 2(|S| - 1)$ as large as possible, we want to pick a state where $|S|$ and $a(S)$ are large but $b(S)$ is small.
  - For $|S|$ to be large, we need there to be many disjoint circles in the link corresponding to S.
  - For $a(S)$ to be large and $b(S)$ to be small, we want as many of the splits as possible to be A-splits, and, consequently, as few of the splits as possible to be B-splits.

Wr try taking all A-splits and no B-splits.

Since we have n crossings, this means $a(S) = n$ and $b(S) = 0$.

We look at what happens to $|S|$.

Since the knot is alternating, when we place A's and B's around a crossing, the vertices in any region of the projection are either all labeled with A's or all labeled with B's.

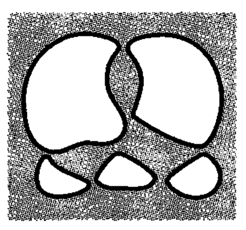


We shade the A regions gray while leaving the B regions white.

Span of Reduced Alternating Projection (Cont'd)

- What happens when we open all of the A-channels?

The gray waters flood the projection, leaving only a set of white islands in the middle of the gray lake.



Each circle is either the boundary of an island or the boundary of the lake (if the outermost region is white).

Thus, if W is the number of white regions in the original projection, including possibly the outer region, then $|S| = W$.

So the highest power of A in the term of this particular state is

$$a(S) - b(S) + 2(|S| - 1) = n + 2(W - 1).$$

Claim: Every other state has a highest power that is lower than this.

Span of Reduced Alternating Projection (Cont'd)

- Any other state has some B-splits.

  We show that if we have a state S_1 and we go to a state S_2 by changing one A-split to a B-split, the highest power cannot go up.

  Since, going from S_1 to S_2, we decrease the number of A-splits by one and increase the number of B splits by one.

  So the highest power in the term corresponding to:
  - S_1 is $a(S_1) - b(S_1) + 2(|S_1| - 1)$;
  - S_2 is $(a(S_1) - 1) - (b(S_1) + 1) + 2(|S_2| - 1)$.

  So the question remaining is how different $|S_2|$ can be from $|S_1|$.

  But S_2 differs from S_1 in only one split. So the number of circles either increases by one or it decreases by one.

  Hence, $|S_2| = |S_1| \pm 1$.

  Thus the highest power of the term corresponding to S_2 is

  $$a(S_1) - b(S_1) - 2 + 2((|S_1| \pm 1) - 1).$$

  This is certainly no greater than the highest term corresponding to S_1.

Span of Reduced Alternating Projection (Cont'd)

- Thus, any time we change an A-split to a B-split, we do not increase the highest power.

  Since every state can be obtained from the all-A-split state by a sequence of such changes, no other state has a higher power than the all-A-split state.

  Therefore, the highest power that occurs in the bracket polynomial for K is in fact $n + 2(W - 1)$.

  By a similar argument, we can also show that the lowest power that occurs is $-n - 2(D - 1)$, where D is the number of darkened regions.

  This lowest power occurs in the term of the polynomial coming from the all-B-split state.

Span of Reduced Alternating Projection (Conclusion)

- We reasoned that:
  - Highest power is $n+2(W-1)$;
  - Lowest power is $-n-2(D-1)$.

  We therefore have

$$\begin{aligned} \mathrm{span}(\langle K\rangle) &= \text{highest power} - \text{lowest power} \\ &= n+2(W-1)-(-n-2(D-1)) \\ &= 2n+2(W+D-2). \end{aligned}$$

  But $W+D$ is the total number of regions in the projection.

  The total number of regions is $n+2$.

  Hence,

$$\mathrm{span}(\langle K\rangle) = 2n+2n = 4n.$$

Reduced Alternating Projections and Number of Crossings

Theorem

Two reduced alternating projections of the same knot have the same number of crossings.

- Suppose the first projection has n crossings.

  By the lemma, the span of its bracket polynomial is $4n$.

  But the span of the bracket polynomial is an invariant of the knot.

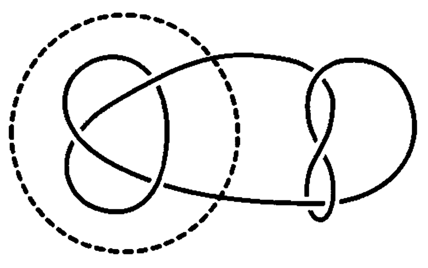
  So the span of the bracket polynomial of the second projection is $4n$.

  The lemma implies that the number of crossings in the second projection is also n.

  Hence, both projections have the same number of crossings.

Alternating Projections and Compositeness

- Menasco proved that if $K_1 \# K_2$ is an alternating knot, then it appears composite in any alternating projection.
- This means that there is a circle in the projection plane that intersects the knot twice, such that the factor knots on either side of the circle are themselves alternating.
- In particular, $K_1 \# K_2$ looks something like the following figure:



Crossing Number of a Composite Alternating Knot

Corollary

If $K_1 \# K_2$ is an alternating knot, then $c(K_1 \# K_2) = c(K_1) + c(K_2)$.

- Choose a reduced alternating projection for $K_1 \# K_2$.

  By Menasco's result, K_1 appears as part of this projection.

  Hence, we have a reduced alternating projection of K_1.

  By the Kauffman, Thistlethwaite, Murasugi Theorem, the least number of crossings for K_1 is the number appearing in this picture.

  Since K_2 is alternating, by the same result, its least number of crossings is the number appearing in this picture.

  But $K_1 \# K_2$ is alternating.

  So its least number of crossings also occurs in this picture.

  We conclude that $c(K_1 \# K_2) = c(K_1) + c(K_2)$.

Subsection 3

The Alexander and HOMFLY Polynomials

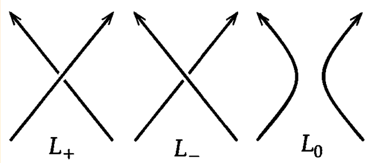
The Alexander Polynomial Δ

- The first rule is that the trivial knot has trivial polynomial equal to 1.

$$\text{Rule 1:} \qquad \Delta(\bigcirc) = 1.$$

This holds true for any projection of the trivial knot, not just the usual one.

- For the second rule: We take three projections of links L_+, L_- and L_0, such that they are identical except in the region depicted in the figure.



Then the polynomials of these three links are related through our second rule:

$$\text{Rule 2:} \qquad \Delta(L_+) - \Delta(L_-) + (t^{1/2} - t^{-1/2})\Delta(L_0) = 0.$$

Invariance of the Alexander Polynomial

- The two rules are enough to ensure that the Alexander polynomial is an invariant for knots and links.
- In particular, the following hold:
  - If we are given a projection of a knot, we can compute the Alexander polynomial of the knot in that projection;
  - We will get the same answer as in any projection.
- So we do not need to keep the projections frozen throughout the calculation, as we had to do with the bracket polynomial.

Example

- We compute the Alexander polynomial of the trefoil knot.
- Treating the trefoil knot as L_+, with the circled crossing as the one to which Rule 2 is to be applied, we obtain

  $\Delta($$) - \Delta($$) + (t^{1/2} - t^{-1/2})\Delta($$) = 0.$

- Now, we have

  $\Delta($$) = \Delta(\bigcirc) = 1.$

- Moreover,

  $\Delta($$) - \Delta($$) + (t^{1/2} - t^{-1/2})\Delta($$) = 0.$

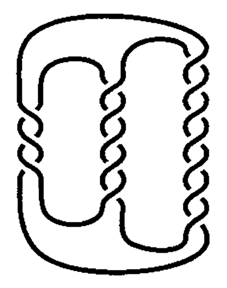
- The Alexander polynomial of any splittable link is 0, $\Delta($$) = 0.$
- Therefore, $\Delta($$) = -t^{1/2} + t^{-1/2}.$
- So we get

  $\Delta($$) = (t^{1/2} - t^{-1/2})^2 + 1 = t - 1 + t^{-1}.$

Alexander Polynomial and the Trivial Knot

- Unlike the Jones polynomial, there are known examples of nontrivial knots with Alexander polynomial equal to 1.
- So one of the disadvantages of the Alexander polynomial is that it cannot distinguish all knots from the trivial knot.

  Example: The $(-3, 5, 7)$-pretzel knot pictured below has Alexander polynomial 1.



The Computation Takes Finitely Many Steps

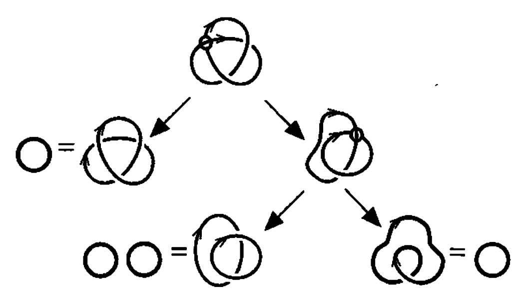
- When we discussed unknotting number, we proved that any projection can be turned into a projection of a trivial link by changing some subset of the crossings.
- Suppose we have a knot or link for which we would like to compute the Alexander polynomial.
  - Given a particular projection, we could choose a crossing, such that it is one of the crossings that we would like to change in order to turn the projection into a trivial projection.
  - Letting the original projection correspond to either L_+ or L_-, we can use Rule 2 in order to obtain the polynomial of our original link in terms of:
    - The polynomial of a link with a projection with one fewer crossing;
    - The polynomial of a link with a projection that is one crossing closer to the trivial projection.

Iterating this procedure allows us to obtain the polynomial of the original link in terms of the polynomials of a set of trivial links.

The Resolving Tree

- This process of repeatedly choosing a crossing, and then applying Rule 2 to reduce the process to two simpler links, yields a tree of links called the **resolving tree**.
  - At the top is our original link.
  - At the bottom, we find all of the trivial links that result from repeatedly applying Rule 2.

Example: The resolving tree for the trefoil knot:



The Depth of a Link

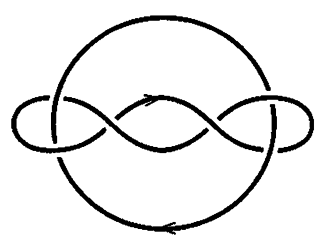
- Define the **depth** of a resolving tree to be the number of levels of links in the tree, not including the initial level at the top.

  Example: The resolving tree shown for the trefoil has depth two.
- Define the **depth of a link** L to be the minimal depth for any resolving tree for that link.
- The depth of a link is an invariant for links that measures the complexity of the calculation of the Alexander polynomial.
- The following facts are known:
  - The only links of depth zero are the trivial links.
  - Bleiler and Scharlemann:
    - A knot of depth one is always a trivial knot;
    - The links of depth one are all Hopf links, possibly with a few extra disentangled trivial components added in.
  - Scharlemann and Thompson: The links of depth two have also been classified.

The HOMFLY Polynomial

- The **HOMFLY polynomial** is a polynomial with two variables m and ℓ that generalizes both the Jones polynomial and the Alexander polynomial.

  Example: The oriented link in the figure



has HOMFLY polynomial

$$P = (-\ell^3 - \ell^5)m^{-1} + (2\ell^3 - \ell^5 - \ell^7)m + (-\ell^3 + \ell^5)m^3.$$

Rule 1 for the HOMFLY Polynomial

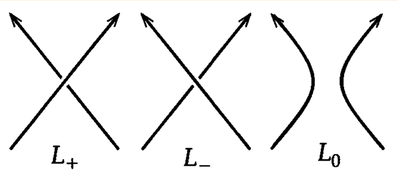
- We look at the rules used to calculate the HOMFLY polynomial.
- The first rule is

$$\text{Rule 1:} \qquad P(\bigcirc) = 1.$$

- The unknot has polynomial 1.
- As with the Alexander polynomial, this holds true for any projection of the unknot.

Rule 2 for the HOMFLY Polynomial

- The second rule: Consider three oriented links that are identical except in the region appearing the figure.

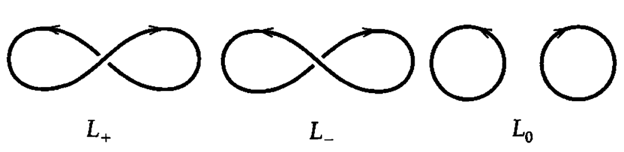


Rule 2: $\ell P(L_+) + \ell^{-1} P(L_-) + mP(L_0) = 0.$

- Notice the similarity of this rule to:
  - Rule 2 for the Alexander polynomial;
  - The relation satisfied by the Jones polynomial in the first section.

Example

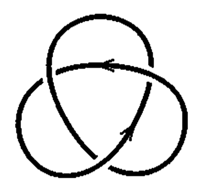
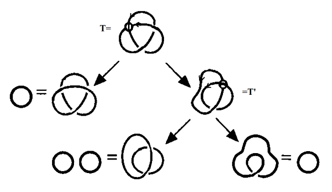
- We use the rules to calculate the HOMFLY polynomials for some links.
- The three links shown are identical except at the one crossing.



- Thus, they form a triple of links L_+, L_- and L_0.
- Hence, we have that $\ell P(L_+) + \ell^{-1} P(L_-) + mP(L_0) = 0$.
- Both L_+ and L_- are simply slightly twisted pictures of the unknot.
- Hence $P(L_+) = P(L_-) = 1$.
- Therefore, $mP(L_0) = -(\ell + \ell^{-1})$.
- Thus, we have shown that $P(L_0) = -m^{-1}(\ell + \ell^{-1})$.

Example

- Determine the polynomial of the trefoil in the figure.

- We have

$$\ell^{-1}P(T) + \ell + mP(T') = 0$$
$$\ell(-m^{-1}(\ell + \ell^{-1})) + \ell^{-1}P(T') + m = 0.$$

- The second gives $P(T') = m^{-1}\ell^3 + m^{-1}\ell - m\ell$.
- Plugging into the first, we get

$$\ell^{-1}P(T) = -\ell - m(m^{-1}\ell^3 + m^{-1}\ell - m\ell)$$
$$P(T) = -2\ell^2 - \ell^4 + m^2\ell^2.$$

HOMFLY Polynomial and Orientation

Claim: The HOMFLY polynomial of a knot is identical to the HOMFLY polynomial of the same knot, but with the opposite orientation.

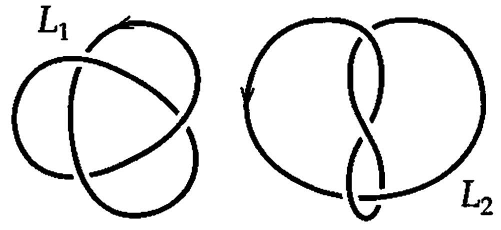
- This shows that we need not distinguish between orientations when we are discussing the HOMFLY polynomial of a knot.
- If we are dealing with a link, however, changing some but not all of the orientations on the components can have an effect on the polynomial.

Computation of the HOMFLY Polynomial

- We can always compute the HOMFLY polynomial of a link.
- As with the Alexander polynomial, all that we need is a resolving tree in order to do the calculation.
- However, the calculation can be very slow, since a link with c crossings would eventually reduce to 2^c links, none of which have any crossings.

HOMFLY Polynomial and Split Union

- Suppose $L_1 \cup L_2$ is the **split union** of the two links L_1 and L_2.
- This is the link obtained by moving L_1 over near L_2, but not overlapping them or linking them in any way.



- L_1 and L_2 can be separated by a sphere, so $L_1 \cup L_2$ is splittable.
- Then
$$P(L_1 \cup L_2) = -(\ell + \ell^{-1})m^{-1}P(L_1)P(L_2).$$

Example

- Consider the trivial knot on two components.
- It is the split union of two unknots L_1 and L_2.
- Using the split union rule for the HOMFLY polynomial, we get

$$\begin{aligned} P(L_1 \cup L_2) &= -(\ell + \ell^{-1})m^{-1}P(L_1)P(L_2) \\ &= -m^{-1}(\ell + \ell^{-1}). \end{aligned}$$

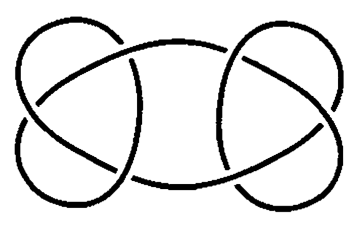
- This coincides with the expression obtained previously.

HOMFLY Polynomial and Composition

- A second interesting property of the HOMFLY polynomial is the following:
$$P(L_1 \# L_2) = P(L_1)P(L_2).$$
- So the polynomial of the composition of two links is simply the product of the polynomials of the factor links.

Example

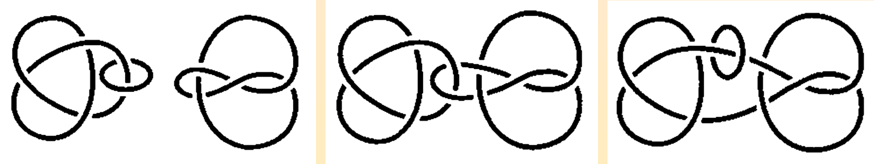
- Consider the composite of two trefoils L.



- We showed that the HOMFLY polynomial of a trefoil is
$$-2\ell^2 - \ell^4 + \ell^2 m^2.$$
- So we get
$$P(L) = (-2\ell^2 - \ell^4 + \ell^2 m^2)^2.$$

HOMFLY Polynomials of Composites

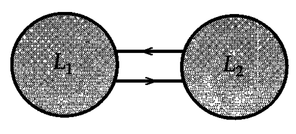
- We did not specify how to take the composition of a link.
- That is, we did not say which component of the first link should be connected up to which component of the second link.
- In fact, it does not matter.
- All (possibly distinct) composite links will have the same polynomial.



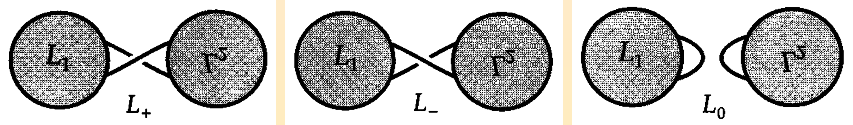
- This is our first example of links that are certainly distinct, but that cannot be distinguished by the HOMFLY polynomial.

Formula for $L_1 \# L_2$ from Formula for $L_1 \cup L_2$

- The composite link $L_1 \# L_2$ has a projection that appears as in the figure on the right.



Without cutting the strands to L_1, we flip that part of the projection corresponding to L_2 in two different ways, to get the two links L_+ and L_-.



Note that both of these projections are still projections of $L_1 \# L_2$.

In addition, L_0 is simply the disjoint union $L_1 \cup L_2$.

Formula for $L_1 \# L_2$ from Formula for $L_1 \cup L_2$ (Cont'd)

- The second rule for calculation of the P polynomial then says

$$\ell P(L_1 \# L_2) + \ell^{-1} P(L_1 \# L_2) + mP(L_l \cup L_2) = 0.$$

But we know that

$$P(L_1 \cup L_2) = -(\ell + \ell^{-1}) m^{-1} P(L_1) P(L_2).$$

Hence we have

$$\begin{gathered} \ell P(L_1 \# L_2) + \ell^{-1} P(L_1 \# L_2) + m(-(\ell + \ell^{-1}) m^{-1} P(L_1) P(L_2)) = 0 \\ (\ell + \ell^{-1}) P(L_1 \# L_2) + (-(\ell + \ell^{-1}) P(L_1) P(L_2)) = 0 \\ P(L_1 \# L_2) = P(L_1) P(L_2). \end{gathered}$$

- So the polynomials of the knots behave exactly as the integers do.

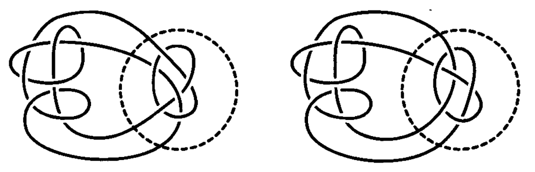
  The polynomial of a composite knot factors into the polynomials of all of its factor knots.

HOMFLY Polynomial and Distinguishability of Links

- The HOMFLY polynomial is better than either the Jones polynomial or the Alexander polynomial at telling apart knots and links, since we will see that both of those are simply special cases of this polynomial.
- But we have already seen examples of links that it will not distinguish, particularly composite variants of two links.
- The HOMFLY polynomial is not even for knots what is called a **complete invariant**, i.e., it cannot distinguish all knots.

Example

- A pair of mutant knots will always have the same HOMFLY polynomial.



- Mutants are troublesome in general.
- They cannot be distinguished by hyperbolic volume.
- They cannot be distinguished by genus either.

The HOMFLY Polynomial and the Jones Polynomial

- Take the HOMFLY Polynomial.
- Let $i = \sqrt{-1}$ be the imaginary unit.
- Perform the replacements:

$$\ell \leftarrow it^{-1} \quad \text{and} \quad m \leftarrow i(t^{-1/2} - t^{1/2}).$$

- Recall Rule 2 for the HOMFLY polynomial

$$\ell P(L_+) + \ell^{-1} P(L_-) + mP(L_0) = 0.$$

- It gives

$$it^{-1} P(L_+) + (it^{-1})^{-1} P(L_-) + i(t^{-1/2} - t^{1/2}) P(L_0) = 0.$$

- Upon multiplication by $-i$, we get

$$t^{-1} P(L_+) - tP(L_-) + (t^{-1/2} - t^{1/2}) P(L_0) = 0.$$

- This is the rule proven previously for the Jones polynomial.

Example

- We have seen that the HOMFLY polynomial of the trefoil knot is

$$P(K) = -2\ell^2 - \ell^4 + \ell^2 m^2.$$

- Substituting for m and ℓ, we have that

$$\begin{aligned} V(K) &= -2(it^{-1})^2 - (it^{-1})^4 + (it^{-1})^2(t^{-1/2} - t^{1/2})^2 \\ &= 2t^{-2} - t^{-4} - t^{-2}(t^{-1} - 2 + t) \\ &= t^{-4} - t^{-3} + t^{-1}. \end{aligned}$$

- This is exactly the Jones polynomial for the trefoil.

The HOMFLY Polynomial and the Alexander Polynomial

- In a similar way, take the HOMFLY Polynomial.
- Perform the replacements:

$$\ell \leftarrow i \quad \text{and} \quad m \leftarrow i(t^{1/2} - t^{-1/2}).$$

- Then one obtains the Alexander polynomial.
- This can be proven by showing that the resulting polynomial obeys the rules for the Alexander polynomial.
- Thus, the HOMFLY polynomial carries the information of both the Jones and the Alexander polynomials within it.
- So it is a more powerful invariant than either the Jones polynomial or the Alexander polynomial.

Subsection 4

Amphichirality

Amphichiral Knots

- Recall that an **amphichiral knot** is a knot that is ambient isotopic to its mirror image.
- That is to say, a knot is amphichiral if it can be deformed through space to the knot obtained by changing every crossing in the projection of the knot to the opposite crossing.
- We also insist that an orientation on the knot is taken to the corresponding orientation on the mirror image of the knot under the ambient isotopy.
- We denote by K^* the mirror image of K.

Amphichiral Knots and the Bracket Polynomial

- We show that the bracket polynomial of K^* is just the bracket polynomial of K where the variable A is replaced by A^{-1}.
- Consider K and K^* and fix a crossing of K and its mirror image in K^*.
- Denote by $K_{)(}$, $K_{\asymp}$, $K^*_{)(}$ and $K^*_{\asymp}$ the knots with one less crossing obtained by opening up the fixed crossing in K and K^*, respectively, in the way suggested by the notation.
- Then, denoting by $B(K)(A) := \langle K \rangle$, we have:

$$\begin{aligned} B(K)(A) &= AB(K_{)(})(A) + A^{-1}B(K_{\asymp})(A), \\ B(K^*)(A) &= AB(K^*_{\asymp})(A) + A^{-1}B(K^*_{)(})(A). \end{aligned}$$

- Now we compute, using induction,

$$\begin{aligned} B(K)(A^{-1}) &= AB(K_{\asymp})(A^{-1}) + A^{-1}B(K_{)(})(A^{-1}) \\ &= AB(K^*_{\asymp})(A) + A^{-1}B(K^*_{)(})(A) \\ &= B(K^*)(A). \end{aligned}$$

Amphichiral Knots and the X Polynomial

- We show that the X polynomial of K^* is just the X polynomial of K where the variable A is replaced by A^{-1}.
- Keep the same notation as in the preceding slide.
- Note that $w(K^*) = -w(K)$.
- So we have

$$\begin{aligned} X(K^*)(A) &= (-A^3)^{-w(K^*)} B(K^*)(A) \\ &= (-A^3)^{w(K)} B(K)(A^{-1}) \\ &= (-(A^{-1})^3)^{-w(K)} B(K)(A^{-1}) \\ &= X(K)(A^{-1}). \end{aligned}$$

Amphichiral Knots and Palindromic X Polynomial

- If K is an amphichiral knot, then K is in fact the same knot as K^*, since they are simply in distinct projections.
- Hence, it must be the case that $X_K(A) = X_{K^*}(A)$, where $X_K(A)$ means the X polynomial of K with variable A.
- By the preceding claim, $X_K(A) = X_{K^*}(A^{-1})$.
- Thus, if K is an amphichiral knot, it must be that

$$X_K(A) = X_{K*}(A^{-1}) = X_K(A^{-1}).$$

- Hence the polynomial of an amphichiral knot must be **palindromic**, i.e., the coefficients must be the same backwards or forwards, where we list *all* of the coefficients, including all the zeros.

Example: The Figure-Eight Knot

- We showed that the figure-eight knot was amphichiral utilizing the Reidemeister moves.
- Therefore its polynomial should be palindromic.
- Its polynomial is
$$A^8 - A^4 + 1 - A^{-4} + A^{-8}.$$
- We see that, replacing every A by an A^{-1} gives us the same polynomial back again.
- So the polynomial is indeed palindromic.

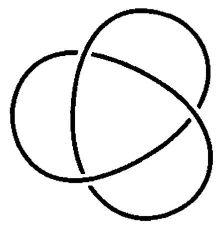
Example: The Trefoil Knot

- The trefoil knot has polynomial $A^4 + A^{12} - A^{16}$.
- This polynomial is not palindromic.
- If we replace every A by an A^{-1}, we get
$$A^{-4} + A^{-12} - A^{-16}.$$
- This is not the same polynomial.
- This shows that the trefoil knot is not amphichiral, i.e., the trefoil knot is distinct from its mirror image.

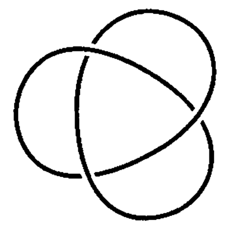
Example: The Trefoil Knot (Cont'd)

- All along we have been discussing the trefoil as if it were a single knot.
- But it turns out it is actually two knots.

The **left-hand trefoil**:



The **right-hand trefoil**:



Amphichiral Alternating Knots

- We have already seen that if K is an alternating knot in a reduced alternating projection of n crossings, then

$$\begin{aligned} \max\deg\langle K\rangle &= n+2(W-1), \\ \min\deg\langle K\rangle &= -n-2(D-1). \end{aligned}$$

- Since $X(K)=(-A)^{-3w(K)}\langle K\rangle$, we have that

$$\begin{aligned} \max\deg X(K) &= n+2(W-1)-3w(K), \\ \min\deg X(K) &= -n-2(D-1)-3w(K). \end{aligned}$$

Amphichiral Alternating Knots (Cont'd)

- But we have already seen that for an amphichiral knot
$$X_K(A) = X_K(A^{-1}).$$
- So
$$\max \deg X(K) = -\min \deg X(K).$$
- Thus,
$$\begin{aligned} n + 2(W-1) - 3w(K) &= -(-n - 2(D-1) - 3w(K)) \\ n + 2W - 2 - 3w(K) &= n + 2D - 2 + 3w(K) \\ 6w(K) &= 2(W-D) \\ 3w(K) &= W - D. \end{aligned}$$
- For an amphichiral alternating knot, the difference in the number of white regions and darkened regions in any reduced alternating projection is exactly three times the writhe.

Amphichiral Knots and the HOMFLY Polynomial

- Since the information of the Jones and X polynomials is embedded within the HOMFLY polynomial, it should also provide us with information about amphichirality.
- In fact, the following holds.

  Claim: The HOMFLY polynomial of K^* is obtained by replacing each ℓ in the HOMFLY polynomial of K with an ℓ^{-1}.

  Corollary: The left-hand and right-hand trefoil knots are distinct.

Remarks

- Although surprisingly effective at determining the amphichirality of knots, the HOMFLY polynomial is not infallible.
- Consider the knot K in the figure.
- It has HOMFLY polynomial

$$P(K) = (-2\ell^{-2} - 3 - 2\ell^2) + (\ell^{-2} + 4 + \ell^2)m^2 - m^4.$$

- Note that the polynomial is unchanged when every ℓ is replaced by an ℓ^{-1}.
- Hence, $P(K) = P(K^*)$.
- However, there exists a "signature" invariant coming out of algebraic topology that proves that K is not amphichiral.