Introduction to Lattices and Order

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1 A Brief Topological Detour

Topological Spaces

- A **topological space** $(X;\mathcal{T})$ consists of a set X and a family $\mathcal{T}$ of subsets of X, such that:
  - (T1) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
  - (T2) a finite intersection of members of T is in T;
  - (T3) an arbitrary union of members of T is in T.
- The family $\mathcal{T}$ is called a **topology** on X;
- The members of $\mathcal{T}$ are called **open sets**.
- We write X in place of $(X;\mathcal{T})$ when $\mathcal{T}$ is the only topology under consideration.

A Standard Example

- The standard topology $\mathcal{T}_{\mathbb{R}}$ on $\mathbb{R}$ consists of

$$\{U \subseteq \mathbb{R} : (\forall x \in U)(\exists \delta > 0)\ (x - \delta, x + \delta) \subseteq U\},$$

  where δ may depend on x.
- Equivalently, $\mathcal{T}_{\mathbb{R}}$ consists of those sets which can be expressed as unions of open intervals, together with $\varnothing$.
- The equation

$$\bigcap_{n \geq 1} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$$

  exhibits an intersection of open sets which is not open.

Closed Sets, Clopen Sets, Connected Spaces

- Given a topological space $(X;\mathcal{T})$, we define a subset of X to be **closed** if it belongs to $\Gamma(X) := \{X \backslash U : U \in \mathcal{T}\}$.
- The family $\Gamma(X)$ is closed under arbitrary intersections and finite unions.
- For every $A \subseteq X$, there exists a smallest closed set $\overline{A}$ containing A, called the **closure** of A.
- Sets which are both open and closed are called **clopen**.
- A topological space is **connected** if its only clopen subsets are the whole space and the empty set.
  - Many of the topological spaces encountered in elementary analysis and geometry are connected;
  - By contrast, the spaces that will be used in our representation theory have an ample supply of clopen sets.

Subspaces, Bases and Subbases

- Let $(X;\mathcal{T})$ be a topological space. Any subset Y of X inherits a **subspace topology** given by

$$\mathcal{T}_Y := \{V \subseteq Y : V = U \cap Y, \text{ for some } U \subseteq \mathcal{T}\}.$$

- To create a topology on a set X in which a specified family $\mathcal{S}$ of subsets of X, including $\varnothing$ and X, are open sets, we do the following:
  - If $\mathcal{S}$ is already closed under finite intersections, we define $\mathcal{T}$ to be those sets which are unions of sets in $\mathcal{S}$.
    Then $\mathcal{T}$ satisfies (T1), (T2) and (T3) and $\mathcal{S}$ is said to be a **basis** for $\mathcal{T}$.
  - In general,
    - we first form $\mathcal{B}$, the family of sets which are finite intersections of members of $\mathcal{S}$,
    - and then define $\mathcal{T}$ to be all arbitrary unions of members of $\mathcal{B}$.

  In this case $\mathcal{S}$ is called a **subbasis** for $\mathcal{T}$.

Continuity and Homeomorphisms

- Let $(X;\mathcal{T})$ and $(X';\mathcal{T}')$ be topological spaces and $f: X \to X'$ a map. Then the following conditions are equivalent:
  - (i) $f^{-1}(U)$ is open in X whenever U is open in X';
  - (i)′ $f^{-1}(V)$ is closed in X whenever V is closed in X';
  - (ii) $f^{-1}(U)$ is open in X, for every $U \in \mathcal{S}$, where $\mathcal{S}$ is a given basis or subbasis for $\mathcal{T}'$.

  When f satisfies any of these conditions it is said to be **continuous**.

  Example: If $(X;\mathcal{T}) = (X';\mathcal{T}') = (\mathbb{R};\mathcal{T}_{\mathbb{R}})$ and $\mathcal{S}$ is the family of subintervals (a,b) (for $-\infty < a < b < \infty$), plus $\mathbb{R}$ and $\varnothing$, (ii) is just a restatement of the ϵ-δ definition of continuity.
- The map $f: X \to X'$ is said to be a **homeomorphism** if f is bijective and both f and f^{-1} are continuous.

  Homeomorphisms are topology's isomorphisms.

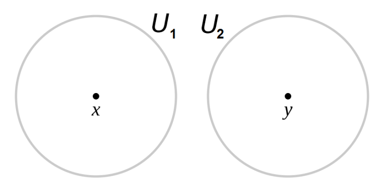
Hausdorff Spaces

- There is a hierarchy of separation conditions, one of which is the *Hausdorff condition*.

  The topological space $(X;\mathcal{T})$ is said to be **Hausdorff** if, given $x, y \in X$, with $x \neq y$, there exist open sets U_1, U_2, such that

$$x \in U_1, \quad y \in U_2, \quad \text{and} \quad U_1 \cap U_2 = \varnothing.$$

- Mnemonically, X is Hausdorff if distinct points can be "housed off" in disjoint open sets.



Singletons are Closed in Hausdorff Spaces

Lemma

Let $(X;\mathcal{T})$ be a Hausdorff space. Then, for all $x \in X$, $\{x\}$ is closed.

- For every $y \in X$, with $y \neq x$, there exist open sets U_y, V_y, such that

$$x \in U_y, \quad y \in V_y \quad \text{and} \quad U_y \cap V_y = \varnothing.$$

Set $V = \bigcup_{x \neq y \in X} V_y$. Since V is the union of open sets, it is open.
To show that $\{x\}$ is closed, it suffices to show that $\{x\} = X \backslash V$.

  - $x \in \bigcap_{y \neq x} U_y \subseteq \bigcap_{y \neq x}(X \backslash V_y) = X \backslash \bigcup_{y \neq x} V_y = X \backslash V$.
  - If $y \neq x$, then $y \in V_y$, whence $y \in V$. So $y \notin X \backslash V$.

We conclude that $X \backslash V = \{x\}$.

Compactness

- Let $(X;\mathcal{T})$ be a topological space and let $\mathcal{U} := \{U_i\}_{i\in I} \subseteq \mathcal{T}$.

  The family $\mathcal{U}$ is called an **open cover** of $Y \subseteq X$ if $Y \subseteq \bigcup_{i\in I} U_i$.

  A finite subset of $\mathcal{U}$ whose union still contains Y is a **finite subcover**.
- We say Y is **compact** if every open cover of Y has a finite subcover.

  Example: The famous **Heine-Borel Theorem** states that a subset of $\mathbb{R}$ is closed and bounded if and only if it is compact.
- Compactness is a fundamental topological concept and may be regarded as a substitute for finiteness.

  It frequently compensates for the restriction to finite intersections in axiom (T2) by allowing arbitrary families of open sets to be reduced to finite families.
- All the spaces we use in our representation theory are compact.

Compact Hausdorff Spaces and Continuous Maps

- We present two basic results about compact Hausdorff spaces.
- The first relates compactness and closedness and shows that continuous maps behave well:

Lemma

Let $(X; \mathcal{T})$ be a compact Hausdorff space.

(i) A subset C of X is compact if and only if it is closed.

(ii) Let $f: X \to X'$ be a continuous map, where $(X'; \mathcal{T}')$ is any topological space.

   (a) $f(X)$ is a compact subset of X'.
   (b) If $(X'; \mathcal{T}')$ is Hausdorff and $f: X \to X'$ is bijective, then f is a homeomorphism.

Proof of (i)

- Suppose, first, that C is a compact subset of a Hausdorff space X.

  Let x be some fixed point in $X \backslash C$. We show that there exists an open set U_x containing x and with $U_x \subseteq X \backslash C$. This will show that $X \backslash C$ is open, whence C is closed.

  For each $c \in C$, there exist disjoint open sets U_c, V_c, with $x \in U_c$, $c \in V_c$. The collection $\{V_c : c \in C\}$ is an open cover of C. By compactness there is a finite subcover, say $\{V_{c_1}, \ldots, C_{c_r}\}$. Let $U_x = \bigcap_{i=1}^r U_{c_i}$. As a finite intersection of open sets, U_x is open in X. Clearly $x \in U_x$, since $x \in U_{c_i}$, for all i.

  We finally show that $U_x \subseteq X \backslash C$.

  For each $i = 1, \ldots, r$, we have $U_x \subseteq U_{c_i}$. So $U_x \cap V_{c_i} \subseteq U_{c_i} \cap V_{c_i} = \varnothing$. Hence $U_x \cap C \subseteq U_x \cap (\bigcup_{i=1}^r V_{c_i}) = \bigcup_{i=1}^r (U_x \cap V_{c_i}) = \varnothing$. So $U_x \subseteq X \backslash C$, as required.

Proof of (i) (Converse)

- Suppose C is a closed subset of a compact space X.

  Let $\mathcal{U}$ be any cover of C by sets open in X. Since C is closed in X, $X \backslash C$ is open in X. If we add it to $\mathcal{U}$ we get an open cover of X. But X is compact, so there is a finite subcover, say $\{U_1, \ldots, U_r\}$. This certainly covers C since it covers all of X.
  - If $X \backslash C$ is one of these U_i then we may throw it out and the remaining $r-1$ sets will still cover C.
  - If $X \backslash C$ is not one of the U_i then we leave $\{U_1, \ldots, U_r\}$ alone.

  In either case we get a finite subcover of $\mathcal{U}$ for C. So C is compact.

Proof of (ii)(a)

- Suppose X is compact and $f : X \to X'$ is continuous.

  Let $\mathcal{U}$ is an open cover of $f(X)$. Since f is continuous, $f^{-1}(U)$ is open in X for every $U \in \mathcal{U}$. The family

  $$\{f^{-1}(U) : U \in \mathcal{U}\}$$

  covers X since $\mathcal{U}$ covers $f(X)$. Hence by compactness of X, there is a finite subcover, say

  $$\{f^{-1}(U_1), \ldots, f^{-1}(U_r)\}.$$

  Then $\{U_1, \ldots, U_r\}$ is a finite subcover of $f(X)$. We conclude that $f(X)$ is compact in X'.

Proof of (ii)(b)

- Let $f : X \to X'$ be a bijective continuous map from a compact Hausdorff space X onto a Hausdorff space X'.
  Since f is bijective, we know that there is an inverse function $f^{-1} : X' \to X$. We just have to prove that f^{-1} is continuous.
  Suppose that V is closed in X. It is enough to show that $(f^{-1})^{-1}(V)$ is closed in X'. Note that $(f^{-1})^{-1}(V) = f(V)$. Then we have:

  $$\begin{array}{lll} V \text{ closed in } X & \Rightarrow & V \text{ is compact} \\ & & \text{closed subset of a compact space is compact} \\ & \Rightarrow & f(V) \text{ is compact} \\ & & \text{the continuous image of a} \\ & & \text{compact space is compact} \\ & \Rightarrow & f(V) \text{ is closed in } X'. \\ & & \text{a compact subspace of a} \\ & & \text{Hausdorff space is closed} \end{array}$$

  So $(f^{-1})^{-1}(V) = f(V)$ is closed in X'.

Strengthening the Hausdorff Separability Axiom

- The following lemma strengthens the Hausdorff condition, which is recaptured by taking the closed sets to be singletons:

Lemma

Let $(X;\mathcal{T})$ be a compact Hausdorff space.

(i) Let V be a closed subset of X and $x \notin V$. Then there exist disjoint open sets W_1 and W_2, such that $x \in W_1$ and $V \subseteq W_2$.

(ii) Let V_1 and V_2 be disjoint closed subsets of X. Then there exist disjoint open sets U_1 and U_2, such that $V_i \subseteq U_i$, for $i = 1, 2$.

- For $y \in V$, by the Hausdorff axiom, there are open sets $U_1^{x,y}$ and $U_2^{x,y}$ containing x and y, respectively. Then $\mathcal{U}_2 := \{U_2^{x,y} : y \in V\}$ is an open cover of V. By the preceding lemma, V is compact. Take a finite subcover $\{U_2^{x,y_j} : j = 1, \ldots, n\}$. Let $U_1^x := \bigcap_{1 \le j \le n} U_1^{x,y_j}$ and $U_2^x := \bigcup_{1 \le j \le n} U_2^{x,y_j}$. Each U_2^{x,y_j} does not intersect the corresponding U_1^{x,y_j}. So, it is disjoint from U_1^x. Hence, U_1^x and U_2^x are disjoint.

Strengthening the Hausdorff Separability Axiom (Cont'd)

- Also U_1^x and U_2^x are open. These sets contain x and V, respectively. Take $W_1 := U_1^x$ and $W_2 := U_2^x$ to obtain (i).
- For (ii) we repeat the process, taking $V := V_2$ and letting x vary over V_1. The family $U_1 := \{U_1^x : x \in V_1\}$ is an open cover of the compact set V_1. Take a finite subcover $\{U_1^{x_i} : i = 1, \ldots, m\}$. Define $U_1 := \bigcup_{1 \leq i \leq m} U_1^{x_i}$ and $U_2 := \bigcap_{1 \leq i \leq m} U_2^{x_i}$.

Finiteness of a Compact Hausdorff Space

- The next lemma enables us to fit our finite representation theory into the general theory:

Lemma

Let $(X;\mathcal{T})$ be a compact Hausdorff space. Then the following conditions are equivalent:

(i) X is finite;

(ii) Every subset of X is open (that is, $\mathcal{T}$ is discrete);

(iii) Every subset of X is clopen.

(ii)$\Leftrightarrow$(iii): trivial.

(iii)$\Rightarrow$(i): Consider the open cover $\{\{x\} : x \in X\}$.

(i)$\Rightarrow$(ii): Finally, assume (i). For $\varnothing \neq Y \subseteq X$, the set $X \setminus Y$ is a finite union of singleton sets, which are closed because X is Hausdorff. So $X \setminus Y$ is closed. Hence Y is open.

Alexander's Subbasis Lemma

- We prove Alexander's Subbasis Lemma using (BPI).

Alexander's Subbasis Lemma

Let $(X;\mathcal{T})$ be a topological space and $\mathcal{S}$ a subbasis for $\mathcal{T}$. Then X is compact if every open cover of X by members of $\mathcal{S}$ has a finite subcover.

- Let $\mathcal{B}$ be the basis formed from all finite intersections of members of $\mathcal{S}$. To prove X is compact it is enough to show that every open cover $\mathcal{U}$ of X by sets in $\mathcal{B}$ has a finite subcover. Suppose this is false and let $\mathcal{U}$ be an open cover of X by sets in $\mathcal{B}$, which does not have a finite subcover. Define J to be the ideal in $\mathcal{P}(X)$ generated by $\mathcal{U}$. So a typical element of J is a subset of $U_1 \cup \cdots \cup U_k$, for some $U_1,\ldots,U_k \in \mathcal{U}$. J is proper, by our hypothesis. Use (BPI) to construct a prime ideal I of $\mathcal{P}(X)$ containing J.

Alexander's Subbasis Lemma (Cont'd)

- For each $x \in X$, there exists $U(x) \in \mathcal{U}$, with $x \in U(x)$. Each $U(x)$ is a finite intersection of members of $\mathcal{S}$ and belongs to I since $\mathcal{U} \subseteq I$. As I is prime we may assume that $U(x)$ itself lies in $\mathcal{S}$. Let

$$\mathcal{V} := \{U(x) : x \in X\}.$$

Then $\mathcal{V}$ is an open cover of X by members of $\mathcal{S}$. So, by assumption, $\mathcal{V}$ has a finite subcover. But then $X = U(x_1) \cup \cdots \cup U(x_n)$, for some finite subset $\{x_1, \ldots, x_n\}$ of X. Therefore $X \in I$, a contradiction.