Introduction to Lattices and Order

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LSSU Math 400

Subsection 1

Ordered sets

Partial Orders

Definition (Order or Partial Order)

Let P be a set. An **order** (or **partial order**) on P is a binary relation $\leq$ on P, such that, for all $x, y, z \in P$,

(i) $x \leq x$; (**Reflexivity**)

(ii) $x \leq y$ and $y \leq x$ imply $x = y$; (**Antisymmetry**)

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$; (**Transitivity**)

- A set P equipped with an order relation $\leq$ is said to be an **ordered set** (or **partially ordered set** or **poset**).
- Usually we say simply "P is an ordered set", but where it is necessary to specify the order relation overtly we write $\langle P; \leq \rangle$.
- On any set, $=$ is an order, the **discrete order**.

Additional Terminology and Notation

- A relation $\leq$ on a set P which is reflexive and transitive but not necessarily antisymmetric is called a **quasi-order** or a **pre-order**.
- An order relation $\leq$ on P gives rise to a relation $<$ of strict inequality: $x < y$ in P if and only if $x \leq y$ and $x \neq y$.
- Other notation associated with $\leq$:
  - We use $x \leq y$ and $y \geq x$ interchangeably;
  - We write $x \nleq y$ to mean "$x \leq y$ is false".
- We use $\parallel$ to denote non-comparability:

$$x \parallel y \quad \text{if} \quad x \nleq y \text{ and } y \nleq x.$$

- Let P be an ordered set and let Q be a subset of P. Then Q *inherits an order relation from* P; Given $x, y \in Q$, $x \leq y$ in Q if and only if $x \leq y$ in P. We say in these circumstances that Q has the **induced order**, or the **order inherited from** P.

Chains and Antichains

Definition

Let P be an ordered set.

- P is a **chain** if, for all $x, y \in P$, either $x \leq y$ or $y \leq x$ (i.e., if any two elements of P are comparable).

  Chains are called **linearly ordered sets** and **totally ordered sets**.
- The ordered set P is an **antichain** if $x \leq y$ in P implies $x = y$.

- Clearly, with the induced order, any subset of a chain (an antichain) is a chain (an antichain).
- Let P be the n-element set $\{0, 1, \ldots, n-1\}$.
  - We write $\boldsymbol{n}$ to denote the chain obtained by giving P the order in which $0 < 1 < \cdots < n-1$;
  - We write $\overline{\boldsymbol{n}}$ for P regarded as an antichain.
- Any set S may be converted into an antichain S by giving S the discrete order.

Order-Isomorphisms

Definition

Two ordered sets, P and Q, are (**order-**)**isomorphic**, written $P \cong Q$, if there exists a map φ from P onto Q, such that $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q. Then φ is called an **order-isomorphism**.

- Such a map φ is necessarily bijective (that is, one-to-one and onto):

$$\begin{array}{rcl} \varphi(x) = \varphi(y) & \Leftrightarrow & \varphi(x) \leq \varphi(y) \ \& \ \varphi(y) \leq \varphi(x) \\ & & \text{(Reflexivity \& Antisymmetry in } Q\text{)} \\ & \Leftrightarrow & x \leq y \ \& \ y \leq x \quad \text{(Order Isomorphism)} \\ & \Leftrightarrow & x = y. \quad \text{(Refl. \& Antisym. in } P\text{)} \end{array}$$

- Not every bijective map between ordered sets is an order-isomorphism: Consider $P = Q = \mathbf{2}$ and define φ by $\varphi(0) = 1, \varphi(1) = 0$.
- Being a bijection, an order-isomorphism $\varphi : P \to Q$ has a well defined inverse, $\varphi^{-1} : Q \to P$.

  It is easily seen that this is also an order-isomorphism.

Example: Number Systems

- The set $\mathbb{R}$ of real numbers, with its usual order, forms a chain.
- Each of $\mathbb{N}$ (the natural numbers $\{1, 2, 3, \ldots\}$), $\mathbb{Z}$ (the integers) and $\mathbb{Q}$ (the rational numbers) also has a natural order making it a chain.
- In each case this order relation is compatible with the arithmetic structure in the sense that the sum and product of two elements strictly greater than zero is also greater than zero.
- We denote the set $\mathbb{N} \cup \{0\} (= \{0, 1, 2, \ldots\})$ by $\mathbb{N}_0$. Endowed with the order in which $0 < 1 < 2 < \ldots$, the set $\mathbb{N}_0$ becomes the chain known in set theory as ω.

  It is order-isomorphic to $\mathbb{N}$: The successor function $n \mapsto n^+ := n + 1$ from $\mathbb{N}_0$ to $\mathbb{N}$ is an order-isomorphism.
- A different order on $\mathbb{N}_0$ is defined as follows: Write $m \preccurlyeq n$ if and only if there exists $k \in \mathbb{N}_0$, such that $km = n$ (that is, m divides n).

  Then $\preccurlyeq$ is an order relation. Of course, $\langle \mathbb{N}_0; \preccurlyeq \rangle$ is not a chain.

Families of Sets

Definition

Let X be any set. The **powerset** $\mathcal{P}(X)$ consists of all subsets of X. It is ordered by set inclusion: For $A, B \in \mathcal{P}(X)$, we define $A \leq B$ if and only if $A \subseteq B$.

- Any subset of $\mathcal{P}(X)$ inherits the inclusion order.
  - Such a family of sets might be specified set-theoretically; E.g., it might consist of all finite subsets of an infinite set X.
  - More commonly, families of sets arise where X carries some additional structure; E.g., X might have an algebraic structure:
    - The set of all subgroups of a group G (denoted $\mathrm{Sub}G$), and the set of all normal subgroups of G (denoted $\mathcal{N}\text{-}\mathrm{Sub}G$);
  - Families of sets also occur in other mathematical contexts; E.g.:
    - For a topological space $(X; \mathcal{T})$, we may consider the families of open, closed, and clopen (meaning simultaneously closed and open) subsets of X as ordered sets under inclusion.
    - For an ordered ser P, consider the family $\mathcal{O}(P)$ of its *down-sets*, revisited later.

The Poset of Predicates on a Set X

- The ordered set $\langle\mathcal{P}(X);\subseteq\rangle$ manifests itself in a different form, as the set of predicates on X.
- A **predicate** is a statement taking value T (true) or value F (false).
  More precisely, a **predicate on** X is a function from X to $\{\mathrm{T},\mathrm{F}\}$.
  Example: The map $p:\mathbb{R}\to\{\mathrm{T},\mathrm{F}\}$, given by $p(x)=\begin{cases}\mathrm{T}, & \text{if } x\geq 0\\ \mathrm{F}, & \text{if } x<0\end{cases}$ is a predicate on $\mathbb{R}$.
- We write $\mathbb{P}(X)$ for the set of predicates on X and order it by implication: for $p,q\in\mathbb{P}(X)$,
  $$p\Rightarrow q \text{ if and only if } \{x\in X:p(x)=\mathrm{T}\}\subseteq\{x\in X:q(x)=\mathrm{T}\}.$$
- Define a map $\varphi:\mathbb{P}(X)\to\mathcal{P}(X)$ by $\varphi(p)=\{x\in X:p(x)=\mathrm{T}\}$.
  Then φ is an order-isomorphism between $\langle\mathbb{P}(X);\Rightarrow\rangle$ and $\langle\mathcal{P}(X);\subseteq\rangle$.

Subsection 2

Diagrams: The Art of Drawing Ordered Sets

The Covering Relation

Definition

Let P be an ordered set and let $x, y \in P$. We say x is **covered by** y (or y **covers** x), and write $x \lessdot y$ or $y \gtrdot x$, if $x < y$ and $x \leq z < y$ implies $z = x$. The latter condition is demanding that there be no element z of P with $x < z < y$.

- Observe that, *if P is finite*, $x < y$ if and only if there exists a finite sequence of covering relations $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y$.

  Thus, *in the finite case*, the order relation determines, and is determined by, the covering relation.

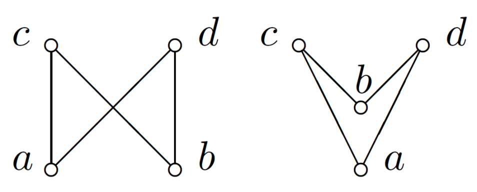
  Example:
  - In the chain $\mathbb{N}$, we have $m \lessdot n$ if and only if $n = m + 1$.
  - In $\mathbb{R}$, there are no pairs x, y, such that $x \lessdot y$.
  - In $\mathcal{P}(X)$, we have $A \lessdot B$ if and only if $B = A \cup \{b\}$, for some $b \in X \backslash A$.

Hasse Diagrams of Posets

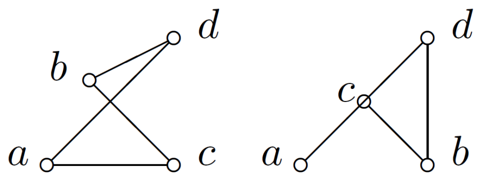
- Let P be a finite ordered set. We can represent P by a configuration of circles (representing the elements of P) and interconnecting lines (indicating the covering relation):
  (1) To each point $x \in P$, associate a point $p(x)$ of the Euclidean plane $\mathbb{R}^2$, depicted by a small circle with center at $p(x)$.
  (2) For each covering pair $x \lessdot y$ in P, take a line segment $\ell(x,y)$ joining the circle at $p(x)$ to the circle at $p(y)$.
  (3) Carry out (1) and (2) in such a way that
    (a) if $x \lessdot y$, then $p(x)$ is "lower" than $p(y)$ (that is, in standard Cartesian coordinates, has a strictly smaller second coordinate);
    (b) the circle at $p(z)$ does not intersect the line segment $\ell(x,y)$ if $z \neq x$ and $z \neq y$.
- It is easily proved by induction on the size, $|P|$, of P that Condition (3) can be achieved.
- A configuration satisfying Conditions (1)-(3) is called a **diagram** (or **Hasse diagram**) of P.

Using Hasse Diagrams

- A diagram may be used to define a finite ordered set.
- The figure shows two alternative diagrams for the ordered set $P = \{a, b, c, d\}$ in which $a < c$, $a < d$, $b < c$ and $b < d$.

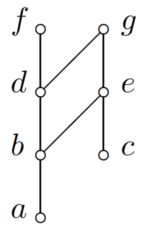


- The figure has drawings that are not legitimate diagrams for P; in the first, Condition (3)(a) is violated, in the second, Condition (3)(b) is.

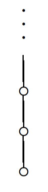


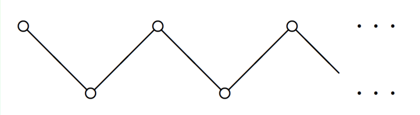
Mining Information from Hasse Diagrams

- It is easy to tell from a diagram whether one element of an ordered set is less than another: $x < y$ if and only if there is a sequence of connected line segments moving upwards from x to y.
  In the ordered set on the right, $e \parallel f$ and $a < g$.



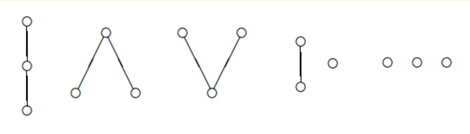
- It is not possible to represent the whole of an infinite ordered set by a diagram;
  But if its structure is sufficiently regular it can often be suggested diagrammatically:



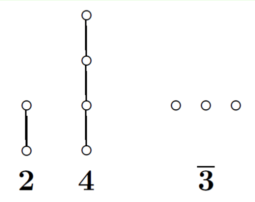


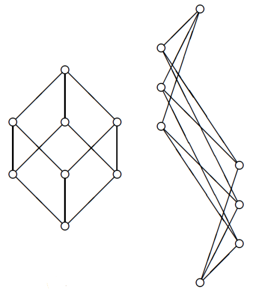
Some Examples

- All possible ordered sets with three elements:



- The posets $\mathbf{2}, \mathbf{4}$ and $\overline{\mathbf{3}}$.

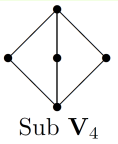


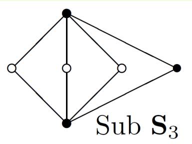


- Two different drawings of $\mathcal{P}(\{1,2,3\})$ (known as the **cube**).

Some More Examples

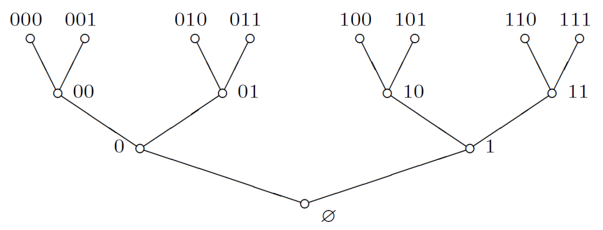
- A less Diagrams for Sub$G$ for $G = V_4$, the Klein 4-group, and $G = S_3$, the symmetric group on 3 letters.





  In each case the subset $\mathcal{N}$-Sub$G$ is shaded.

- The subset of Σ^*, consisting of strings of length not more than 3.



Order Isomorphisms and Coverings

Lemma

Let P and Q be finite ordered sets and let $\varphi : P \to Q$ be a bijective map. Then the following are equivalent:

(i) φ is an order-isomorphism;

(ii) $x < y$ in P if and only if $\varphi(x) < \varphi(y)$ in Q;

(iii) $x \lessdot y$ in P if and only if $\varphi(x) \lessdot \varphi(y)$ in Q.

- The equivalence of (i) and (ii) is immediate from the definitions. Now assume (ii) holds. Take $x \lessdot y$ in P. Then $x < y$, so $\varphi(x) < \varphi(y)$ in Q. Suppose there exists $w \in Q$, with $\varphi(x) < w < \varphi(y)$. Since φ is onto, there exists $u \in P$, such that $w = \varphi(u)$. By (ii), $x < u < y$, a contradiction. Hence $\varphi(x) \lessdot \varphi(y)$.
  Conversely, assume $\varphi(x) \lessdot \varphi(y)$. Then $\varphi(x) < \varphi(y)$. Hence, $x < y$. Suppose, there exists $w \in P$, such that $x < w < y$. Then $\varphi(x) < \varphi(w) < \varphi(y)$. This contradicts $\varphi(x) \lessdot \varphi(y)$. Hence (iii) holds.

Order Isomorphisms and Coverings (Cont'd)

- Now assume (iii). Let $x < y$ in P. Then there exist elements $x_1, x_2, \ldots, x_{n-1} \in P$, such that $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y$. By (iii), $\varphi(x) = \varphi(x_0) \lessdot \varphi(x_1) \lessdot \cdots \lessdot \varphi(x_n) = \varphi(y)$. Hence $\varphi(x) < \varphi(y)$.

  Suppose, conversely, that $\varphi(x) < \varphi(y)$ in Q.

  Then there exist $w_1, w_2, \ldots, w_{n-1} \in Q$, such that

  $$\varphi(x) \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{n-1} \lessdot \varphi(y).$$

  By surjectivity, there exist $x_1, x_2, \ldots, x_{n-1} \in P$, such that $\varphi(x_i) = w_i$, $i = 1, \ldots, n-1$. Hence, we get

  $$\varphi(x) \lessdot \varphi(x_1) \lessdot \varphi(x_2) \lessdot \cdots \lessdot \varphi(x_{n-1}) \lessdot \varphi(y).$$

  Thus, by Condition (iii), $x \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot x_{n-1} \lessdot y$ and, therefore, $x < y$, by transitivity.

Order Isomorphisms and Hasse Diagrams

Proposition

Two finite ordered sets P and Q are order-isomorphic if and only if they can be drawn with identical diagrams.

- Assume there exists an order-isomorphism $\varphi : P \to Q$. To show that the same diagram represents both P and Q, note that the diagram is determined by the covering relation and invoke the lemma.

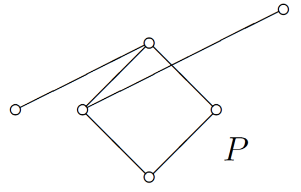
  Conversely, assume P and Q can both be represented by the same diagram, D. Then there exist bijective maps f and g from P and Q onto the points of D. The composite map $\varphi = g^{-1} \circ f$ is bijective and satisfies Condition (iii) of the lemma. So it is an order-isomorphism.

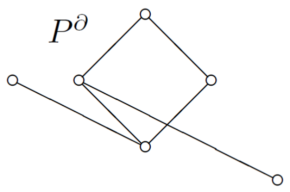
Subsection 3

Constructing and De-Constructing Ordered Sets

The Dual Partial Ordering

- Given any ordered set P we can form a new ordered set P^∂ (the **dual** of P) by defining $x \leq y$ to hold in P^∂ if and only if $y \leq x$ holds in P.
- For P finite, we obtain a diagram for P^∂ simply by "turning upside down" a diagram for P.





- To each statement about the ordered set P there corresponds a statement about P^∂.

  Example: In P, there exists a unique element covering just three other elements. In P^∂ there exists a unique element covered by just three other elements.

The Duality Principle

- In general, given any statement Φ about ordered sets, we obtain the **dual statement** Φ^∂ by replacing each occurrence of $\leq$ by $\geq$:

The Duality Principle

Given a statement Φ about ordered sets which is true in all ordered sets, the dual statement Φ^∂ is also true in all ordered sets.

Bottom and Top

- Let P be an ordered set. We say P has a **bottom element** if there exists $\bot \in P$ (called **bottom**) with the property that $\bot \leq x$, for all $x \in P$.
- Dually, P has a **top element** if there exists $\top \in P$, such that $x \leq \top$, for all $x \in P$.
- By the Duality Principle the true statement "$\bot$ is unique when it exists" (by the antisymmetry of $\leq$) has as its dual version the statement "$\top$ is unique when it exists".

  Example:
  - In $\langle \mathcal{P}(X); \subseteq \rangle$, we have $\bot = \varnothing$ and $\top = X$.
  - A finite chain always has bottom and top elements, but an infinite chain need not have.

    For example, the chain $\mathbb{N}$ has bottom element 1, but no top, while the chain $\mathbb{Z}$ of integers possesses neither bottom nor top.
  - Bottom and top do not exist in any antichain with more than one element.

Lifting

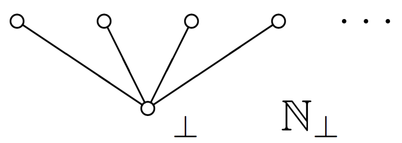
- Lack of a bottom element can be easily remedied by adding one.
- Given an ordered set P (with or without $\perp$), we form $P_\perp$ (called P "**lifted**") as follows.

  Take an element $\mathbf{0} \notin P$ and define $\leq$ on $P_\perp = P \cup \{\mathbf{0}\}$ by

  $$x \leq y \quad \text{if and only if} \quad x = \mathbf{0} \text{ or } x \leq y \text{ in } P.$$

- Any set S gives rise to an ordered set with $\perp$, as follows.
  - Order S by making it an antichain, $\overline{S}$;
  - Then form $\overline{S}_\perp$.

  Ordered sets obtained in this way are called **flat**.
- In applications it is likely that $S \subseteq \mathbb{R}$ and, for simplicity, we write $S_\perp$ instead of the more correct $\overline{S}_\perp$.



Maximal and Minimal Elements

- Let P be an ordered set and let $Q \subseteq P$.
- $a \in Q$ is a **maximal element of** Q if $a \leq x$ and $x \in Q$ imply $a = x$.

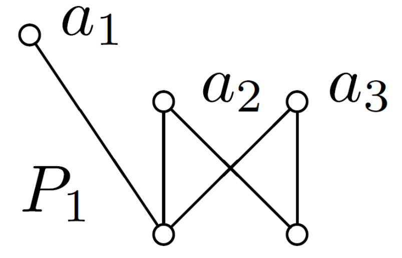
  We denote the set of maximal elements of Q by $\text{Max}Q$.
- If Q (with the order inherited from P) has a top element, $\top_Q$, then $\text{Max}Q = \{\top_Q\}$.

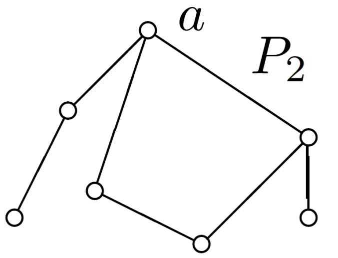
  In this case $\top_Q$ is called the **greatest** (or **maximum**) element of Q, and we write $\top_Q = \max Q$.
- $b \in Q$ is a **minimal element of** Q if $x \leq b$ and $x \in Q$ imply $b = x$.

  We denote the set of maximal elements of Q by $\text{Min}Q$.
- If Q (with the order inherited from P) has a bottom element, $\bot_Q$, then $\text{Min}Q = \{\bot_Q\}$.

  In this case $\bot_Q$ is called the **least** (or **minimum**) element of Q, and we write $\bot_Q = \min Q$.

Maximal and Minimal Elements: Examples





P_1 has maximal elements a_1, a_2, a_3, but no greatest element.

a is the greatest element of P_2.

Remarks on the Existence of Maximal Elements

- Let P be a finite ordered set.

  Then any non-empty subset of P has at least one maximal element.

  Moreover, for each $x \in P$, there exists $y \in \mathrm{Max}P$, with $x \leq y$.
- In general a subset Q of an ordered set P may have many maximal elements, just one, or none.
  - A subset of the chain $\mathbb{N}$ has a maximal element if and only if it is finite and non-empty.
  - In the subset Q of $\mathcal{P}(\mathbb{N})$ consisting of all subsets of $\mathbb{N}$ other than $\mathbb{N}$ itself, there is no top element, but $\mathbb{N} \setminus \{n\} \in \mathrm{Max}Q$ for each $n \in \mathbb{N}$.
  - The subset of $\mathcal{P}(\mathbb{N})$ consisting of all finite subsets of $\mathbb{N}$ has no maximal elements.
- An important set-theorists' tool, known as **Zorn's Lemma**, discussed in a later set, guarantees the existence of maximal elements, under suitable conditions.

Disjoint Union of Ordered Sets

- In subsequent "sum constructions" we require that the sets being joined are disjoint.
- Suppose that P and Q are (disjoint) ordered sets.
  The **disjoint union** $P \dot{\cup} Q$ of P and Q is the ordered set formed by defining $x \leq y$ in $P \dot{\cup} Q$ if and only if either:
  - $x, y \in P$ and $x \leq y$ in P or
  - $x, y \in Q$ and $x \leq y$ in Q.
- A diagram for $P \dot{\cup} Q$ is formed by placing side by side diagrams for P and Q.

The Linear Sum of Ordered Sets

- Again let P and Q be (disjoint) ordered sets.

  The **linear sum** $P \oplus Q$ is defined by taking the following order relation on $P \cup Q$:

$$x \leq y \quad \text{if and only if} \quad \begin{cases} & x, y \in P \text{ and } x \leq y \text{ in } P \\ \text{or} & x, y \in Q \text{ and } x \leq y \text{ in } Q \\ \text{or} & x \in P \text{ and } y \in Q. \end{cases}$$

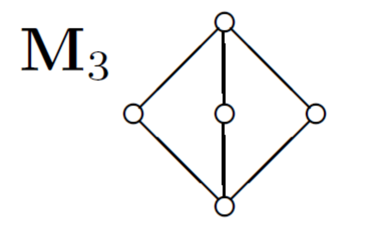
- A diagram for $P \oplus Q$ (when P and Q are finite) is obtained by placing a diagram for P directly below a diagram for Q and then adding a line segment from each maximal element of P to each minimal element of Q.
- The lifting construction is a special case of a linear sum: $P_\perp = \mathbf{1} \oplus P$.

  Similarly, $P \oplus \mathbf{1}$ represents P with a (new) top element added.

Remarks of the Sum Operations

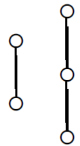
- Each of the operations $\dot\cup$ and $\oplus$ is associative:
  For (pairwise disjoint) ordered sets P, Q and R,
  - $P \dot\cup (Q \dot\cup R) = (P \dot\cup Q) \dot\cup R$;
  - $P \oplus (Q \oplus R) = (P \oplus Q) \oplus R$.

  This allows us to write iterated disjoint unions and linear sums unambiguously without brackets.
- We denote by $\boldsymbol{M}_n$ the sum $\mathbf{1} \oplus \overline{\boldsymbol{n}} \oplus \mathbf{1}$.



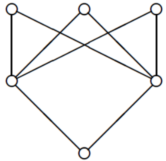
Examples of the Sum Operations

- Examples of sums:

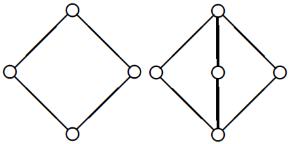


$\mathbf{2} \mathbin{\dot\cup} \mathbf{3}$

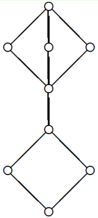
$\mathbf{2} \oplus \mathbf{3} \cong \mathbf{5}$



$\overline{\mathbf{1}} \oplus \overline{\mathbf{2}} \oplus \overline{\mathbf{3}}$



$\mathbf{M}_2 \mathbin{\dot\cup} \mathbf{M}_3$



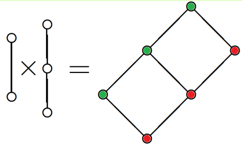
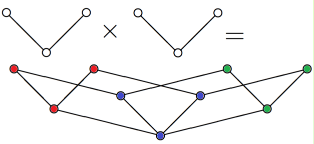
$\mathbf{M}_2 \oplus \mathbf{M}_3$

Products

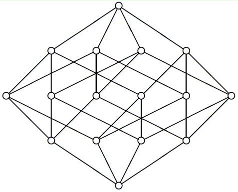
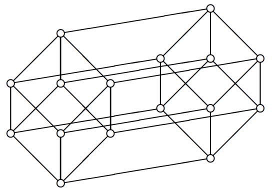
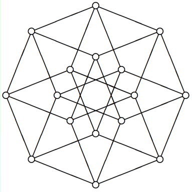
- Let $P_1, \ldots, P_n$ be ordered sets.
  The **Cartesian product** $P_1 \times \cdots \times P_n$ can be made into an ordered set by imposing the **coordinate wise order** defined by
  $$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \quad \Longleftrightarrow \quad (\forall i)\ x_i \leq y_i \text{ in } P_i.$$
- Given an ordered set P, the notation P^n is used as shorthand for the n-fold product $P \times \cdots \times P$.
- There is another way to order the product of ordered sets P and Q:
  Define the **lexicographic order** by
  $$(x_1, x_2) \leq (y_1, y_2) \quad \text{if} \quad x_1 < y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 \leq y_2).$$
  By iteration a lexicographic order can be defined on any finite product of ordered sets.
- Unless otherwise stated we shall always equip a product with the coordinate wise order.

Drawing a Product

- A product $P \times Q$ is drawn by replacing each point of a diagram of P by a copy of a diagram for Q, and connecting "corresponding" points (assuming the rules for diagram-drawing are obeyed).

- The four-dimensional hypercube $\mathbf{2}^4$ drawn in various ways:

- Note that $\mathbf{2}^3 \cong \mathcal{P}(\{1,2,3\})$ (they have the same diagram).

Order Isomorphism Between $\mathcal{P}(X)$ and $\mathbf{2}^n$

Proposition

Let $X = \{1, 2, \ldots, n\}$ and define $\varphi : \mathcal{P}(X) \to \mathbf{2}^n$ by $\varphi(A) = (\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i = \begin{cases} 1, & \text{if } i \in A \\ 0, & \text{if } i \notin A \end{cases}$. Then φ is an order-isomorphism.

- Given $A, B \in \mathcal{P}(X)$, let $\varphi(A) = (\varepsilon_1, \ldots, \varepsilon_n)$ and $\varphi(B) = (\delta_1, \ldots, \delta_n)$. Then

$$\begin{array}{rcl} A \subseteq B & \iff & (\forall i)\ i \in A \text{ implies } i \in B \\ & \iff & (\forall i)\ \varepsilon_i = 1 \text{ implies } \delta_i = 1 \\ & \iff & (\forall i)\ \varepsilon_i \leq \delta_i \\ & \iff & \varphi(A) \leq \varphi(B) \text{ in } \mathbf{2}^n. \end{array}$$

  To show φ is onto, take $x = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbf{2}^n$. Then $x = \varphi(A)$, where $A = \{i : \varepsilon_i = 1\}$, so φ is onto.

Subsection 4

Down-Sets and Up-Sets

Down-Sets and Up-Sets

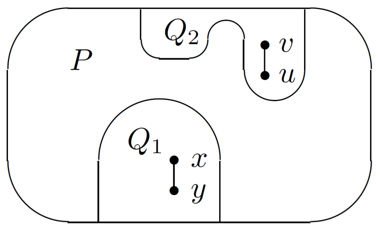
Definition

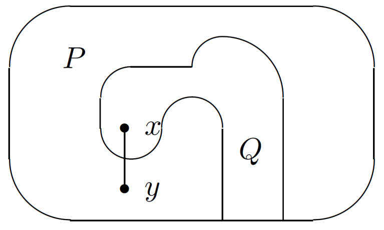
Let P be an ordered set and $Q \subseteq P$.

(i) Q is a **down-set** (alternative terms include **decreasing set** and **order ideal**) if, whenever $x \in Q$, $y \in P$ and $y \leq x$, we have $y \in Q$.

(ii) Dually, Q is an **up-set** (alternative terms are **increasing set** and **order filter**) if, whenever $x \in Q$, $y \in P$ and $x \leq y$, we have $y \in Q$.

- We think of a down-set as one which is "closed under going down".





- Q_1 is a down-set and Q_2 an upset. Q is not a down-set.

Generated Down-Sets and Up-Sets

- Given an arbitrary subset Q of P, we define

$$\downarrow Q = \{y \in P : (\exists x \in Q)\ y \le x\} \quad \text{and} \quad \uparrow Q = \{y \in P : (\exists x \in Q)\ y \ge x\}.$$

These are read "down Q" and "up Q".

- Given an arbitrary element $x \in P$, we define

$$\downarrow x = \{y \in P : y \le x\} \quad \text{and} \quad \uparrow x = \{y \in P : y \ge x\}.$$

These are read "down x" and "up x".

- Down-sets (up-sets) of the form $\downarrow x$ ($\uparrow x$) are called **principal**.

Properties of Generated Down-Sets and Up-Sets

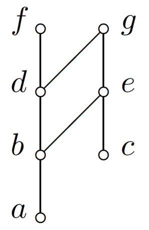
- It is easily checked that:
  - $\downarrow Q$ is the smallest down-set containing Q.
  - Q is a down-set if and only if $Q = \downarrow Q$.
- Dually, it is also easily checked that:
  - $\uparrow Q$ is the smallest up-set containing Q.
  - Q is an up-set if and only if $Q = \uparrow Q$.
- We show the first property, which consists of three parts:
  - $\downarrow Q$ is a downset: Suppose $x \in \downarrow Q$ and $y \leq x$. By definition, there exists $z \in Q$, such that $x \leq z$. By transitivity, $y \leq z$. By definition again, $y \in \downarrow Q$.
  - $Q \subseteq \downarrow Q$: Suppose $x \in Q$. By reflexivity, $x \leq x$. By definition, $x \in \downarrow Q$.
  - If P is a downset containing Q, then $\downarrow Q \subseteq P$: Suppose P is a downset containing Q. Let $x \in \downarrow Q$. By definition, there exists $y \in Q$, such that $x \leq y$. Since P contains Q, $y \in P$. Since P is a downset, $x \in P$. Therefore, $\downarrow Q \subseteq P$.

The Ordered Set $\mathcal{O}(P)$ of Down-Sets

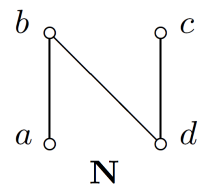
- The family of all down-sets of P is denoted by $\mathcal{O}(P)$.
- $\mathcal{O}(P)$ is itself an ordered set, under the inclusion order.
- When P is finite, every non-empty down-set Q of P is expressible in the form
$$\bigcup_{i=1}^{k} \downarrow x_i,$$
where $\{x_1, \ldots, x_k\} = \mathrm{Max} Q$ is an antichain.
- This provides a recipe for finding $\mathcal{O}(P)$, though one which is practical only when P is small.

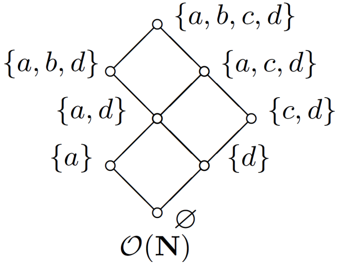
Examples I

(1) Consider the set on the right. The sets $\{c\}$, $\{a,b,c,d,e\}$ and $\{a,b,d,f\}$ are all down-sets. The set $\{b,d,e\}$ is not a down-set. We have $\downarrow\{b,d,e\} = \{a,b,c,d,e\}$. The set $\{e,f,g\}$ is an up-set. The set $\{a,b,d,f\}$ is not an up-set.



(2) $\mathcal{O}(P)$ in a simple case:





Examples II

(3) If P is an antichain, then $\mathcal{O}(P) = \mathcal{P}(P)$.

(4) If P is the chain $\boldsymbol{n}$, then $\mathcal{O}(P)$ consists of all the sets $\downarrow x$ for $x \in P$, together with the empty set.

Hence $\mathcal{O}(P)$ is an $(n+1)$-element chain.

(5) If P is the chain $\mathbb{Q}$ of rational numbers, then $\mathcal{O}(P)$ contains the empty set, $\mathbb{Q}$ itself and all sets $\downarrow x$ (for $x \in \mathbb{Q}$).

There are other sets in $\mathcal{O}(P)$ too, e.g., $\downarrow x \backslash \{x\}$ (for $x \in \mathbb{Q}$) and $\{y \in \mathbb{Q} : y < a\}$ (for $a \in \mathbb{R} \backslash \mathbb{Q}$).

Order Relation and Down-Sets

Lemma

Let P be an ordered set and $x, y \in P$. Then the following are equivalent:

(i) $x \leq y$;

(ii) $\downarrow x \subseteq \downarrow y$;

(iii) $(\forall Q \in \mathcal{O}(P))\ y \in Q \Rightarrow x \in Q$.

(i)⇒(ii): Suppose $x \leq y$. Let $z \in \downarrow x$. By definition, $z \leq x$. By transitivity, $z \leq y$. Again by definition, $z \in \downarrow y$. Hence, $\downarrow x \subseteq \downarrow y$.

(ii)⇒(iii): Suppose $\downarrow x \subseteq \downarrow y$. Let $Q \in \mathcal{O}(P)$, such that $y \in Q$. Since $x \in \downarrow x$, we get, by hypothesis, $x \in \downarrow y$. Thus, by definition, $x \leq y$. Since $Q \in \mathcal{O}(P)$ and $y \in Q$, $x \in Q$. This proves the implication.

(iii)⇒(i): Suppose that, for every down-set Q, $y \in Q$ implies $x \in Q$. Take, in particular, $Q := \downarrow y \in \mathcal{O}(P)$. Since $y \in \downarrow y$, we get, by hypothesis, $x \in \downarrow y$. Hence, by definition, $x \leq y$.

$\mathcal{O}(P)$ and Duality

- Besides being related by duality, down-sets and up-sets are related by complementation:

  > Q is a down-set of P if and only if $P \backslash Q$ is an up-set of P (equivalently, a down-set of P^∂).

  Suppose Q is a down-set. Let $x \in P \backslash Q$ and $x \leq y$. Then $x \notin Q$. Thus, since Q is a down-set and $x \notin Q$, we get $y \notin Q$. So $y \in P \backslash Q$. Therefore, $P \backslash Q$ is an up-set.

  The converse can be shown similarly.

- For subsets A, B of P, we have $A \subseteq B$ if and only if $P \backslash A \supseteq P \backslash B$.

  It follows that

  $$\mathcal{O}(P)^\partial \cong \mathcal{O}(P^\partial),$$

  the order-isomorphism being the complementation map.

Poset of Down-Sets in Sums

Proposition

Let P be an ordered set. Then:

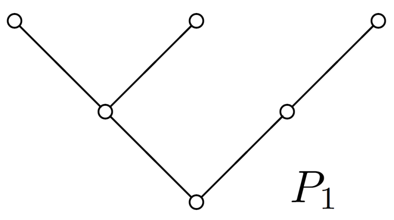
(i) $\mathcal{O}(P \oplus \mathbf{1}) \cong \mathcal{O}(P) \oplus \mathbf{1}$ and $\mathcal{O}(\mathbf{1} \oplus P) \cong \mathbf{1} \oplus \mathcal{O}(P)$;

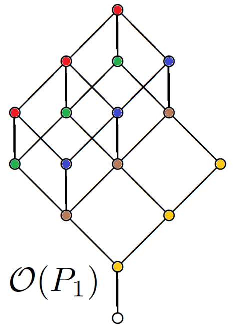
(ii) $\mathcal{O}(P_1 \dot\cup P_2) \cong \mathcal{O}(P_1) \times \mathcal{O}(P_2)$.

(i) The down-sets of $P \oplus \mathbf{1}$ are the down-sets of P together with $P \oplus \mathbf{1}$ itself. The down-sets of $\mathbf{1} \oplus P$ are the empty set and all down-sets of P with the least element of $\mathbf{1} \oplus P$ adjoined. The required isomorphisms are set up using these observations.

(ii) It can be verified that the map $U \mapsto (U \cap P_1, U \cap P_2)$ defines an order-isomorphism from $\mathcal{O}(P_1 \dot\cup P_2)$ to $\mathcal{O}(P_1) \times \mathcal{O}(P_2)$.

Example I



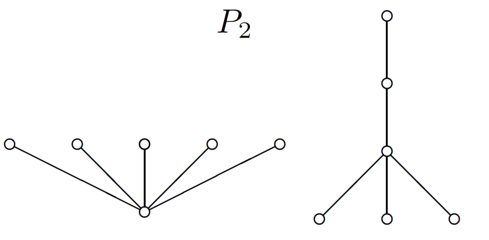


- We have

$$
\begin{aligned}
\mathcal{O}(P_1) &\cong \mathcal{O}(\mathbf{1} \oplus ((\mathbf{1} \oplus \bar{\mathbf{2}}) \,\dot{\cup}\, \mathbf{2})) \\
&\cong \mathbf{1} \oplus \mathcal{O}((\mathbf{1} \oplus \bar{\mathbf{2}}) \,\dot{\cup}\, \mathbf{2}) \\
&\cong \mathbf{1} \oplus (\mathcal{O}(\mathbf{1} \oplus \bar{\mathbf{2}}) \times \mathcal{O}(\mathbf{2})) \\
&\cong \mathbf{1} \oplus ((\mathbf{1} \oplus \mathcal{O}(\bar{\mathbf{2}})) \times \mathcal{O}(\mathbf{2})) \\
&\cong \mathbf{1} \oplus ((\mathbf{1} \oplus \mathbf{2}^2) \times \mathbf{3}).
\end{aligned}
$$

Example II

- Consider the partially ordered set P_2.



How many elements does $\mathcal{O}(P_2)$ have?

- We have $P_2 \cong (\mathbf{1} \oplus \overline{\mathbf{5}}) \mathbin{\dot\cup} (\overline{\mathbf{3}} \oplus \mathbf{3})$.

  Therefore, $\mathcal{O}(P_2) \cong (\mathbf{1} \oplus \mathbf{2}^5) \times (\mathbf{2}^3 \oplus \mathbf{3})$.

  Hence, its size is $|\mathcal{O}(P_2)| = (1 + 2^5) \times (2^3 + 3) = 363$.

Subsection 5

Maps Between Ordered Sets

Maps Between Ordered Sets

- We have already defined order-isomorphisms.
- We consider more general structure-preserving maps:

Definition

Let P and Q be ordered sets. A map $\varphi : P \to Q$ is said to be:

(i) **order-preserving** (or, alternatively, **monotone**) if

$$x \leq y \text{ in } P \text{ implies } \varphi(x) \leq \varphi(y) \text{ in } Q;$$

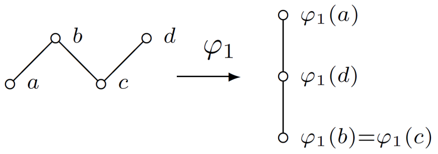
(ii) an **order-embedding** (and we write $\varphi : P \hookrightarrow Q$) if

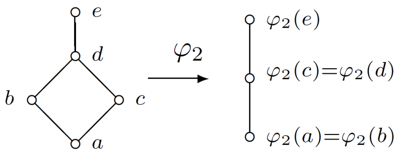
$$x \leq y \text{ in } P \text{ if and only if } \varphi(x) \leq \varphi(y) \text{ in } Q;$$

(iii) an **order-isomorphism** if it is an order-embedding which maps P onto Q.

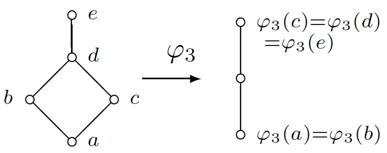
Examples

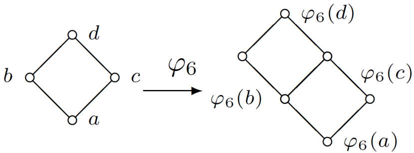
(1) The map φ_1 is not order-preserving. φ_2 is order-preserving, but not an order-embedding.





φ_3 is order-preserving, but not an order-embedding. φ_6 is an order embedding, but not an order-isomorphism.





(2) Let P be any ordered set. The map $x \mapsto {\downarrow}x$ sets up an order embedding from P into $\mathcal{O}(P)$.

Remarks on Maps Between Ordered Sets

(1) Let $\varphi : P \to Q$ and $\psi : Q \to R$ be order-preserving maps. Then the composite map $\psi \circ \varphi$, given by $(\psi \circ \varphi)(x) = \psi(\varphi(x))$, for $x \in P$, is order-preserving.

More generally the composite of a finite number of order-preserving maps is order-preserving, if it is defined.

(2) Let $\varphi : P \hookrightarrow Q$ and let $\varphi(P)$ (defined to be $\{\varphi(x) : x \in P\}$) be the image of φ. Then $\varphi(P) \cong P$. This justifies the use of the term embedding.

(3) An order-embedding is automatically a one-to-one map.

Suppose $\varphi : P \to Q$ is an order embedding. Let $x, y \in P$, such that $\varphi(x) = \varphi(y)$. Then we get $\varphi(x) \leq \varphi(y)$ and $\varphi(y) \leq \varphi(x)$. Since φ is an order embedding, $x \leq y$ and $y \leq x$. By antisymmetry, $x = y$. Therefore, φ is one-to-one.

Remarks on Maps Between Ordered Sets (Cont'd)

(4) Ordered sets P and Q are order-isomorphic if and only if there exist order-preserving maps $\varphi : P \to Q$ and $\psi : Q \to P$, such that $\varphi \circ \psi = \mathrm{id}_Q$ and $\psi \circ \varphi = \mathrm{id}_P$ (where $\mathrm{id}_S : S \to S$ denotes the **identity map on** S given by $\mathrm{id}_S(x) = x$, for all $x \in S$).

Suppose that P and Q are order isomorphic. Then, there exists a surjective order embedding $\varphi : P \to Q$. Consider $x' \in Q$. Since φ is surjective, there exists $x \in P$, such that $\varphi(x) = x'$. Since φ is an order embedding, it is one-to-one. Thus, there exists a unique $x \in P$, such that $\varphi(x) = x'$. Define $\psi : Q \to P$ by mapping $x' \in Q$ to the unique $\psi(x') = x \in P$, such that $\varphi(x) = x'$. We have $\varphi(\psi(x')) = \varphi(x) = x'$. We also have $\psi(\varphi(x)) = \psi(x') = x$.

Remarks on Maps Between Ordered Sets (Cont'd)

Assume, conversely, that there exist order-preserving maps $\varphi : P \to Q$ and $\psi : Q \to P$, such that $\varphi \circ \psi = \mathrm{id}_Q$ and $\psi \circ \varphi = \mathrm{id}_P$.

We show that φ is a surjective order embedding.

It is surjective because it has a two-sided inverse.

It is an order embedding since, for all $x, y \in P$:

- By the monotonicity of φ,

$$x \leq y \quad \text{implies} \quad \varphi(x) \leq \varphi(y);$$

- By the monotonicity of ψ,

$$\begin{array}{rcl} \varphi(x) \leq \varphi(y) & \Rightarrow & \psi(\varphi(x)) \leq \psi(\varphi(y)) \\ & \Rightarrow & x \leq y. \end{array}$$

Pointwise Ordered Sets of Maps

- Example: Consider the statement from elementary analysis

$$\sin x \leq |x|, \text{ on } \mathbb{R}.$$

  The order relation implicit here is the **pointwise order**:

  For functions $f, g : \mathbb{R} \to \mathbb{R}$, the relation

$$f \leq g \text{ means } f(x) \leq g(x), \text{ for all } x \in \mathbb{R}.$$

- Suppose X is any set and Y an ordered set. We may order the set Y^X of all maps from X to Y as follows:

  $f \leq g$ if and only if $f(x) \leq g(x)$ in Y, for all $x \in X$.
- When X is an n-element set, then Y^X is really just Y^n.
- Any subset Q of Y^X inherits the pointwise order.

Order Preserving Sets of Maps

- When both X and Y are ordered sets, we may take Q to be the set of all order-preserving maps from X to Y;

  This is a set with order inherited from that of Y^X and is denoted $Y^{\langle X \rangle}$.
- We sometimes write $(X \to Y)$ in place of Y^X and $\langle X \to Y \rangle$ in place of $Y^{\langle X \rangle}$.

  This alternative notation is needed because the notation Y^X and $Y^{\langle X \rangle}$ becomes unwieldy when X or Y is of the form $P_\perp$ or when higher-order functions are involved (i.e., functions which map functions to functions).

Example

- We have, for every ordered set X,
$$\langle X \to \mathbf{2} \rangle \cong \mathcal{O}(X)^\partial.$$
For every monotone map $f : X \to \mathbf{2}$, define $\varphi(f) = \{x \in X : f(x) = 0\}$.
  - $\varphi(f)$ is a down-set of X: Suppose $x \in \varphi(f)$ and $y \le x$. By definition of $\varphi(f)$, $f(x) = 0$. Since $f : X \to \mathbf{2}$ is order-preserving and $y \le x$, $f(y) = 0$. Hence, by definition of $\varphi(f)$, $y \in \varphi(f)$.
  - φ is an order-embedding:
$$\begin{array}{lll} f \le g & \text{iff} & f(x) \le g(x), \text{ all } x \in X \\ & \text{iff} & g(x) = 0 \text{ implies } f(x) = 0, \text{ all } x \in X \\ & \text{iff} & \varphi(g) \subseteq \varphi(f). \end{array}$$
  - φ is onto: Let $D \in \mathcal{O}(P)$. Define $f_D : X \to \mathbf{2}$, by setting $f_D(x) = \begin{cases} 0, & \text{if } x \in D \\ 1, & \text{if } x \notin D \end{cases}$. Then, $\varphi(f_D) = D$. We must show that f_D is monotone. Let $x, y \in X$, such that $x \le y$. Suppose $f_D(x) = 1$. Then $x \notin D$. Since $D \in \mathcal{O}(P)$ and $x \le y$, $y \notin D$. Hence, $f_D(y) = 1$. This shows that f_D is monotone.