Introduction to Lattices and Order

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1. Maximality Principles
   - Zorn's Lemma and the Axiom of Choice
   - Prime and Maximal Ideals
   - Power Set Algebras and Down-Set Lattices

Subsection 1

Zorn's Lemma and the Axiom of Choice

The Axiom of Choice and Maximality Axioms

- The **Axiom of Choice** may be stated as follows:

  (AC) Given a non-empty family $\mathcal{A} = \{A_i\}_{i \in I}$ of non-empty sets, there exists a **choice function for** $\mathcal{A}$, that is, a map

  $$f : I \to \bigcup_{i \in I} A_i, \text{ such that } (\forall i \in I)\ f(i) \in A_i.$$

- Alternately, we may take the following statement as a postulate:

  (ZL) Let P be a non-empty ordered set in which every nonempty chain has an upper bound. Then P has a maximal element.

- We shall also need the following three axioms concerning the existence of maximal elements:

  (ZL)′ Let $\mathcal{E}$ be a non-empty family of sets such that $\bigcup_{i \in I} A_i \in \mathcal{E}$ whenever $\{A_i\}_{i \in I}$ is a non-empty chain in $\langle \mathcal{E}; \subseteq \rangle$. Then $\mathcal{E}$ has a maximal element.

  (ZL)″ Let P be a CPO. Then P has a maximal element.

  (KL) Let P be an ordered set. Then every chain in P is contained in a maximal chain.

Equivalence of Maximality Axioms

- (ZL)′ is just the restriction of (ZL) to families of sets.
- We now show that the five assertions (AC), (ZL), (ZL)′, (ZL)″ and (KL) are all equivalent.
  - The implication (AC)⇒(ZL) is **Zorn's Lemma**.
    Some authors use Zorn's Lemma to mean the statement (ZL) instead.
  - Similarly, the implication (AC)⇒(KL) is **Kuratowski's Lemma**.

Theorem

The conditions (AC), (ZL), (ZL)′, (ZL)″ and (KL) are equivalent.

- We prove (AC)⇒(ZL)″⇒(KL)⇒(ZL)⇒(ZL)′⇒(AC).

Equivalence of Maximality Axioms: (AC)⇒(ZL)″

- (AC)⇒(ZL)″: Suppose (AC) holds and let P be a CPO, such that every element is not maximal. This says that for every $x \in P$, the set $A_x = \{y \in P : y > x\}$ is nonempty. By (AC), for every x, there exists $F(x) \in A_x$, i.e., $F(x) \in P$, such that $F(x) > x$. By the fixed-point theorem for CPO's, $F : P \to P$ has a fixed-point, i.e., there exists $x \in P$, such that $F(x) = x$, a contradiction. Therefore, P has a maximal element.
- (ZL)⇒(ZL)′ is trivial, since (ZL)′ is a restricted form of (ZL).

Equivalence of Maximality Axioms: (ZL)$''$ $\Rightarrow$(KL)

- (ZL)$''$ $\Rightarrow$(KL): Take an ordered set P and let $\mathcal{P}$ denote the family of all chains in P which contain a fixed chain C^0. Order this family of sets by inclusion.

  Claim: $\mathcal{P}$ is a CPO.

  It suffices to show that every chain in P has a least upper bound in P. Let $\mathcal{C} = \{C_i\}_{i \in I}$ be a chain in $\mathcal{P}$. If I is empty, then $\bigvee_{\mathcal{P}} \mathcal{C} = C^0$, since C^0 is the bottom of P. Now assume that I is non-empty. Let $C = \bigcup_{i \in I} C_i$. We claim that $C \in \mathcal{P}$, that is, C is a chain. Then, $\bigvee_{\mathcal{P}} \mathcal{C} = C$. Let $x, y \in C$. We are required to show that x and y are comparable. There exist $i, j \in I$, such that $x \in C_i$ and $y \in C_j$. Since $\mathcal{C}$ is a chain, we have $C_i \subseteq C_j$ or $C_j \subseteq C_i$. Assume, without loss of generality, that $C_i \subseteq C_j$. Then x, y both belong to the chain C_j, and hence x and y are comparable, whence $\mathcal{C}$ is a chain, as required. We may therefore apply (ZL)$''$ to $\mathcal{P}$ to obtain a maximal element C^* in $\mathcal{P}$.

Equivalence of Maximality Axioms: (KL)⇒(ZL)

- (KL)⇒(ZL): Let P be a nonempty ordered set in which every non-empty chain has an upper bound. By (KL), an arbitrarily chosen chain C in P is contained in a maximal chain C^*. By hypothesis, C^* has an upper bound u in P. If u were not a maximal element of P, we could find $v > u$. Clearly $v \notin C^*$, since $u \geq c$, for all $c \in C^*$. Thus, $C^* \cup \{v\}$ would be a chain strictly containing the maximal chain C^*, a contradiction.

Equivalence of Maximality Axioms: (ZL)′ ⇒(AC)

- (ZL)′ ⇒(AC): Consider the ordered set P of partial maps from I to $\bigcup_{i\in I} A_i$. By identifying maps with their graphs we may regard P as a family of sets ordered by inclusion. Let

$$\mathcal{E} = \{\pi \in P : (\forall i \in \mathrm{dom}\pi)\ \pi(i) \in A_i\}.$$

  Certainly $\mathcal{E} \neq \varnothing$, since the partial map with empty domain vacuously belongs to $\mathcal{E}$. Now let $\mathcal{C} = \{\pi_j\}_{j\in J}$ be a non-empty chain in $\mathcal{E}$. Because $\mathcal{C}$ is a chain, the partial maps π_j are consistent and the union of their graphs is the graph of a partial map, which necessarily belongs to $\mathcal{E}$. By (ZL)′, $\mathcal{E}$ has a maximal element, $f : \mathrm{dom} f \to \bigcup A_i$, say.
  - If f is a total map, it serves as the required choice function.
  - Suppose f is not total. Then there exists $k \in I \backslash \mathrm{dom} f$. Because $A_k \neq \varnothing$, there exists $a_k \in A_k$. Define g by $g(j) = \begin{cases} a_k, & \text{if } j = k \\ f(j), & \text{if } j \in \mathrm{dom} f \end{cases}$. Then $g \in \mathcal{E}$ and $g > f$. But this contradicts the maximality of f.

Inductive Ordered Sets

- An ordered set P in which every nonempty chain has an upper bound is often referred to as **inductive**.
- Contrast this with the earlier definition of P being **completely inductive**: every chain in P has a least upper bound.
- In the definition of "inductive" it is convenient to exclude the empty chain (which, of course, has every element of P as an upper bound). (ZL) and (ZL)$''$ can be restated as:

(ZL) Every non-empty inductive ordered set has a maximal element;

(ZL)$''$ Every completely inductive ordered set has a maximal element.

(ZL) In Action

- Axiom (ZL) (or more usually (ZL)$'$) is used to assert the existence of an object X which cannot be directly constructed.
- Proofs involving (ZL)$'$ follow a pattern. We let X be an object whose existence we wish to establish. We proceed as follows:
  (i) Take a non-empty family $\mathcal{E}$ of sets ordered by inclusion, in which X is a (hypothetical) maximal element;
  (ii) Check that (ZL)$'$ is applicable;
  (iii) Verify that the maximal element supplied by (ZL)$'$ has all the properties demanded of X.
- A quick review of these steps:
  - Choosing $\mathcal{E}$ is usually straightforward.
    We then have to exhibit an element of $\mathcal{E}$ to ensure $\mathcal{E} \neq \varnothing$.
  - To confirm that (ZL)$'$ applies, we need to show that the union of a non-empty chain of sets in $\mathcal{E}$ is itself in $\mathcal{E}$.
    In many (ZL) applications, $\mathcal{E}$ is an algebraic $\bigcap$-structure, and it is this fact that ensures success in this step.
  - If (iii) is non-trivial, we usually argue by contradiction.

Subsection 2

Prime and Maximal Ideals

Prime Ideals

- Let L be a lattice. Recall that a non-empty subset J of L is called an **ideal** if:
  (i) $a, b \in J$ implies $a \vee b \in J$;
  (ii) $a \in L, b \in J$ and $a \leq b$ imply $a \in J$.
- J is **proper** if $J \neq L$.
- A proper ideal J of L is said to be **prime** if $a, b \in L$ and $a \wedge b \in J$ imply $a \in J$ or $b \in J$.
- The set of prime ideals of L is denoted $\mathcal{I}_p(L)$.

  It is ordered by set inclusion.

Prime Filters

- Let L be a lattice. Recall that a non-empty subset F of L is called a **filter** if:
  - (i) $a, b \in F$ implies $a \wedge b \in F$;
  - (ii) $a \in F, b \in L$ and $a \leq b$ imply $b \in F$.
- F is **proper** if $F \neq L$.
- A proper filter F of L is said to be **prime** if $a, b \in L$ and $a \vee b \in F$ imply $a \in F$ or $b \in F$.
- The set of prime filters of L is denoted $\mathcal{F}_p(L)$.

  It is ordered by set inclusion.
- A subset J of a lattice L is a prime ideal if and only if $L \backslash J$ is a prime filter.

  Thus, it is easy to switch between $\mathcal{I}_p(L)$ and $\mathcal{F}_p(L)$.

Join Irreducibles and Prime Ideals

Lemma

Let L be a finite distributive lattice and let $a \in L$. Then the map $x \mapsto L \backslash \uparrow x$ is an order-isomorphism of $\mathcal{J}(L)$ onto $\mathcal{I}_p(L)$ that maps $\{x \in \mathcal{J}(L) : x \leq a\}$ onto $\{I \in \mathcal{I}_p(L) : a \notin I\}$.

- We have

$$\begin{array}{rll} L \backslash \uparrow x \in \mathcal{I}_p(L) & \text{iff} & \uparrow x \in \mathcal{F}_p(L) \\ & \text{iff} & (\forall y, z \in L) y \vee z \in \uparrow x \Rightarrow y \in \uparrow x \text{ or } z \in \uparrow x \\ & \text{iff} & (\forall y, z \in L) x \leq y \vee z \Rightarrow x \leq y \text{ or } x \leq z \\ & \text{iff} & x \in \mathcal{J}(L). \end{array}$$

  Hence, $\mathcal{I}_p(L) = \{L \backslash \uparrow x : x \in \mathcal{J}(L)\}$. We now know that φ maps $\mathcal{J}(L)$ onto $\mathcal{I}_p(L)$. Since $x \leq y$ if and only if $\uparrow x \supseteq \uparrow y$, φ is an order-embedding.

  We also have $(x \in \mathcal{J}(L)$ & $x \leq a)$ iff $(x \in \mathcal{J}(L)$ & $a \in \uparrow x)$ iff $(x \in \mathcal{J}(L)$ & $a \notin L \backslash \uparrow x)$ iff $a \notin I \in \mathcal{I}_p(L)$.

Join Irreducibles and Prime Ideals (Cont'd)

Corollary

Let L be a finite distributive lattice and let $a \nleq b$ in L. Then there exists $I \in \mathcal{I}_p(L)$, such that $a \notin I$ and $b \in I$.

- $a \nleq b$ iff there exists $x \in \mathcal{J}(L)$, such that $x \leq a$ and $x \nleq b$ iff there exists $x \in \mathcal{J}(L)$, such that $a \in \uparrow x$ and $b \notin \uparrow x$ iff there exists $x \in \mathcal{J}(L)$, such that $a \notin L \backslash \uparrow x$ and $b \in L \backslash \uparrow x$ iff there exists $I \in \mathcal{I}_p(L)$, such that $a \notin I$ and $b \in I$.

Corollary

Let B be a finite Boolean algebra and let $a \in B$. Then the map $x \mapsto B \backslash \uparrow x$ is a bijection of $\mathcal{A}(L)$ onto $\mathcal{I}_p(B)$ that maps $\{x \in \mathcal{A}(L) : x \leq a\}$ onto $\{I \in \mathcal{I}_p(B) : a \notin I\}$.

Maximal Ideals and Maximal Filters

- Let L be a lattice and I a proper ideal of L. Then I is said to be a **maximal ideal** if the only ideal properly containing I is L.

  In other words, I is a maximal ideal if and only if it is a maximal element in $\langle \mathcal{I}(L) \backslash \{L\}; \subseteq \rangle$.
- A **maximal filter**, also known as an **ultrafilter**, is defined dually.

Theorem

Let L be a distributive lattice with 1. Then every maximal ideal in L is prime. Dually, in a distributive lattice with 0, every ultrafilter is a prime filter.

- Let I be a maximal ideal in L and let $a, b \in L$. Assume $a \wedge b \in I$ and $a \notin I$. Define $I_a = \downarrow\{a \vee c : c \in I\}$. Then I_a is an ideal containing I and a. Because I is maximal, we have $I_a = L$. In particular $1 \in I_a$, so $1 = a \vee d$, for some $d \in I$. Then $I \ni (a \wedge b) \vee d = (a \vee d) \wedge (b \vee d)$ $= b \vee d$. Since $b \leq b \vee d$, we have $b \in I$.

Prime and Maximal Ideals in Boolean Lattices

- The preceding theorem is true whether or not L has any bounds.
- In a Boolean lattice we can do better:

Theorem

Let B be a Boolean lattice and let I be a proper ideal in B. Then the following are equivalent:

(i) I is a maximal ideal;

(ii) I is a prime ideal;

(iii) for all $a \in B$, it is the case that $a \in I$ if and only if $a' \notin I$.

(i)⇒(ii): By the preceding theorem.

(ii)⇒(iii): Note that, for any $a \in B$, we have $a \wedge a' = 0$. Because I is prime, $a \in I$ or $a' \in I$. If both a and a' belong to I then $1 = a \vee a' \in I$, a contradiction.

(iii)⇒(i): Let J be an ideal properly containing I. Fix $a \in J \backslash I$. Then $a' \in I \subseteq J$, so $1 = a \vee a' \in J$. Therefore $J = B$. Thus, I is maximal.

Ultrafilters on a Set

- Let S be a non-empty set. An ultrafilter of the Boolean lattice $\mathcal{P}(S)$ is called an **ultrafilter on** S.
- An ultrafilter on S is said to be **principal**, if it is a principal filter, and **non-principal**, otherwise.
- For each $s \in S$, the set $\{A \in \mathcal{P}(S) : s \in A\}$ is a principal ultrafilter on S, and every principal ultrafilter is of this form.
- All ultrafilters on a finite set are, of course, principal.

Characterizations of Ultrafilters on a Set

Theorem

Let $\mathcal{F}$ be a proper filter in $\mathcal{P}(S)$. Then the following are equivalent:

(i) $\mathcal{F}$ is an ultrafilter;

(ii) $\mathcal{F}$ is a prime filter;

(iii) for each $A \subseteq S$, either $A \in \mathcal{F}$ or $S \backslash A \in \mathcal{F}$;

(iv) for each $B \subseteq S$, if $A \cap B \neq \varnothing$, for all $A \in \mathcal{F}$, then $B \in \mathcal{F}$;

(v) given pairwise disjoint sets $A_1, \ldots, A_n$, such that $A_1 \cup \cdots \cup A_n = S$, there exists a unique j, such that $A_j \in \mathcal{F}$.

- For the proof, one shows (ii)⇒(v)⇒(iii)⇒(iv)⇒(i)⇒(ii).

Characterizations of Ultrafilters on a Set (Proof)

(ii)⇒(v): Suppose that $\mathcal{F}$ is prime. Let $A_1, \ldots, A_n \subseteq S$ be disjoint subsets, such that $A_1 \cup \cdots \cup A_n = S$. Since $S \in \mathcal{F}$ and $\mathcal{F}$ is prime, there exists $j < n$, such that $A_j \in \mathcal{F}$. If there exist $i, j < n$, $i \neq j$, such that $A_i, A_j \in \mathcal{F}$, then $\varnothing = A_i \cap A_j \in \mathcal{F}$, a contradiction. Thus, there exists unique $j < n$, such that $A_j \in \mathcal{F}$.

(v)⇒(iii): Suppose (v) holds. Let $A \subseteq S$, such that $A \notin \mathcal{F}$. Since $A \cup (S \backslash A) = S$, we get, by hypothesis, $S \backslash A \in \mathcal{F}$.

(iii)⇒(iv): Assume that (iii) holds. Let $B \subseteq S$, such that $A \cap B \neq \varnothing$, for all $A \in \mathcal{F}$. Then $S \backslash B \notin \mathcal{F}$, since $(S \backslash B) \cap B = \varnothing$. Thus, by hypothesis, $B \in \mathcal{F}$.

(iv)⇒(i): Suppose (iv) holds. Let $B \subseteq S$, such that $B \notin \mathcal{F}$. Then, for all $A \in \mathcal{F}$, $A \nsubseteq B$. Hence, for all $A \in \mathcal{F}$, $A \cap (S \backslash B) \neq \varnothing$. By hypothesis, $S \backslash B \in \mathcal{F}$.

(i)⇒(ii): This has already been shown.

Existence of Prime Ideals: (BPI) and (DPI)

- Consider a Boolean lattice B.
- The preceding theorem implies that a prime ideal in B is just a maximal element of $\langle \mathcal{I}(B) \backslash \{B\}; \subseteq \rangle$.
- The existence of maximal elements has closer affinities with set theory than with lattice theory. To circumvent such a treatment, we resort to the following:
  - The statements (BPI) and (DPI) introduced below assert the existence of certain prime ideals.
    - On one level, (BPI) and (DPI) may be taken as axioms, whose lattice-theoretic implications we pursue.
    - At a deeper level, we show how (BPI) and (DPI) may be derived from (ZL).

  The difference between these two philosophies is less than might appear, as will be indicated.

(DPI) and (BPI)

(DPI) Given a distributive lattice L and an ideal J and a filter G of L, such that $J \cap G = \varnothing$, there exist $I \in \mathcal{I}_p(L)$ and $F = L \backslash I \in \mathcal{F}_p(L)$, such that $J \subseteq I$ and $G \subseteq F$.

(BPI) Given a proper ideal J of a Boolean lattice B, there exists $I \in \mathcal{I}_p(B)$, such that $J \subseteq I$.

Theorem

(ZL) implies (BPI).

- Let B be a Boolean lattice and J be a proper ideal of B. We apply the special case (ZL)′ of (ZL) to the set $\mathcal{E} := \{K \in \mathcal{I}(B) : B \neq K \supseteq J\}$, ordered by inclusion.
  - The set $\mathcal{E}$ contains J, and so is non-empty.
  - Let $C = \{K_\lambda : \lambda \in \Lambda\}$ be a chain in $\mathcal{E}$. We require $K := \bigcup_{\lambda \in \Lambda} K_\lambda \in \mathcal{E}$. Certainly $K \neq B$, $K \supseteq J$ and K is a down-set. If $a, b \in K$, $a \in K_\lambda$ and $b \in K_\mu$, for some $\lambda, \mu \in \Lambda$. Since $\mathcal{C}$ is a chain, assume $K_\lambda \subseteq K_\mu$. Then $a, b \in K_\mu$, so $a \vee b \in K_\mu \subseteq K$.
  - The maximal element of $\mathcal{E}$ given by (ZL)′ is the required maximal ideal.

(ZL) Implies (DPI)

- For distributive lattices, we have:

Theorem

(ZL) implies (DPI).

- Take L, G and J as in the statement (DPI). Define $\mathcal{E} = \{K \in \mathcal{I}(L) : K \supseteq J \text{ and } K \cap G = \varnothing\}$. We use a similar argument to the one in the preceding theorem to show that $\langle \mathcal{E}; \subseteq \rangle$ has a maximal element I.

  Let $\mathcal{K} = \{K_\lambda : \lambda \in \Lambda\}$ be a chain in $\mathcal{E}$. Set $K = \bigcup_{\lambda \in \Lambda} K_\lambda$. Clearly, $J \subseteq K$. Moreover, $K \cap G = \bigcup_{\lambda \in \Lambda} K_\lambda \cap G = \bigcup_{\lambda \in \Lambda} (K_\lambda \cap G) = \varnothing$. Since every set in $\mathcal{E}$ is a down-set, the same holds for K. To see that K is an ideal, let $a, b \in K$. Then, there exist $\lambda, \mu \in \Lambda$, such that $a \in K_\lambda$ and $b \in K_\mu$. Since $\mathcal{K}$ is a chain, either $K_\lambda \subseteq K_\mu$ or $K_\mu \subseteq K_\lambda$. Assume, without loss of generality, that the former holds. Then, $a, b \in K_\mu$. Since K_μ is an ideal, $a \vee b \in K_\mu$. Therefore, $a \vee b \in K$. By (ZL)$'$, we conclude that $\langle \mathcal{E}, \subseteq \rangle$ has a maximal element.

(ZL) Implies (DPI) (Cont'd)

- We showed that $\langle \mathcal{E}; \subseteq \rangle$ has a maximal element I.

  It remains to prove that I is prime.

  Suppose $a, b \in L \backslash I$, but $a \wedge b \in I$. Because I is maximal, any ideal properly containing I is not in $\mathcal{E}$. Consequently, $I_a = \downarrow\{a \vee c : c \in I\}$ (the smallest ideal containing I and a) intersects G. Therefore there exists $c_a \in I$, such that $a \vee c_a$ is above an element of G. Hence, since G is an up-set, $a \vee c_a \in G$. Similarly, we can find $c_b \in I$, such that $b \vee c_b \in G$. Now consider

$$(a \wedge b) \vee (c_a \vee c_b) = ((a \vee c_a) \vee c_b) \wedge ((b \vee c_b) \vee c_a).$$

  The right-hand side is in G, since G is a filter, while the left is in I, since I is an ideal. Thus, $I \cap G \neq \varnothing$, a contradiction.

(BPI) and (DPI) in Distributive Lattices with 1

- When L is a distributive lattice with 1, we may take $G = \{1\}$ in (DPI). Then (DPI) implies the existence of a maximal ideal of L containing a given proper ideal J. So (DPI), restricted to Boolean lattices, yields (BPI) as a special case.

  Much less obviously, (BPI)$\Rightarrow$(DPI). This is proved by constructing an embedding of a given distributive lattice into a Boolean lattice, to which (BPI) is applied.

  Hence (BPI) and (DPI) are equivalent.

A Choice of Axioms

- We proved that (ZL) is equivalent to the Axiom of Choice (AC) Another equivalent statement is:

(DMI) every distributive lattice with 1, which has more than one element, contains a maximal ideal.

  It is easy to derive (DMI) from (ZL).

  Conversely it can be proved that (AC) can be derived from (DMI), applied to a suitable lattice of sets.

- By contrast, (BPI) and (DPI) belong to a family of conditions known to be equivalent to the choice principle $(AC)_F$ (asserting that every family of non-empty finite sets has a choice function).
  - It is known that $(AC)_F$ is strictly weaker than (AC), so that it is not true that (DPI) implies (DMI).
  - However, $(AC)_F$ is not derivable within Zermelo-Fraenkel set theory. To obtain results such as (DPI) and (BPI) some additional axiom must be added (whether (AC), (ZL), or (DPI) itself, is a matter of choice). Thus, our suggestion that readers ignorant of (ZL) should take (DPI) as a hypothesis has a sound logical basis.

(BUF) and Relations Between Axioms

- We finally introduce

(BUF) Given a proper filter G of a Boolean lattice B, there exists $F \in \mathcal{F}_p(B)$, such that $G \subseteq F$.

- A proper filter (an ultrafilter) of a Boolean lattice B is a proper ideal (a maximal ideal) of B^∂ (which is also a Boolean lattice).

  Thus, the statements (BPI) and (BUF) are equivalent.

- We summarize the established relations between the various conditions:

$$\begin{array}{ccccccc}
\text{(AC)} & \Longleftrightarrow & \text{(ZL)} & \Longrightarrow & \text{(BPI)} & \Longleftrightarrow & \text{(BUF)} \\
/ & & & & / & & / \\
/ & & & & / & & / \\
\text{(DMI)} & & \Longrightarrow & & \text{(DPI)} & \Longleftrightarrow & \text{(AC)}_F
\end{array}$$

Subsection 3

Power Set Algebras and Down-Set Lattices

Extended the Representations to the Infinite Case

- The representation theorems in the finite case show that:
  - Any finite Boolean algebra is isomorphic to a powerset;
  - Any finite distributive lattice is isomorphic to the lattice of down-sets of an ordered set.
- We cannot expect these statements to remain universally true when we delete the word "finite": We already gave an example of a Boolean algebra which is not isomorphic to a powerset algebra.
- We will use the results of the preceding section to show that every distributive lattice has a concrete representation as a lattice of sets, or, in a Boolean case, an algebra of sets.
- Then we will characterize among Boolean algebras and bounded distributive lattices those which are, respectively, powerset algebras and down-set lattices.

Lattice and Power Set of Prime Ideals

Lemma

Let L be a lattice and let $X = \mathcal{I}_p(L)$. Then the map $\eta : L \to \mathcal{P}(X)$ defined by $\eta : a \mapsto X_a := \{I \in \mathcal{I}_p(L) : a \notin I\}$ is a lattice homomorphism.

- We must show $X_{a \vee b} = X_a \cup X_b$ and $X_{a \wedge b} = X_a \cap X_b$, for all $a, b \in L$. Take $I \in \mathcal{I}_p(L)$. Since I is an ideal, $a \vee b \in I$ if and only if $a \in I$ and $b \in I$. Since I is prime, $a \wedge b \in I$ if and only if $a \in I$ or $b \in I$. Thus, we have

$$\begin{aligned} X_{a \vee b} &= \{I \in \mathcal{I}_p(L) : a \vee b \notin I\} \\ &= \{I \in \mathcal{I}_p(L) : a \notin I \text{ or } b \notin I\} \\ &= X_a \cup X_b. \end{aligned}$$

  Similarly,

$$\begin{aligned} X_{a \wedge b} &= \{I \in \mathcal{I}_p(L) : a \wedge b \notin I\} \\ &= \{I \in \mathcal{I}_p(L) : a \notin I \text{ and } b \notin I\} \\ &= X_a \cap X_b. \end{aligned}$$

Characterization of Distributivity

- We would like η to give a faithful copy of L in the lattice $\mathcal{P}(\mathcal{I}_p(L))$:
  - This cannot be proven without the additional hypothesis of distributivity, because a lattice of sets must be distributive.
  - It turns out (DPI) is exactly what is needed to ensure that a distributive lattice L has enough prime ideals for $\eta : L \to \mathcal{P}(\mathcal{I}_p(L))$ to be an embedding.

Theorem

Let L be a lattice. Then the following are equivalent:

(i) L is distributive;

(ii) given an ideal J of L and a filter G of L with $J \cap G = \varnothing$, there exists a prime ideal I, such that $J \subseteq I$ and $I \cap G = \varnothing$;

(iii) given $a, b \in L$, with $a \nleq b$, there exists a prime ideal I, such that $a \notin I$, $b \in I$;

(iv) the map $\eta : a \mapsto X_a := \{I \in \mathcal{I}_p(L) : a \notin I\}$ is an embedding of L into $\mathcal{P}(\mathcal{I}_p(L))$;

(v) L is isomorphic to a lattice of sets.

Proving the Characterization of Distributivity

(i)⇒(ii): By (DPI).

(ii)⇒(iii): Suppose (ii) holds. Let $a, b \in L$, such that $a \nleq b$. Then, $\uparrow a$ is a filter of L, $\downarrow b$ is an ideal of L and $\uparrow a \cap \downarrow b = \varnothing$. By hypothesis, there exists a prime ideal $I \in \mathcal{I}_p(L)$, such that $\uparrow a \cap I = \varnothing$ and $\downarrow b \subseteq I$. Thus, $a \notin I$ and $b \in I$.

(iii)⇒(iv): By the preceding lemma, it suffices to show that, for all $a, b \in L$, $a \nleq b$ implies $X_a \nsubseteq X_b$. But, if $a \nleq b$, then, by hypothesis, there exists $I \in \mathcal{I}_p(L)$, such that $a \notin I$ and $b \in I$. Thus, $I \in X_a$, but $I \notin X_b$, whence $X_a \nsubseteq X_b$.

(iv)⇒(v): Trivial.

(v)⇒(i): Trivial.

The Case of Boolean Algebras

Theorem

Let B be a Boolean algebra. Then:

(i) Given a proper ideal J of B, there exists a maximal ideal $I \in \mathcal{I}_p(B)$ with $J \subseteq I$;

(ii) Given $a \neq b$ in B, there exists a maximal ideal $I \in \mathcal{I}_p(B)$, such that I contains one and only one of a and b;

(iii) The map $\eta : a \mapsto X_a := \{I \in \mathcal{I}_p(B) : a \notin I\}$ is a Boolean algebra embedding of B into the powerset algebra $\mathcal{P}(\mathcal{I}_p(L))$.

(ii) (i) holds by the (BPI). Take $a, b \in B$, with $a \neq b$. We may assume $a \nleq b$. This gives $1 \neq a' \vee b$. Apply (i) with $J = \downarrow(a' \vee b)$. Any prime ideal I containing J contains b, but not a.

(iii) The map $\eta : a \mapsto X_a := \{I \in \mathcal{I}_p(B) : a \notin I\}$ is a lattice homomorphism. $X_0 = \varnothing$ because each prime ideal contains 0. $X_1 = X$, since each prime ideal is proper. So, η is a Boolean algebra homomorphism. Since (ii) holds, η is also one-to-one.

Infinite Distributive Laws

- Note that since lattices of sets are distributive, complete lattices of sets, and in particular powersets, must satisfy a very strong distributive law:
- **Infinite Distributive Laws**: A complete lattice L is said to be **completely distributive** if, for any doubly indexed subset $\{x_{ij}\}_{i\in I, j\in J}$ of L, we have

$$\bigwedge_{i\in I}(\bigvee_{j\in J} x_{ij}) = \bigvee_{\alpha:I\to J}(\bigwedge_{i\in I} x_{i\alpha(i)}). \qquad \text{(CD)}$$

- The formulation of (CD) is simply a formal way of saying that any meet of joins is converted into the join of all possible elements obtained by taking the meet over $i \in I$ of elements x_{ik}, where k depends on i; the functions $\alpha : I \to J$ do the job of picking out the indices k.
- The law (CD) can be shown to be self-dual, as distributivity is: L satisfies (CD) if and only if L^∂ does.

Join- and Meet-Infinite Distributive Laws

- Certainly any powerset $\langle \mathcal{P}(X); \subseteq \rangle$ satisfies (CD).
- So does any complete lattice of sets, and in particular any lattice $\langle \mathcal{O}(P); \subseteq \rangle$, where P is an ordered set.
- As an instance of (CD), obtained by taking $I = \{1, 2\}, x_{1j} = x$ and $x_{2j} = y_j$, for all $j \in J$, we have the **Join-Infinite Distributive Law**: for any subset $\{y_j\}_{j \in J}$ of L and any $x \in L$,

$$x \wedge \bigvee_{j \in J} y_j = \bigvee_{j \in J} x \wedge y_j. \qquad \text{(JID)}$$

- The dual condition is the **Meet-Infinite Distributive Law**, (MID), and it too holds in any completely distributive lattice.

$$x \vee \bigwedge_{j \in J} y_j = \bigwedge_{j \in J} x \vee y_j. \qquad \text{(MID)}$$

Power Set Boolean Algebras

Theorem

Let B be a Boolean algebra. Then the following are equivalent:

(i) $B \cong \mathcal{P}(X)$, for some set X;

(ii) B is complete and atomic;

(iii) B is complete and completely distributive.

(i)⇒(ii) & (iii): is clear.

(ii)⇒(i): The map $\eta : a \mapsto \{x \in \mathcal{A}(B) : x \leq a\}$ is a Boolean algebra isomorphism mapping B onto $\mathcal{P}(\mathcal{A}(B))$. Thus, (ii) implies (i).

(iii)⇒(ii): We apply (CD) with $I = B$ and $J = \{\pm 1\}$, with

$$x_{ij} = \begin{cases} i, \text{if } j = 1 \\ i', \text{if } j = -1 \end{cases}.$$

Note that, for any i, we have $\bigvee_{j \in J} x_{ij} = i \vee i' = 1$.

Power Set Boolean Algebras (Cont'd)

- We saw that $\bigvee_{j \in J} x_{ij} = 1$.

  Therefore, by (CD),

$$\bigvee_{\alpha:I \to J} (\bigwedge_{i \in I} x_{i\alpha(i)}) = \bigwedge_{i \in I} (\bigvee_{j \in J} x_{ij}) = 1.$$

  Let $y \in B$. Then by (JID) we have

$$\bigvee_{\alpha:I \to J} (y \wedge \bigwedge_{i \in I} x_{i\alpha(i)}) = y \wedge \bigvee_{\alpha:I \to J} (\bigwedge_{i \in I} x_{i\alpha(i)}) = y.$$

  Claim: $z_\alpha := y \wedge \bigwedge_{i \in I} x_{i\alpha(i)}$ is an atom whenever it is nonzero.

  Suppose $0 < u \le z_\alpha$. Then $u \le x_{u\alpha(u)}$. This forces $\alpha(u) = 1$ since otherwise $u \le u'$ in contradiction to $u \ne 0$. But $\alpha(u) = 1$ gives $x_{u\alpha(u)} = u$, so that $u \ge z_\alpha$. Therefore $u = z_\alpha$, so that $z_\alpha \in \mathcal{A}(B)$, as claimed.

  We conclude that B is atomic.

Instances of Algebraic Lattices

- Characterizing down-set lattices requires substantially more work.
- We note that any lattice $\mathcal{O}(P)$ is algebraic.
- $\langle \mathbb{N}_0; \preccurlyeq \rangle$ fails (JID).
- By contrast, any bounded distributive lattice which satisfies (ACC) (respectively (DCC)) does satisfy (JID) (respectively (MID));

Properties of Algebraic Lattices

Proposition

Let L be an algebraic lattice.

(i) Meet distributes over directed joins in L, that is, $x \wedge \bigsqcup\{y_i : i \in I\} = \bigsqcup\{x \wedge y_i : i \in I\}$.

(ii) If L is distributive, then it satisfies (JID).

(i) Let $D = \{y_i\}_{i \in I}$ be directed. It is easy to see that $\{x \wedge y_i\}_{i \in I}$ is also directed. Note that $x \wedge \bigsqcup\{y_i : i \in I\} \geq \bigsqcup\{x \wedge y_i : i \in I\}$, since the left-hand side is an upper bound for $\{x \wedge y_i\}_{i \in I}$. Suppose for a contradiction that the inequality is strict. Because L is algebraic, this implies that there exists $k \in F(L)$, such that $k \leq x \wedge \bigsqcup\{y_i : i \in I\}$ but $k \nleq \bigsqcup\{x \wedge y_i : i \in I\}$. Then $k \leq x$ and $k \leq \bigsqcup D$, from which we get $k \leq y_j$, for some j. But then $k \leq x \wedge y_j \leq \bigsqcup\{x \wedge y_i\}$, a contradiction.

(ii) For any non-empty set S, $\bigvee S = \bigsqcup\{\bigvee F : \varnothing \neq F \subseteq S\}$. But meet distributes over directed joins and over finite joins. Hence, meet distributes over arbitrary joins.

Completely Join-Irreducible Elements

- After seeing various analogues, (CD), (JID) and (MID), of the distributive laws (D) and $(D)^\partial$ that a complete lattice may satisfy, we visit analogues of join- and meet-irreducible elements:
- An element a of a complete lattice is called **completely join-irreducible** if $a = \bigvee S$ implies that $a \in S$, for every subset S of L; in particular, $a \neq 0$ (take $S = \varnothing$).
- The element a is called **completely join-prime** if $a \leq \bigvee S$ implies $a \leq s$, for some $s \in S$.
- We denote the set of completely join-prime elements in L by $\mathcal{J}_p(L)$.

Completely Join-Prime and Completely Join-Irreducibles

Lemma

Let L be a complete lattice.

(a) Every completely join-prime element is completely join-irreducible;

(b) In the presence of (JID), every completely join-irreducible element is completely join-prime.

(a) Assume that a is completely join-prime. Let $S \subseteq L$, such that $a = \bigvee S$. Then $a \leq \bigvee S$. Since a is completely join-prime, there exists $s \in S$, such that $a \leq s$. But, by hypothesis, $s \leq a$, whence $a = s \in S$. Therefore, a is completely join-irreducible.

(b) Assume L satisfies the (JID) and a is completely join-irreducible in L. Let $S \subseteq L$, such that $a \leq \bigvee S$. Then $a = a \wedge \bigvee S = \bigvee_{s \in S}(a \wedge s)$. Since a is completely join-irreducible, there exists $s \in S$, such that $a = a \wedge s$, i.e., $a \leq s$. Therefore, a is completely join-prime.

Completely Meet-Irreducible Elements

- Let L be a complete lattice.
- An element a of a complete lattice is called **completely meet-irreducible** if $a = \bigwedge S$ implies that $a \in S$, for every subset S of L; in particular, $a \neq 1$ (take $S = \varnothing$).
- The element a is called **completely meet-prime** if $\bigwedge S \leq a$ implies $s \leq a$, for some $s \in S$.
- We denote the set of completely meet-prime elements in L by $\mathcal{M}_p(L)$.
- It is easy to see that every completely meet-prime element is completely meet-irreducible.

  In the presence of (MID), every completely meet-irreducible element is completely meet-prime.

Weak Atomicity

- We say that a lattice L is **weakly atomic** if, given $x < y$ in L, there exist $a, b \in L$, such that $x \leq b \lessdot a \leq y$.
- This condition is satisfied in any down-set lattice and it is self-dual.

Proposition

Let L be a complete lattice.

(i) Assume that L is algebraic. Then the completely meet-irreducible elements are meet-dense in L.

(ii) Assume that L satisfies (JID) and is weakly atomic. Then the completely meet-irreducible elements are meet-dense in L.

(ii) To prove meet-density of a set Q it suffices to show that if $s, t \in L$, with $t > s$, then there exists $m \in Q$, with $m \geq s$ and $m \ngeq t$.
Assume L satisfies (JID) and is weakly atomic. Take $t > s$. Then there exist $p, q \in L$, such that $t \geq q \gtrdot p \geq s$. Define $P = \{x \in L : x \geq p \text{ and } x \ngeq q\}$.

Weak Atomicity (Cont'd)

- We set $P = \{x \in L : x \geq p \text{ and } x \ngeq q\}$.
  - The set P contains p and so $P \neq \varnothing$.
  - Let C be a non-empty chain in P, and suppose for a contradiction that $\bigvee C \notin P$. This means that $\bigvee C \geq q$. Invoking (JID) we have $\bigvee_{x \in C}(x \wedge q) = q$. If we had $x \wedge q \leq p$, for all $x \in C$, then $\bigvee_{x \in C}(x \wedge q) \leq p$, a contradiction. Pick $x \in C$, such that $x \wedge q \nleq p$. Then, using the contrapositive of the Connecting Lemma, $p < (x \wedge q) \vee p$. By distributivity, which is implied by (JID), $p < (x \wedge q) \vee p = (x \vee p) \wedge (q \vee p) = x \wedge q \leq q$. Hence, because $q \succ p$, we have $x \wedge q = q$, a contradiction.

  By (ZL), P has a maximal element, m say, and this satisfies $m \geq p$ and $m \ngeq q$. By transitivity, $m \geq s$ and $m \ngeq t$.

  Finally suppose for a contradiction that $m = \bigwedge S$, but that $m \neq y$, for every $y \in S$. Because m is maximal in P, every $y \in S$ lies outside P. But $y \geq m \geq p$, so we must have $y \geq q$, for all $y \in S$. But then $m = \bigwedge S \geq q$, a contradiction. Hence m is completely meet-irreducible.

Algebraicity Implies Weak Atomicity

Proposition

Every algebraic lattice L is weakly atomic.

- Let $x < y$ in L. Recall that $K := [x, y]$ is an algebraic lattice.

  Claim: If $a \in K$ is finite and $x < a$, then there exists $b \in K$, such that $x \leq b \prec a$.

  Let $a \in K$ be finite and $x < a$. Consider the set $P = \{b \in K : x \leq b < a\}$. Since $x \in P$, $P \neq \varnothing$. Let $C \subseteq P$ be a nonempty chain in P. Since, for all $c \in C$, $x \leq c < a$, we get that $x \leq \bigvee C \leq a$. If $\bigvee C = a$, then, since a is finite, there would exist $c \in C$, such that $a = c$, a contradiction. Hence, $x \leq \bigvee C < a$. Thus, every nonempty chain C in P has an upper bound in P. By (ZL), P has a maximal element b.

  To see that b is a lower cover of a, suppose that there exists $z \in K$, such that $b \leq z < a$. Then $z \in P$. Since b is maximal in P, we get that $z = b$. Thus, b is indeed a lower cover of a.

Characterization of Down-Set Lattices

Theorem (Characterization of Down-Set Lattices)

Let L be a lattice. Then the following are equivalent:

(i) L is isomorphic to $\mathcal{O}(P)$ for some ordered set P;

(ii) L is isomorphic to a complete lattice of sets;

(iii) L is distributive and both L and L^∂ are algebraic;

(iv) L is complete, L satisfies (JID) and the completely join-irreducible elements are join-dense;

(v) the map $\eta : x \mapsto \{x \in \mathcal{J}_p(L) : x \leq a\}$ is an isomorphism from L onto $\mathcal{O}(\mathcal{J}_p(L))$;

(vi) L is completely distributive and L is algebraic;

(vii) L is complete,satisfies (JID) and (MID) and is weakly atomic.

Proof of the Chain of Implications

(i)⇒(ii): Trivially;

(ii)⇒(iii): Trivially;

(iii)⇒(iv): We proved that, if an algebraic lattice L is distributive, then it satisfies the (JID). We also proved that, if L is algebraic, then the completely meet-irreducible elements of L are meet-dense in L. By the dual, we get (iv).

(iv)⇒(v): Analogous to the Birkhoff Representation Theorem;

(v)⇒(i): Trivially;

(ii)⇒(vi): We have seen that any complete lattice of sets is completely distributive and algebraic;

(vi)⇒(vii): By the preceding proposition;

(vii)⇒(iv): We have seen that, if a complete lattice satisfies the (JID) and is weakly atomic, then the completely meet-irreducible elements are meet-dense. The dual gives (iv).